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Powers of a rational number modulo 1 cannot lie in a small interval

by

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1. Introduction. Let throughout $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of real numbers, integers and positive integers, respectively. We will denote by $[x]$ and $\{x\}$ the integral part and the fractional part of $x \in \mathbb{R}$, respectively. For an interval $[s, s+t) \subset [0,1)$ and two integers p, q, where $1 < q < p$, put

$$Z_{p/q}(s, s+t) = \{\xi \neq 0 : s \leq \{\xi(p/q)^n\} < s+t \text{ for all integer } n \geq 0\}.$$

In [14] Mahler asked whether the set $Z_{3/2}(0, 1/2)$ is empty or not. A hypothetical $\xi \in Z_{3/2}(0, 1/2)$ is called a *Z-number.* It seems very likely that Z-numbers do not exist. An important step towards solution of this problem has been made by Flatto, Lagarias and Pollington [12] (see also [11]). It was proved in [12] that for coprime positive integers $p > q > 1$ and any $\xi \neq 0$ the inequality

$$\limsup_{n\to\infty}\{\xi(p/q)^n\} - \liminf_{n\to\infty}\{\xi(p/q)^n\} \geq 1/p \tag{1}$$

holds. A generalization of (1) to powers of algebraic numbers is given in [9]. The case of positive integers, namely, $p \geq 2$, $q = 1$ was studied in [7].

Inequality (1) implies that the fractional parts $\{\xi(p/q)^n\}$, $n = 0, 1, 2, \ldots,$ cannot lie in an interval of length strictly smaller than $1/p$. Can they all lie in an interval of length $1/p$? This small step towards Mahler's problem turns out to be very difficult. It was shown in [12] that the set of $s \in [0, 1-1/p]$ for which $Z_{p/q}(s, s+1/p)$ is empty is everywhere dense in $[0, 1-1/p]$. Naturally, it was conjectured that $Z_{p/q}(s, s+1/p)$ is empty for *each* $s \in [0, 1-1/p]$ (see p. 138 in [12]).

This problem is still open, although Bugeaud has made some progress in this direction in [6]. He was able to prove that $Z_{p/q}(s, s+1/p)$ is empty for *almost all* $s \in [0, 1-1/p]$. Moreover, he showed that the set $Z_{3/2}(s, s+1/3)$

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is empty for

$$s \in \{0\} \cup [8/57, 4/19] \cup [4/15, 2/5] \cup [26/57, 10/19] \cup \{2/3\}.$$

In this paper, we prove the set $Z_{p/q}(s, s + 1/p)$ to be indeed empty for *each* $s \in [0, 1-1/p]$ provided that p, q are integers satisfying $1 < q < p < q^2$. More precisely, we prove the following:

Theorem 1. *Let p, q be two coprime integers satisfying $1 < q < p < q^2$, and let I be a closed subinterval of length $1/p$ of the torus $\mathbb{R}/\mathbb{Z}$. Then for each real number $\xi \neq 0$ we have $\{\xi(p/q)^n\} \notin I$ for infinitely many $n \in \mathbb{N}$.*

Of course, Theorem 1 implies that the set $Z_{p/q}(s, s+1/p)$ is empty if $1 < q < p < q^2$ and $s \in [0, 1-1/p]$. In particular, the number $p/q = 3/2$ satisfies the condition $p < q^2$. So the most interesting application of Theorem 1 is that the set $Z_{3/2}(s, s + 1/3)$ is empty for every $s \in [0, 2/3]$. This solves the problem considered in Corollary 1.4a of [12] and Corollary 1 of [6].

2. Auxiliary results. We shall need some terminology which is usually used in combinatorics on words (see, e.g., [2], [4], [13]). Any sequence (finite or infinite) of letters of an alphabet A is called a *word*. Any string of consecutive letters of a word is called its *factor*. A string of letters starting from the first letter is called a *prefix*. Let $p(\mathbf{w}, m)$ be the number of distinct factors of length m occurring in the word $\mathbf{w}$. By an old result of Morse and Hedlund [15], every infinite word $\mathbf{w} = w_1w_2w_3\ldots$ is either periodic (which means that there exist $n_0, t \in \mathbb{N}$ such that $w_{n+t} = w_n$ for every $n \geq n_0$) or $p(\mathbf{w}, m) \geq m + 1$ for each $m \in \mathbb{N}$. Every infinite word which is not periodic is called *aperiodic*.

An infinite word $\mathbf{w}$ is called *Sturmian* if $p(\mathbf{w}, m) = m + 1$ for every $m \in \mathbb{N}$. In particular, every Sturmian word is over two letters. Below we shall use the fact that an aperiodic word on the alphabet $A = \{U, V\}$ is Sturmian if and only if for any finite word $\mathbf{u}$ on A either $U\mathbf{u}U$ or $V\mathbf{u}V$ is not a factor of $\mathbf{w}$.

Fix two relatively prime integers $p > q > 1$ and two real numbers $\xi \neq 0$ and ν. Set

$$x_n = [\xi(p/q)^n + \nu] \quad \text{and} \quad y_n = \{\xi(p/q)^n + \nu\}.$$

Let also

$$s_n = qx_{n+1} - px_n \quad \text{and} \quad t_n = -qy_{n+1} + py_n. \tag{2}$$

From $(p/q)(x_n + y_n - \nu) = x_{n+1} + y_{n+1} - \nu$ it follows that

$$s_n = t_n - (p-q)\nu. \tag{3}$$

Using $-q < t_n < p$ we derive that

$$-q + (p-q)\nu < s_n < p - (p-q)\nu$$

for each $n \geq 0$. Let

$$\mathcal{A} = \mathcal{A}(p, q, \nu) = \mathbb{Z} \cap (-q + (p-q)\nu, p - (p-q)\nu).$$

With this notation, we have the following:

LEMMA 2. *For relatively prime $p > q > 1$ and arbitrary $\xi \neq 0$ and ν, the word $\mathbf{w} = s_0 s_1 s_2 s_3 \ldots$ is an aperiodic word in the finite alphabet $\mathcal{A}$.*

This is exactly Lemma 2 of [10]. (The fact that in [10] it is stated only for $\xi > 0$ is irrelevant.) Lemma 2 combined with (3) implies that the word $t_0 t_1 t_2 t_3 \ldots$ is an aperiodic word on the finite alphabet $\mathcal{A} + (p-q)\nu$.

It is easy to see that, for any $m \in \mathbb{N}$, we have $x_{n+m} = (p/q)x_{n+m-1} + s_{n+m-1}/q$. Expressing x_{n+m-1} by x_{n+m-2} and so on (up to x_n), we find that

$$x_{n+m} = (p/q)^m x_n + ((p/q)^{m-1} s_n + (p/q)^{m-2} s_{n+1} + \cdots + s_{n+m-1})/q. \tag{4}$$

Analogously, using $t_n = s_n + (p-q)\nu$ for $n \geq 0$, we derive that

$$y_{n+m} = (p/q)^m y_n - ((p/q)^{m-1} t_n + (p/q)^{m-2} t_{n+1} + \cdots + t_{n+m-1})/q. \tag{5}$$

Our next statement also holds for arbitrary coprime integers $p > q > 1$, but ν will be selected in a special way:

THEOREM 3. *Let $p > q > 1$ be two coprime integers, and let $\xi \neq 0$ be a real number. Suppose $I = [s, s + 1/p] \pmod 1$, where $0 \leq s < 1$, is a closed subinterval of the torus $\mathbb{R}/\mathbb{Z}$ such that $\{\xi(p/q)^n\} \in I$ for each integer $n \geq 0$. Then the word $s_0 s_1 s_2 s_3 \ldots$, where*

$$s_n = q[\xi(p/q)^{n+1} - s] - p[\xi(p/q)^n - s]$$

for $n \geq 0$, is a Sturmian word on the two-symbol alphabet $\{k, k+1\}$ for some $k \in \mathbb{Z}$.

Proof. Take $\nu = -s$. Then

$$0 \leq y_n = \{\xi(p/q)^n + \nu\} = \{\xi(p/q)^n - s\} \leq 1/p$$

for each $n \geq 0$. Suppose first that the word $\mathbf{w} = s_0 s_1 s_2 s_3 \ldots$ contains at least three distinct letters (in this case simply integers). Then there exist $u, v \in \mathbb{Z}$ such that, for some $i, j \geq 0$, we have $s_i = u$, $s_j = v$ with $u + 2 \leq v$. Then, using (2), (3) with $n = i$ and the inequalities $y_i \geq 0$ and $y_{i+1} \leq 1/p$, we deduce that

$$-u = -py_i + qy_{i+1} + (p-q)\nu \leq q/p + (p-q)\nu.$$

Similarly, from (2), (3) with $n = j$, we have

$$v = py_j - qy_{j+1} - (p-q)\nu \leq 1 - (p-q)\nu.$$

Adding these two inequalities, we obtain

$$2 \leq v - u \leq q/p + (p-q)\nu + 1 - (p-q)\nu = q/p + 1 < 2,$$

which is impossible. Hence $\mathbf{w}$ must be a word on an alphabet of two letters which are consecutive integers, say, k and $k+1$, where $k \in \mathbb{Z}$.

By Lemma 2, we already know that the word $\mathbf{w}$ is aperiodic. If $\mathbf{w}$ is not Sturmian, then there is a word $\mathbf{u}$ on the alphabet $\{k, k+1\}$ such that $(k+1)\mathbf{u}(k+1)$ and $k\mathbf{u}k$ are both factors of $\mathbf{w}$ (see, e.g., Proposition 2.1.3 and Theorem 2.1.5 in [13]). Suppose that the factors $k\mathbf{u}k$ and $(k+1)\mathbf{u}(k+1)$ of length m start at the ith and jth places of $\mathbf{w} = s_0s_1s_2s_3\ldots$, where $i, j \geq 0$. Setting in (5) $n = j$ and $n = i$ and subtracting the second equality from the first, we obtain

$$y_{j+m} - y_{i+m} = (p/q)^m(y_j - y_i) - ((p/q)^{m-1} + 1)/q.$$

Since $y_i, y_{j+m} \geq 0$ and $y_j, y_{i+m} \leq 1/p$, this implies that

$$((p/q)^{m-1} + 1)/q = (p/q)^m(y_j - y_i) - y_{j+m} + y_{i+m} \leq ((p/q)^m + 1)/p.$$

Multiplying by p we obtain $(p/q)^m + p/q \leq (p/q)^m + 1$, i.e., $p \leq q$, a contradiction. Hence $\mathbf{w}$ must be a Sturmian word over the alphabet $\{k, k+1\}$. ∎

3. Proof of Theorem 1. Suppose there is a closed subinterval $I = [s, s+1/p] \pmod 1$, where $0 \leq s < 1$, of the torus $\mathbb{R}/\mathbb{Z}$ and some $\xi \neq 0$ such that $\{\xi(p/q)^n\} \in I$ for each $n \geq n_1$. On replacing ξ by $\xi(p/q)^{n_1}$, we can assume that $\{\xi(p/q)^n\} \in I$ for each integer $n \geq 0$. By Theorem 3, $\mathbf{w} = s_0s_1s_2s_3\ldots$, where

$$s_n = q[\xi(p/q)^{n+1} - s] - p[\xi(p/q)^n - s] = qx_{n+1} - px_n$$

for $n \geq 0$, must be a Sturmian word on an alphabet $\{k, k+1\}$. Using (4) we will show that this is not the case.

Consider $m+2$ words of length m each, namely, $s_ns_{n+1}\ldots s_{n+m-1}$, where $n = 0, \ldots, m+1$. Since $\mathbf{w}$ is Sturmian, we have $p(\mathbf{w}, m) = m+1$, so at least two of these $m+2$ words must be equal, say, $s_is_{i+1}\ldots s_{i+m-1} = s_js_{j+1}\ldots s_{j+m-1}$, where $0 \leq i = i(m) < j = j(m) \leq m+1$. Selecting in (4) first $n = i$ then $n = j$ and subtracting the first equality from the second, we obtain

$$x_{m+j} - x_{m+i} = (p/q)^m(x_j - x_i). \tag{6}$$

Since $\xi \neq 0$, there is a positive integer m_0 such that the sequence $x_m = [\xi(p/q)^m - s]$ is increasing for $m \geq m_0$ if $\xi > 0$ or decreasing for $m \geq m_0$ if $\xi < 0$. In both cases, the difference $x_{m+j} - x_{m+i}$ is nonzero for each $m \geq m_0$. Hence, by (6), we have $x_j \neq x_i$ for each $m \geq m_0$. Clearly, $x_n \in \mathbb{Z}$ for each integer $n \geq 0$, so (6) implies that the number q^m divides the difference $x_j - x_i$. In particular,

$$q^m \leq |x_j - x_i| \leq |x_i| + |x_j|,$$

because $x_j \neq x_i$. Recall that $s_n \in \mathcal{A}(p, q, -s)$, so $|s_n| < 2p$. Using (4) with $n = 0$ we deduce that $|x_i| \leq c(p/q)^i$ and $|x_j| \leq c(p/q)^j$, where the constant c

is equal to, say, $|x_0| + 2p$. It follows that

$$q^m \le |x_i| + |x_j| \le c(p/q)^i + c(p/q)^j$$
$$\le 2c(p/q)^{m+1} = 2c(p/q)(p/q)^m \le cp(p/q)^m,$$

because $i < j \le m+1$. Taking mth roots and using $p \le q^2 - 1$, we derive that

$$q^2 \le (cp)^{1/m} p \le (cp)^{1/m}(q^2 - 1).$$

This is impossible if $m \ge m_0$ is so large that $(cp)^{1/m} < 1 + 1/q^2$. This proves the theorem. ∎

4. Concluding remarks. Note that Theorem 1 only holds under the condition $p < q^2$. The question of whether the set $Z_{p/q}(s, s+1/p)$ is also empty if $p > q^2$ remains open. The condition $p < q^2$ can be made less restrictive in case the constant I_S described below is greater than 2.

Before giving a formal definition of the constant I_S, let us recall that every Sturmian word $\mathbf{w}$ (in this context more often called *Sturmian sequence*) begins in arbitrarily long squares, namely, there exist arbitrarily long prefixes of $\mathbf{w}$ of the form $\mathbf{v}^2$ (see, e.g., [1], [8]). Also, every Sturmian sequence contains arbitrarily long cubes, i.e., the prefixes of $\mathbf{w}$ are of the form $\mathbf{uv}^3$ with arbitrarily large $|\mathbf{v}|$ (see [3], [16]). Given a Sturmian sequence $\mathbf{w}$, we define $I(\mathbf{w})$ as

$$I(\mathbf{w}) = \sup_{\sigma \ge 0,\, \tau \ge 2} \frac{\tau + \sigma}{1 + \sigma},$$

where the supremum is taken over real numbers $\sigma \ge 0$ and $\tau \ge 2$ such that, for every positive integer k, there exist two words $\mathbf{u}_k$ and $\mathbf{v}_k$ satisfying the following three conditions:

- $\mathbf{u}_k \mathbf{v}_k^{\tau}$ is a prefix of $\mathbf{w}$ for each $k \in \mathbb{N}$,
- $|\mathbf{u}_k| \le \sigma |\mathbf{v}_k|$ for each $k \in \mathbb{N}$,
- $|\mathbf{v}_k| \to \infty$ as $k \to \infty$.

Throughout, $\mathbf{v}^{\tau}$ is defined as a word of length $[\tau|\mathbf{v}|]$ consisting of $[\tau]$ words $\mathbf{v}$ and the prefix of $\mathbf{v}$ of length $[\{\tau\}|\mathbf{v}|]$. The constant I_S is then defined by the formula

$$I_S = \inf_{\mathbf{w}\ \text{Sturmian}} I(\mathbf{w}).$$

By the above-mentioned results, one can select $\sigma = 0$ and $\tau = 2$ for every Sturmian sequence $\mathbf{w}$, hence $I_S \ge 2$. We do not know whether I_S is strictly greater than 2 or not. However, the result of Berthé, Holton and Zamboni [5] implies that τ must be at most 3 for some Sturmian sequences $\mathbf{w}$. For those $\mathbf{w}$ we have $I(\mathbf{w}) \le 3$, hence $I_S \le 3$. Consequently, $2 \le I_S \le 3$.

We claim that the conclusion of Theorem 1 holds for any coprime integers p,q satisfying $1<q<p<q^{I_S}$. Our argument follows the same line as that in the proof of Theorem 1. It is based on Theorem 3 and repeated application of (4).

Indeed, suppose that two integers p,q satisfy $1<q<p<q^{I_S}$. As we already observed above, $I_S \le 3$, so $I_S<\infty$. Fix $\varepsilon>0$ so small that $p<q^{I_S-3\varepsilon}$ and fix two real numbers $\sigma \ge 0$, $\tau \ge 2$ for which the above three conditions are satisfied for a Sturmian word $\mathbf{w}=s_0s_1s_2s_3\dots$ and $(\tau+\sigma)/(1+\sigma)>I(\mathbf{w})-\varepsilon$. Here s_n, $n=0,1,2,\dots$, and $\nu=-s$ are defined as in Theorem 3. Note that $I(\mathbf{w}) \ge I_S$, so $(\tau+\sigma)/(1+\sigma)>I_S-\varepsilon$. We will show that this is impossible.

Set $\mathbf{w}_k=\mathbf{v}_k^{\{\tau\}}$, so that $\mathbf{v}_k^{\tau}=\mathbf{v}_k^{[\tau]}\mathbf{w}_k$. Note that $\mathbf{u}_k\mathbf{v}_k^{[\tau]-1}\mathbf{w}_k$ and $\mathbf{u}_k\mathbf{v}_k^{[\tau]}\mathbf{w}_k$ $=\mathbf{u}_k\mathbf{v}_k\mathbf{v}_k^{[\tau]-1}\mathbf{w}_k$ both are prefixes of $\mathbf{w}$. So the words $s_is_{i+1}\dots s_{i+m-1}$ and $s_js_{j+1}\dots s_{j+m-1}$, where $i=|\mathbf{u}_k|$, $j=|\mathbf{u}_k|+|\mathbf{v}_k|$ and $m=|\mathbf{v}_k^{[\tau]-1}\mathbf{w}_k|$, are equal. Selecting in (4) first $n=i$ then $n=j$ and subtracting the first equality from the second, we obtain $x_{m+j}-x_{m+i}=(p/q)^m(x_j-x_i)$ as in (6). As above, $q^m \mid (x_j-x_i)$, where x_j-x_i is a nonzero integer for m large enough. From (4) it follows that $q^m \le |x_j|+|x_i|<c_1(p/q)^j$, giving $q^{m+j}<c_1p^j$ with some positive number $c_1=c_1(p,q,\xi,-s)$. Hence $q^{m/j+1}<c_1^{1/j}p$.

By the definition of $\mathbf{u}_k$ and $\mathbf{v}_k$, we have

$$\frac{m}{j}=\frac{|\mathbf{v}_k^{[\tau]-1}\mathbf{w}_k|}{|\mathbf{u}_k|+|\mathbf{v}_k|}=\frac{([\tau]-1)|\mathbf{v}_k|+[\{\tau\}|\mathbf{v}_k|]}{|\mathbf{u}_k|+|\mathbf{v}_k|}$$
$$>\frac{(\tau-1)|\mathbf{v}_k|-1}{|\mathbf{u}_k|+|\mathbf{v}_k|}\ge\frac{(\tau-1)|\mathbf{v}_k|-1}{\sigma|\mathbf{v}_k|+|\mathbf{v}_k|}.$$

Hence $m/j \ge (\tau-1)/(1+\sigma)-\varepsilon$ for k large enough, because $1/|\mathbf{v}_k|\to 0$ as $k\to\infty$. Adding 1 to both sides gives

$$m/j+1 \ge (\tau+\sigma)/(1+\sigma)-\varepsilon>I_S-2\varepsilon.$$

Therefore $q^{m/j+1}>q^{I_S-2\varepsilon}$. It follows that $q^{I_S-2\varepsilon}<c_1^{1/j}p<c_1^{1/j}q^{I_S-3\varepsilon}$, so $q^{\varepsilon}<c_1^{1/j}$. Clearly, if $k\to\infty$, then $|\mathbf{v}_k|\to\infty$, so $m\to\infty$ and $j\to\infty$. By taking k so large that $c_1^{1/j}<1+\varepsilon/2$, we deduce that $q^{\varepsilon}<1+\varepsilon/2$. However, this is impossible, because q^{ε} is greater than $1+\varepsilon\log q \ge 1+\varepsilon\log 2>1+0.69\varepsilon$. This proves our assertion.

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