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On the logarithmic derivatives of Dirichlet L-functions at $s = 1$

by

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1. Introduction

1.1. The distribution of values of Dirichlet L-functions at $s = 1$ (i.e., $L(\chi, 1)$ for variable χ) has been studied since long, and has a vast literature. However, the case of the logarithmic derivatives $L'(\chi, 1)/L(\chi, 1)$ has not been as much studied. Motivated by their connections with the Euler–Kronecker invariants of global fields (especially the cyclotomic fields), we have started studying this distribution. In this article, we shall restrict our attention to the maximal absolute values (for a given (large) conductor) and to the "moments". Further studies including the construction of the density function for the distribution of $L'(\chi, s)/L(\chi, s)$ on $\mathbb{C}$, where s is fixed and χ varies, has been left to [8] and to future publications. We note first that the basic feature of the logarithmic derivative case is quite different from the case of L-functions themselves. Instead of a Dirichlet series over the positive integers with periodic coefficients we have a Dirichlet series over the prime powers, and instead of holomorphic L-functions we have meromorphic L'/L-functions with poles at each zero of $L(\chi, s)$. Our method is based on suitably chosen "explicit formulas" (Theorems 1, 2 of §2), and on the study of distribution of zeros of L-functions.

1.2. We summarize our main results. For each prime number m, let X_m denote the set of all non-principal multiplicative characters $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$, and X_m^+ (resp. X_m^-) the subset consisting of even (resp. odd) characters. For each $\chi \in X_m$, $L(\chi, s)$ will denote the corresponding Dirichlet L-function. As is well-known, $L(\chi, 1) \neq 0$.

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First, concerning the maximal value

$$\max_{\chi \in X_m} \left| \frac{L'(\chi,1)}{L(\chi,1)} \right|, \tag{1.2.1}$$

we shall assume the Generalized Riemann Hypothesis (GRH) and prove that this is $\leq (2+\mathbf{o}(1)) \log\log m$. In fact, our argument is given more generally for any Dirichlet characters (that is, finite order Hecke characters) of any number field (§3, Theorem 3 and Cor. 3.3.2). Numerical data for the quantity (1.2.1) for $m \leq 10^5$ are also provided (§3.1, Remark 1); they suggest that the actual bound may be $(1+\mathbf{o}(1)) \log\log m$, though the slow growth of $\log\log m$ leaves room for doubt.

Secondly, for each pair (a,b) of non-negative integers, consider the monomial $P^{(a,b)}(z) = z^a \bar{z}^b$. We shall prove, unconditionally, that

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)} \left(\frac{L'(\chi,1)}{L(\chi,1)} \right) = (-1)^{a+b} \mu^{(a,b)} + \mathbf{O}(m^{\varepsilon-1}) \tag{1.2.2}$$

for any $\varepsilon > 0$ (§5, Theorem 5). Here, $\mu^{(a,b)}$ is some non-negative real number defined in §4. This remains valid if X_m is replaced by $X_m^{\pm}$. A preliminary result (Theorem 4) and the conditional version of Theorem 5 are given in the preceding §4. If $(a,b) = (1,0)$, then $\mu^{(1,0)} = 0$, and the left hand side of (1.2.2) is

$$(\gamma_{\mathbb{Q}(\mu_m)} - \gamma)/(m-2).$$

Here, $\gamma_{\mathbb{Q}(\mu_m)}$ denotes the Euler–Kronecker invariant of the cyclotomic field studied in [6], [7], and $\gamma = \gamma_{\mathbb{Q}}$, the Euler constant. So, we obtain some (conditional and also unconditional) estimates for $\gamma_{\mathbb{Q}(\mu_m)}$, and similarly for $\gamma_{\mathbb{Q}(\cos(2\pi/m))}$ (see §6.1). For further discussions and examples, see §6.

1.3. The corresponding classical results on the values of Dirichlet L-functions at $s = 1$ are as follows. Assuming the GRH, Littlewood proved in [11] that

$$|L(\chi,1)| \leq (2+\mathbf{o}(1)) e^{\gamma} \log\log m.$$

He also showed that for infinitely many *real* characters χ we have

$$L(\chi,1) \geq (1+\mathbf{o}(1)) e^{\gamma} \log\log m.$$

Walfisz and Chowla [2] independently showed that this lower bound holds unconditionally. This method does not attempt to produce prime discriminants for which the lower bound holds. But note that Littlewood's upper bound is for all sufficiently large conductors m.

The classical result

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(1,1)}(L(\chi,1)) = \zeta(2) + \mathbf{O}((\log m)^2/m) \tag{1.3.1}$$

by Paley and Selberg has been improved by several authors: W. Zhang [19] sharpened and generalized it to the case of $P^{(k,k)}$, and Katsurada–Matsumoto [9] obtained an ultimate asymptotic expansion (for the case $P^{(1,1)}$). However, methods used for these do not seem to be applicable to our case.

Other more or less related references include [14], [16], and [4], [12], [15], [17, Ch. IX].

1.4. Here, we explain the general features and individual subtleties relating to the formulas (1.2.2), (1.3.1). Let $\alpha(n) \in \mathbb{C}$ $(n = 1, 2, \dots)$ be such that $\alpha(n) = \mathbf{O}(n^{\varepsilon})$ for any $\varepsilon > 0$, and consider the Dirichlet series $\phi(s) = \sum_n \alpha(n) n^{-s}$ (which is absolutely convergent for $\mathrm{Re}(s) > 1$). For each $\chi \in X_m^\star = X_m \cup \{\chi_0\}$, consider also the associated Dirichlet series $\phi_\chi(s) = \sum_n \chi(n)\alpha(n) n^{-s}$. Then the orthogonality relation for characters leads almost directly to the asymptotic formula

$$\frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} P^{(a,b)}(\phi_\chi(s)) = \sum_{n=1}^{m-1} \frac{\alpha_a(n)\,\overline{\alpha_b(n)}}{n^{2\sigma}} + \mathbf{O}_{a,b}(m^{1+\varepsilon-\sigma})$$

for any $\sigma = \mathrm{Re}(s) > 1 + \varepsilon$. Here, $\alpha_k(n)$ denotes the Dirichlet coefficient of $\phi(s)^k$ $(k = 0, 1, 2, \dots)$. In particular,

$$\lim_{m\to\infty} \frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} P^{(a,b)}(\phi_\chi(s)) = \sum_{n=1}^{\infty} \frac{\alpha_a(n)\,\overline{\alpha_b(n)}}{n^{2\sigma}}. \tag{1.4.1}$$

Now we ask whether this also holds for some s with $\mathrm{Re}(s) \le 1$ when $X_m^\star$ is replaced by X_m; in particular, assuming that all of the $\phi_\chi(s)$ $(\chi \in X_m)$ are analytic at $s = 1$, whether the equality

$$\lim_{m\to\infty} \frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(\phi_\chi(1)) = \sum_{n=1}^{\infty} \frac{\alpha_a(n)\,\overline{\alpha_b(n)}}{n^{2}}$$

holds. It is "simply" a question whether one can interchange, in (1.4.1), the order of passage to the limits, $m \to \infty$ and $s \to 1$. But this depends delicately on the analytic behavior of $\phi(s)$ to the left of $\mathrm{Re}(s) = 1$. When $\phi(s) = \zeta(s)$ (resp. $\zeta'(s)/\zeta(s)$), this is in fact valid and gives Zhang's formula (though stated only for (k, k) at the end of [19]) generalizing (1.3.1) (resp. (the formula obtained by taking $\lim_{m\to\infty}$ of) (1.2.2)).

When $\phi(s) = \zeta(s)$, the estimation of the error term using the Pólya–Vinogradov inequality for character sums suffices. On the other hand, when $\phi(s) = \zeta'(s)/\zeta(s)$, it is more delicate; the estimation of the error term for some carefully chosen approximations of $L'(\chi, 1)/L(\chi, 1)$ is reduced to some properties of the set of non-trivial zeros ϱ of $L(\chi, s)$. Under GRH, the error term for *each* χ is small, and this, together with Theorem 4 (§4.1), leads to Corollary 4.1.2. Unconditionally, what we can show instead that still leads to our main result, Theorem 5 (§5.1), is that the *average* of the absolute value of the error term is sufficiently small. To prove this, we use a result of Montgomery [13] on the estimation of the total number of zeros ϱ (for $\chi \in X_m$) with $|\mathrm{Im}(\varrho)| \le T$ and $\mathrm{Re}(\varrho) \ge \sigma$ for (say) $\sigma \ge 4/5$.

The case of $\log \zeta(s)$ is also interesting (joint work of K. Matsumoto and the first named author in progress). Unconditionally, one may get into the region $\mathrm{Re}(s) > 1/2$ in the cases of $\zeta(s)$ and also $\log \zeta(s)$. But in the case of $\zeta'(s)/\zeta(s)$, $\mathrm{Re}(s) = 1$ would be the best possible, except that in the function field case where GRH is valid, the analogue of (1.4.1) holds for any s with $\mathrm{Re}(s) > 1/2$ (cf. [8, Th. 7(iii)]).

We begin with the appropriate approximations of $L'(\chi,1)/L(\chi,1)$ and the associated "explicit formula".

2. Explicit formulas

2.1. *The invariant* $\gamma^*_{K,\chi}$. Let K be a number field and χ be a primitive Dirichlet character (i.e., a primitive Hecke character with finite order) on K. Let $L(\chi,s)$ be the associated L-function. When $\chi = \chi_0$, the principal character, it is the Dedekind zeta function $\zeta_K(s)$ of K. As in [6], write γ_K for the constant term divided by the residue, of the Laurent expansion of $\zeta_K(s)$ at $s = 1$. Put

$$\gamma^*_{K,\chi} = \begin{cases} \gamma_K + 1 & (\chi = \chi_0), \\ L'(\chi,1)/L(\chi,1) & (\chi \ne \chi_0). \end{cases}$$

We shall use the following two basic formulas (Theorems 1, 2) for $\gamma^*_{K,\chi}$.

2.2. *The main formula.* For any $x > 1$, put

$$\Phi_{K,\chi}(x) = \frac{1}{x-1} \sum_{N(P)^k \le x} \left(\frac{x}{N(P)^k} - 1 \right) \chi(P)^k \log N(P) \quad (x > 1),$$

where the summation is over the pairs of a non-archimedean prime P of K and a positive integer k such that $N(P)^k \le x$. (When P divides the conductor $\mathbf{f}_\chi$ of χ, we put $\chi(P) = 0$.) Note that $\Phi_{K,\chi}(x)$ is a continuous function of x, and when $\chi = \chi_0$, it is equal to the function $\Phi_K(x)$ considered in [6]. (Note also that

$$\Phi_{K,\chi}(x) = \frac{1}{x-1} \int_1^x \sum_{N(P)^k \le t} \frac{\chi(P)^k \log N(P)}{N(P)^k}\, dt;$$

hence $\Phi_{K,\chi}(x)$ represents the average of the partial sums of $-L'(\chi,1)/L(\chi,1)$.)

THEOREM 1. *For any $x > 1$, we have*

$$\gamma^*_{K,\chi} = \delta_\chi \log x - \Phi_{K,\chi}(x) + \frac{1}{x-1}\sum_\varrho \frac{x^\varrho - 1}{\varrho(1-\varrho)} + \frac{a}{2}F_1(x) + \frac{a'}{2}F_3(x) + r_2 F_2(x). \tag{2.2.1}$$

Here, $\delta_\chi = 1$ (resp. 0) for $\chi = \chi_0$ (resp. $\chi \neq \chi_0$), ϱ runs over all non-trivial zeros of $L(\chi, s)$ (counted with multiplicities),

$$\sum_\varrho = \lim_{T\to\infty} \sum_{|\mathrm{Im}(\varrho)|<T}, \tag{2.2.2}$$

a (resp. a') is the number of real places of K where χ is unramified (resp. ramified), $r_1 = a + a'$ (resp. r_2) is the number of real (resp. complex) places of K, and

$$\begin{cases} F_1(x) = \log \dfrac{x+1}{x-1} + \dfrac{2}{x-1}\log\dfrac{x+1}{2}, \\ F_3(x) = \log\dfrac{x^2}{x^2-1} + \dfrac{2}{x-1}\log\dfrac{2x}{x+1}, \\ F_2(x) = \dfrac{1}{2}(F_1(x) + F_3(x)) = \log\dfrac{x}{x-1} + \dfrac{1}{x-1}\log x. \end{cases}$$

Note that $F_i(x)$ $(i = 1,2,3)$ are positive real-valued functions of x vanishing at $x = \infty$. When $\chi = \chi_0$, (2.2.1) follows directly from the formulas (1.2.1) and (1.4.1) of [6]. By letting $x \to \infty$ in (2.2.1), by the same argument as in [6, §1.6], we obtain the following result.

COROLLARY 2.2.3. *When $\chi \neq \chi_0$,*

$$\frac{L'(\chi,1)}{L(\chi,1)} = -\lim_{x\to\infty} \Phi_{K,\chi}(x) = -\lim_{x\to\infty} \sum_{N(P)^k \le x} \left(\frac{\chi(P)}{N(P)}\right)^k \log N(P).$$

2.3. *A supplementary formula*

THEOREM 2. *Let d_K be the discriminant of K, and $\mathbf{f}_\chi$ be the conductor of χ. Put $d_\chi = |d_K| N(\mathbf{f}_\chi)$ and*

$$\begin{cases} \alpha_{K,\chi} = \dfrac{1}{2}\log d_\chi, \\ \beta_{K,\chi} = -\dfrac{a + r_2}{2}(\gamma_{\mathbb{Q}} + \log 4\pi) - \dfrac{a' + r_2}{2}(\gamma_{\mathbb{Q}} + \log \pi). \end{cases}$$

Then

$$\gamma^*_{K,\chi} = \sum_\varrho \frac{1}{1-\varrho} - \alpha_{K,\chi} - \beta_{K,\chi}. \tag{2.3.1}$$

The equality for the *real part* of (2.3.1) can be found in some standard texts in analytic number theory, but we could not find any references for

(2.3.1) itself. It is not *a priori* clear that the sum of $(1-\varrho)^{-1}$ over ϱ in the sense of (2.2.2) (which is the same as the sum of ϱ'^{-1} over the non-trivial zeros ϱ' of $L(\bar{\chi}, s)$) converges at all.

We add here that if $\chi = \chi_0$, then $\alpha_{K,\chi} = \alpha_K$, $\beta_{K,\chi} = \beta_K$, and (2.3.1) is nothing but (1.4.1) of [6].

2.4. *Indications for verification of* (2.2.1) *and* (2.3.1). Fix K and χ, and for any $x > 1$ and $\sigma \in \mathbb{R}$ define

$$\Psi^{(\sigma)}(x) = \sum_{N(P)^k < x} \left(\frac{\chi(P)}{N(P)^\sigma}\right)^k \log N(P)$$

when $x \neq N(P)^k$ for any P and k, and $\Psi^{(\sigma)}(x) = \frac{1}{2}(\Psi^{(\sigma)}(x+0) + \Psi^{(\sigma)}(x-0))$ otherwise, so that

$$\Phi_{K,\chi}(x) = \frac{1}{x-1}\,(x\Psi^{(1)}(x) - \Psi^{(0)}(x)). \tag{2.4.1}$$

The "explicit formulas" for $\Psi^{(\sigma)}(x)$ ($\sigma = 0, 1$) obtained from Weil's general formula [18] (specialized as indicated in [6, §1.3]) read as follows:

$$\begin{aligned}\Psi^{(0)}(x) = {} & \delta_\chi(x + \log x - 1) - \sum_\varrho \frac{x^\varrho - 1}{\varrho} + \frac{1}{2}\log d_\chi \\ & - \frac{a + r_2}{2}\,(\gamma_{\mathbb{Q}} + \log \pi + \log(x^2 - 1)) \\ & - \frac{a' + r_2}{2}\left(\gamma_{\mathbb{Q}} + \log 4\pi + \log \frac{x-1}{x+1}\right),\end{aligned} \tag{2.4.2}$$

$$\begin{aligned}\Psi^{(1)}(x) = {} & \delta_\chi(\log x + 1 - x^{-1}) - \sum_\varrho \frac{x^{\varrho-1} - 1}{\varrho - 1} + \frac{1}{2}\log d_\chi \\ & - \frac{a + r_2}{2}\left(\gamma_{\mathbb{Q}} + \log 4\pi - \log \frac{1 + x^{-1}}{1 - x^{-1}}\right) \\ & - \frac{a' + r_2}{2}\,(\gamma_{\mathbb{Q}} + \log \pi + \log(1 - x^{-2})).\end{aligned} \tag{2.4.3}$$

From (2.4.1), (2.4.2) and (2.4.3), it follows directly that

$$\begin{aligned}\Phi_{K,\chi}(x) = {} & \delta_\chi \log x + \frac{1}{x-1}\sum_\varrho \left(\frac{x^\varrho - 1}{\varrho(1-\varrho)} - \frac{x-1}{1-\varrho}\right) + \frac{1}{2}\log d_\chi \\ & - \frac{a + r_2}{2}\,(\gamma_{\mathbb{Q}} + \log 4\pi - F_1(x)) \\ & - \frac{a' + r_2}{2}\,(\gamma_{\mathbb{Q}} + \log \pi - F_3(x)).\end{aligned} \tag{2.4.4}$$

Now, since $\sum_\varrho |\varrho|^{-2}$ converges, so does

$$\sum_\varrho \frac{x^\varrho - 1}{\varrho(1-\varrho)}.$$

Hence by (2.4.4), the sum $\sum_\varrho (1-\varrho)^{-1}$ (in the sense of (2.2.2)) also converges. Note that (2.4.4) differs from (2.2.1) just by (2.3.1).

On the other hand, Lagarias–Odlyzko [10, (7.3)], gives

$$\Psi^{(0)}(x) = \delta_\chi(x + \log x - 1) - \gamma^*_{K,\chi} - \sum_\varrho \frac{x^\varrho}{\varrho} + \sum_\varrho \frac{1}{\varrho(1-\varrho)} - \frac{a+r_2}{2}\log\frac{x^2-1}{4} - \frac{a'+r_2}{2}\log\frac{4(x-1)}{x+1}, \tag{2.4.5}$$

and the same method also gives

$$\Psi^{(1)}(x) = \delta_\chi(\log x + 1 - x^{-1}) - \gamma^*_{K,\chi} - \sum_\varrho \frac{x^{\varrho-1}}{\varrho - 1} - \frac{a+r_2}{2}\log\frac{1-x^{-1}}{1+x^{-1}} - \frac{a'+r_2}{2}\log(1-x^{-2}). \tag{2.4.6}$$

Note that the difference of (2.4.2) and (2.4.5), and also that of (2.4.3) and (2.4.6), are both exactly the same supplementary formula (2.3.1). This (doubly) proves (2.3.1) and hence also (2.2.1).

The method of [3], [10] is based on the formulas

$$\Psi^{(\sigma)}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} Z_{K,\chi}(s+\sigma)\,ds$$

where $c+\sigma > 1$, and

$$Z_{K,\chi}(s) = -\frac{L'(\chi,s)}{L(\chi,s)} = \sum_{P,k}\left(\frac{\chi(P)}{N(P)^s}\right)^k \log N(P)$$

for $\mathrm{Re}(s) > 1$, and on the computation of the integral $\Psi^{(\sigma)}(x)$ in terms of residues. The term $-\gamma^*_{K,\chi}$ appears as the residue at $s = 1-\sigma$.

3. A GRH-bound for $|L'(\chi,1)/L(\chi,1)|$

3.1. By using Theorems 1, 2 above and some result from [6], we can obtain an upper bound for $|L'(\chi,1)/L(\chi,1)|$ under the Generalized Riemann Hypothesis (GRH).

THEOREM 3. (Under GRH) *Let χ be a non-principal primitive Dirichlet character of a number field K with conductor $\mathbf{f}_\chi$. Then*

$$|L'(\chi,1)/L(\chi,1)| < 2(\log\log\sqrt{d_\chi}+1) - \gamma_K^* + \mathbf{O}\left(\frac{\log|d_K| + \log\log d_\chi}{\log d_\chi}\right). \tag{3.1.1}$$

Here, $d_\chi = |d_K|N(\mathbf{f}_\chi)$ and $\gamma_K^* = \gamma_K + 1$, γ_K being the Euler–Kronecker invariant of K.

Note that the RHS of (3.1.1) is *positive* (modulo the error term), by [6, Theorem 1]. As will be made evident by the proof, the implicit constant in the error term can be given explicitly whenever needed. An explicit bound for the case $K = \mathbb{Q}$ will be given later.

REMARK 1. Numerical experiments indicate that the coefficient 2 on the RHS of (3.1.1) can probably be replaced by $1 + \mathbf{o}(1)$. We assume and use GRH in the following proof of Theorem 3, but we are not able to make use of cancellations among the ϱ-terms in the explicit formula for $\Phi_{K,\chi}(x)$, and this seems to be the cause of the difference. Figure 1 for $K = \mathbb{Q}$ plots the points

$$Q_m = (m, \max_{\chi\in X_m} |L'(\chi,1)/L(\chi,1)|) \in \mathbb{R}^2,$$

where X_m is the set of all primitive characters modulo m, and m runs over all prime numbers in the three intervals $[3, 21799]$, $[48619, 49277]$, and $[104743, 104849]$. This is given together with the graph of $\log\log m$.

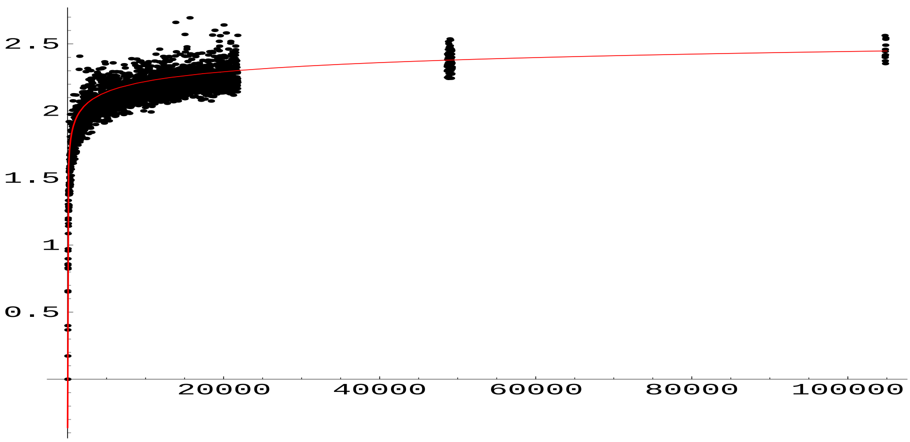

Fig. 1. Q_m and $(m, \log\log m)$

REMARK 2. An interesting case is when $K = \mathbb{Q}$ and χ is a quadratic character such that $L(\chi,1) > 0$ and $L'(\chi,1)/L(\chi,1)$ is large. In particular, suppose that $L'(\chi,1)/L(\chi,1) > 2$ and consider the graph of the real-valued function $t = L(\chi,s)$ on the real s-line. Then the tangent line at $s = 1$ is given by the equation

$$t = L(\chi,1) + L'(\chi,1)(s-1)$$

and it meets the real axis at $1 - L(\chi,1)/L'(\chi,1)$, which lies between 1/2 and 1. So, when we look at the graph at $s = 1$ towards the left, the initial direction of the graph is "going to violate the GRH". The larger the value of $L'(\chi,1)/L(\chi,1)$, the stronger this tendency. But as the following graphs of $t = L(\chi,s)$ show, such violations do not easily happen.

(i) $m = 95717$, $\chi(-1) = 1$, $L'(\chi,1)/L(\chi,1) = 2.16\ldots$.
(ii) $m = 1333963$, $\chi(-1) = -1$, $L'(\chi,1)/L(\chi,1) = 2.736\ldots$.

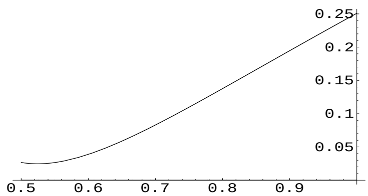

Fig. 2. $m = 95717$

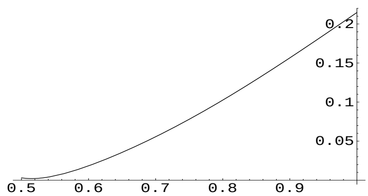

Fig. 3. $m = 1333963$

It may be worthwhile to have more extensive computations of this kind to see if this sheds any light on our perception of the GRH.

3.2. *Proof of Theorem 3.* By Theorem 1,

$$L'(\chi,1)/L(\chi,1) = -\Phi_{K,\chi}(x) + \frac{1}{x-1}\sum_{\varrho}\frac{x^{\varrho}-1}{\varrho(1-\varrho)} + \frac{a+r_2}{2}F_1(x) + \frac{a'+r_2}{2}F_3(x)$$

for any $x > 1$. Under GRH, $\varrho(1-\varrho) = \varrho\bar{\varrho} > 0$ and $|x^{\varrho} - 1| \le \sqrt{x} + 1$; hence by Theorem 2,

$$\begin{aligned}\frac{1}{x-1}\Big|\sum_{\varrho}\frac{x^{\varrho}-1}{\varrho(1-\varrho)}\Big| &\le \frac{1}{\sqrt{x}-1}\sum_{\varrho}\frac{1}{\varrho(1-\varrho)} \\ &= \frac{2}{\sqrt{x}-1}\,(\mathrm{Re}(L'(\chi,1)/L(\chi,1)) + \alpha_{K,\chi} + \beta_{K,\chi}) \\ &\le \frac{2}{\sqrt{x}-1}\,(|L'(\chi,1)/L(\chi,1)| + \alpha_{K,\chi}).\end{aligned}$$

Since

$$\begin{aligned}\frac{a+r_2}{2}F_1(x) + \frac{a'+r_2}{2}F_3(x) &= \mathbf{O}((r_1+r_2)(\log x)/x) \\ &= \mathbf{O}((\alpha_K+1)(\log x)/x),\end{aligned} \tag{3.2.1}$$

and $|\Phi_{K,\chi}(x)| \le \Phi_K(x)$, we obtain

$$\left(1 - \frac{2}{\sqrt{x}-1}\right)\left|\frac{L'(\chi,1)}{L(\chi,1)}\right| < \Phi_K(x) + \frac{2\alpha_{K,\chi}}{\sqrt{x}-1} + \mathbf{O}\left(\frac{(\alpha_K+1)\log x}{x}\right),$$

where the implicit constant is independent of x, K and χ.

On the other hand, by using Theorems 1, 2 for $\chi = \chi_0$, we obtain similarly

$$\Phi_K(x) < \log x - \gamma_K^* + \frac{2(\gamma_K^* + \alpha_K)}{\sqrt{x}-1} + \mathbf{O}\left(\frac{(\alpha_K+1)\log x}{x}\right).$$

Hence

$$\left(1 - \frac{2}{\sqrt{x}-1}\right)\left|\frac{L'(\chi,1)}{L(\chi,1)}\right| < \log x - \gamma_K^* + \frac{2(\gamma_K^* + \alpha_{K,\chi} + \alpha_K)}{\sqrt{x}-1} + \mathbf{O}\left(\frac{(\alpha_K+1)\log x}{x}\right).$$

Now recall that $\alpha_{K,\chi} = \log\sqrt{d_\chi}$, and take $x = \alpha_{K,\chi}^2$. Then the above formula gives

$$\text{(3.2.2)}\quad \frac{\alpha_{K,\chi}-3}{\alpha_{K,\chi}-1}\left|\frac{L'(\chi,1)}{L(\chi,1)}\right| < 2\log\alpha_{K,\chi} - \gamma_K^* + \frac{2\gamma_K^*}{\alpha_{K,\chi}-1} + 2 + \mathbf{O}\left(\frac{\alpha_K+1}{\alpha_{K,\chi}}\right) < 2(\log\alpha_{K,\chi}+1) - \gamma_K^* + \mathbf{O}\left(\frac{\alpha_K+1}{\alpha_{K,\chi}}\right),$$

by Theorem 1 of [6]. To obtain an upper bound for $|L'(\chi,1)/L(\chi,1)|$ itself, we take $\alpha_{K,\chi} > 3$ and add to the RHS of (3.2.2) its multiple by $2(\alpha_{K,\chi}-3)^{-1}$. But since

$$-\gamma_K^* < \alpha_K + \beta_K < \alpha_K$$

([6, Prop. 3]), we have

$$-\gamma_K^*/(\alpha_{K,\chi}-3) < \mathbf{O}((\alpha_K+1)/\alpha_{K,\chi}),$$

and hence

$$\left|\frac{L'(\chi,1)}{L(\chi,1)}\right| < 2(\log\alpha_{K,\chi}+1) - \gamma_K^* + \mathbf{O}\left(\frac{\alpha_K + \log\alpha_{K,\chi}}{\alpha_{K,\chi}}\right),$$

as desired.

3.3. *The case* $K = \mathbb{Q}$. When $K = \mathbb{Q}$, we can use the inequalities $\Phi_{\mathbb{Q}}(x) < \log x$ ([6, (1.6.36)]) and

$$\text{(3.3.1)}\quad \begin{cases} F_1(x) < \dfrac{2}{x-1}(\log x + 1), \\ F_3(x) < \dfrac{1}{x^2-1} + \dfrac{2}{x-1}\log 2 < \dfrac{1/2 + 2\log 2}{x-1}, \end{cases}$$

to obtain the following result.

Corollary 3.3.2. (Under GRH) *When* $K = \mathbb{Q}$, *we have*

$$\frac{\log\sqrt{d_\chi}-3}{\log\sqrt{d_\chi}-1}\left|\frac{L'(\chi,1)}{L(\chi,1)}\right| < 2(\log\log\sqrt{d_\chi}+1) - \frac{b+\chi(-1)\log 2}{\log\sqrt{d_\chi}-1} + \begin{cases} \dfrac{2\log\log\sqrt{d_\chi}+1}{(\log\sqrt{d_\chi})^2-1} & \text{if } \chi(-1)=1, \\ \dfrac{\log 2+1/4}{(\log\sqrt{d_\chi})^2-1} & \text{if } \chi(-1)=-1, \end{cases}$$

where $b = \gamma_{\mathbb{Q}} + \log 2\pi - 2 = 0.415\ldots$.

4. Moments of $L'(\chi,1)/L(\chi,1)$; (I)

4.1. Now let $K = \mathbb{Q}$, and let m run over the odd *prime* numbers. For each such m, denote by X_m the collection of all non-principal primitive Dirichlet characters $\chi : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$. For each pair (a,b) of non-negative integers, let $P^{(a,b)}(z) = z^a\bar{z}^b$. In this section and in §5, we shall study the behavior of the mean value of $P^{(a,b)}(L'(\chi,1)/L(\chi,1))$ (for $\chi \in X_m$) when m is large. The goal of §4 is to obtain an unconditional result (Theorem 4) and some conditional results (Lemma 1, Corollary 4.1.2).

As usual, $\Lambda(n) = \log p$ when n is a positive integral power of a prime number p, and $\Lambda(n) = 0$ if either $n = 1$ or n has at least two prime factors. For each non-negative integer k, define the arithmetic function $\Lambda_k(n)$ by

$$\Lambda_0(n) = \begin{cases} 1 & (n=1), \\ 0 & (n>1), \end{cases}$$

$$\Lambda_k(n) = \sum_{n=n_1\cdots n_k} \Lambda(n_1)\cdots\Lambda(n_k) \qquad (k>0).$$

Note that $\Lambda_k(n) = 0$ unless the number of prime factors of n is at most k *and* the sum of exponents in the prime factorization of n is at least k. It is easy to see that

$$\Lambda_k(p^r) = \binom{r-1}{k-1}(\log p)^k$$

for $1 \le k \le r$, and that

$$\Lambda_k(n) \le (\log n)^k \tag{4.1.1}$$

(cf. [8, §3.8]).

For each pair (a,b) of non-negative integers, put

$$\mu^{(a,b)} = \mu^{(b,a)} = \sum_{n=1}^{\infty} \frac{\Lambda_a(n)\Lambda_b(n)}{n^2}.$$

Note that $\mu^{(0,0)} = 1$, $\mu^{(a,0)} = \mu^{(0,a)} = 0$ for any $a > 0$, $\mu^{(a,b)} > 0$ in all other cases, and

$$\mu^{(a,1)} = \sum_p \frac{(\log p)^{a+1}}{(p^2-1)^a} \qquad (a>0).$$

In particular,

$$\mu^{(1,1)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^2} = \sum_p \frac{(\log p)^2}{p^2-1} = 0.80521\ldots.$$

Put $\Phi_\chi(x) = \Phi_{\mathbb{Q},\chi}(x)$, so that

$$\Phi_\chi(x) = \frac{1}{x-1}\sum_{n\le x}\left(\frac{x}{n}-1\right)\Lambda(n)\chi(n).$$

THEOREM 4. *For each pair* (a,b) *of non-negative integers, and* $x \ge m$, *we have*

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(\Phi_\chi(x)) = \mu^{(a,b)} + \mathbf{O}\left(\frac{(\log x)^d}{m}\right),$$

where $d = a+b+1$ *when* $ab=0$, *and* $d=a+b+2$ *otherwise. This remains valid if* X_m *is replaced by the set of all even* (*resp. odd*) *characters. The implicit constant depends on* a *and* b.

Now, the following conditional Lemma 1 gives an easily observable connection between $\Phi_\chi(x)$ and $-L'(\chi,1)/L(\chi,1)$.

LEMMA 1. (Under GRH) *For* $x>1$, *we have*

$$\frac{L'(\chi,1)}{L(\chi,1)} + \Phi_\chi(x) = \mathbf{O}\left(\frac{\log m}{\sqrt{x}} + \frac{\log x}{x}\right) \qquad (\chi\in X_m),$$

the implicit constant being absolute.

Proofs of Theorem 4 and Lemma 1 will be given in §4.2. By using both these results for $x=m^2$, we easily obtain the following corollary (a similar argument in the more complicated unconditional case is in §5.3 below).

COROLLARY 4.1.2. (Under GRH)

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(L'(\chi,1)/L(\chi,1)) = (-1)^{a+b}\mu^{(a,b)} + \mathbf{O}\left(\frac{(\log m)^d}{m}\right).$$

In particular,

$$\lim_{m\to\infty}\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(L'(\chi,1)/L(\chi,1)) = (-1)^{a+b}\mu^{(a,b)}. \tag{4.1.3}$$

These remain valid if X_m *is replaced by the set of all even* (*resp. odd*) *characters.*

4.2. *Proof of Theorem 4.* First, since $\Phi_{\chi_0}(x) = \mathbf{O}(\log x)$, we may include the principal character in proving the theorem. Write $X_m^\star = X_m \cup \{\chi_0\}$. Put

$$\mu^{(a,b)}(x) = \frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} P^{(a,b)}(\Phi_\chi(x)) = \frac{1}{|X_m^\star|} \sum_{\chi \in X_m^\star} \Phi_\chi(x)^a \Phi_{\bar\chi}(x)^b. \tag{4.2.1}$$

Then the orthogonality relation for characters implies directly that

$$\mu^{(a,b)}(x) = \sum_{c=1}^{m-1} \lambda^{(a)}(c,x)\lambda^{(b)}(c,x), \tag{4.2.2}$$

where

$$\lambda^{(k)}(c,x) = \frac{1}{(x-1)^k} \sum_{\substack{n_1,\dots,n_k < x \\ n_1\cdots n_k \equiv c \,(\mathrm{mod}\, m)}} \prod_{i=1}^{k} \Big(\frac{x}{n_i} - 1\Big) \Lambda(n_i)$$

for $k \ge 1$, and $\lambda^{(0)}(c,x) = 1, 0$ for $c = 1, > 1$ respectively. So, if we put

$$L^{(k)}(N,x) = \frac{1}{(x-1)^k} \sum_{\substack{n_1,\dots,n_k < x \\ n_1\cdots n_k = N}} \prod_{i=1}^{k} \Big(\frac{x}{n_i} - 1\Big) \Lambda(n_i) \tag{4.2.3}$$

for $k, N \ge 1$, and $L^{(0)}(N,x) = 1, 0$ for $N = 1, > 1$ (respectively), then

$$\lambda^{(k)}(c,x) = \sum_{l=0}^{[(x^k - c)/m]} L^{(k)}(c + lm, x). \tag{4.2.4}$$

We shall show that the terms with $l > 0$ are altogether negligible, and that the term with $l = 0$ can be expressed as the sum of a simpler quantity and a negligible one. To see this, we first note that $L^{(k)}(N,x) \ne 0$ only when $N < x^k$, and that in this case

$$\begin{aligned} L^{(k)}(N,x) &\le \frac{1}{N} \sum_{\substack{n_1,\dots,n_k < x \\ n_1\cdots n_k = N}} \Lambda(n_1)\cdots\Lambda(n_k) \\ &\le \frac{1}{N} \Lambda_k(N) \le \frac{(\log N)^k}{N} < k^k \, \frac{(\log x)^k}{N}. \end{aligned}$$

From this, it follows immediately that the sum of terms with $l > 0$ in (4.2.4) is $\mathbf{O}((\log x)^{k+1}/m)$. Therefore,

$$\lambda^{(k)}(c,x) = L^{(k)}(c,x) + \mathbf{O}((\log x)^{k+1}/m). \tag{4.2.5}$$

Note that $\lambda^{(0)}(c,x) = L^{(0)}(c,x)$; hence the above exponent of $\log x$ can be replaced by 0 when $k = 0$. Now we shall show that

$$L^{(k)}(c,x) = \frac{\Lambda_k(c)}{c} + \mathbf{O}((\log m)^k/x) \quad (x \ge m). \tag{4.2.6}$$

This is based on a very simple inequality. For any $x > 0$ and $i, j \geq 1$, we have $(x-i)(x-j) \geq (x-1)(x-ij)$; hence for any $n_1, \ldots, n_k \geq 1$ and $x > n_1 \cdots n_k$,

$$(x-1)^k \geq (x-n_1)\cdots(x-n_k) \geq (x-1)^{k-1}(x-n_1\cdots n_k).$$

This gives directly

$$0 \leq \prod_{i=1}^{k} \frac{1}{n_i} - \frac{1}{(x-1)^k} \prod_{i=1}^{k} \left(\frac{x}{n_i} - 1\right) \leq \frac{c-1}{c(x-1)}$$

for any $n_1, \ldots, n_k \geq 1$ with $n_1 \cdots n_k = c$ and $x \geq m$. Therefore,

$$\frac{1}{(x-1)^k} \prod_{i=1}^{k} \left(\frac{x}{n_i} - 1\right) = \prod_{i=1}^{k} \frac{1}{n_i} + \mathbf{O}\left(\frac{1}{x}\right). \tag{4.2.7}$$

Now, note that for $N = c$ and $x \geq m$, the summation condition $n_1, \ldots, n_k < x$ in (4.2.3) is automatic. Therefore, (4.2.7) and (4.1.1) give (4.2.6).

Now since $\Lambda_k(c) = \mathbf{O}((\log m)^k)$, we obtain, from (4.2.2), (4.2.5) and (4.2.6),

$$\mu^{(a,b)}(x) = \sum_{c=1}^{m-1} \frac{\Lambda_a(c)\Lambda_b(c)}{c^2} + \mathbf{O}\left(\frac{(\log x)^{a+b+2}}{m}\right).$$

(Note that the exponent $a+b+2$ can be replaced by $a+b+1$ when $ab = 0$.) But since

$$\sum_{n \geq m} \frac{\Lambda_a(n)\Lambda_b(n)}{n^2} \leq \sum_{n \geq m} \frac{(\log n)^{a+b}}{n^2} = \mathbf{O}\left(\frac{(\log m)^{a+b}}{m}\right),$$

the first statement of the theorem follows.

Based on this, the additional statement for even characters (and hence also for odd characters) can be reduced to the following simple estimate:

$$\begin{aligned} \sum_{c=1}^{m-1} \frac{\Lambda_a(c)\Lambda_b(m-c)}{c(m-c)} &\leq (\log m)^{a+b} \sum_{c=1}^{m-1} \frac{1}{c(m-c)} \\ &= \mathbf{O}\left(\frac{(\log m)^{a+b+1}}{m}\right). \end{aligned} \tag{4.2.8}$$

Indeed, write $\mu_+^{(a,b)}(x)$ for the modification of (4.2.1) where $X_m^\star$ is replaced by all even characters including χ_0. Then the orthogonality relation for characters gives

$$\mu_{+}^{(a,b)}(x) = \sum_{\substack{1\le c,c'\le m-1\\ c'\equiv \pm c \,(\mathrm{mod}\, m)}} \lambda^{(a)}(c,x)\lambda^{(b)}(c',x)$$
$$= \mu^{(a,b)}(x) + \sum_{c=1}^{m-1} \lambda^{(a)}(c,x)\lambda^{(b)}(m-c,x).$$

Therefore, it remains to prove that

$$\sum_{c=1}^{m-1} \lambda^{(a)}(c,x)\lambda^{(b)}(m-c,x) = \mathbf{O}\biggl(\frac{(\log x)^d}{m}\biggr). \tag{4.2.9}$$

But by (4.2.5) and (4.2.6) we have

$$\lambda^{(k)}(c,x) = \frac{\varLambda_k(c)}{c} + \mathbf{O}\biggl(\frac{(\log x)^{k'}}{m}\biggr) \qquad (x \ge m),$$

where $k' = k+1$ for $k>0$, and $k'=0$ for $k=0$. And since moreover

$$\sum_{c=1}^{m-1} \frac{\varLambda_k(c)}{c} = \mathbf{O}((\log m)^{k'}),$$

(4.2.9) is reduced to (4.2.8).

This completes the proof of Theorem 4.

Proof of Lemma 1. The main explicit formula (2.2.1) gives, for any $x>1$,

$$\frac{L'(\chi,1)}{L(\chi,1)} + \varPhi_\chi(x) = \frac{1}{x-1}\sum_{\varrho} \frac{x^{\varrho}-1}{\varrho(1-\varrho)} + \frac{1}{2} F_{2-\chi(-1)}(x).$$

By (3.3.1), the second term on the RHS has order at most $\mathbf{O}((\log x)/x)$. Moreover, under the GRH, the absolute value of the first term is at most

$$\frac{1}{\sqrt{x}-1}\sum_{\varrho}\frac{1}{\varrho(1-\varrho)} \le \frac{1}{\sqrt{x}-1}\biggl(2\mathrm{Re}\biggl(\frac{L'(\chi,1)}{L(\chi,1)}\biggr) + \log m\biggr) = \mathbf{O}\biggl(\frac{\log m}{\sqrt{x}-1}\biggr)$$

by Theorems 2 and 3. The rest is obvious.

5. Moments of $L'(\chi,1)/L(\chi,1)$; (II)

5.1. *Statement of results.* In this section, we establish the following unconditional version of Corollary 4.1.2.

THEOREM 5. *We have, unconditionally,*

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}(L'(\chi,1)/L(\chi,1)) = (-1)^{a+b}\mu^{(a,b)} + \mathbf{O}(m^{\varepsilon-1}),$$

for any $\varepsilon>0$. In particular, the limit formula (4.1.3) *holds unconditionally. The same remains valid if X_m is replaced by $X_m^{\pm}$.*

5.2. *The key lemma.* Recall that the corresponding conditional result, Corollary 4.1.2, was proved by a direct combination of Theorem 4 and Lemma 1 (§4.1). Among them, Lemma 1 was conditional. Our proof of Theorem 5 is obtained by combining Theorem 4 with the following unconditional substitute for Lemma 1, in which (i) is supplementary and (ii) is more crucial.

Lemma 2.

(i) *Let $\chi \in X_m, x \geq m$, and $\varepsilon > 0$. Then*

$$\left|\frac{L'(\chi,1)}{L(\chi,1)}\right| + \left|\frac{L'(\chi,1)}{L(\chi,1)} + \Phi_\chi(x)\right| \ll \begin{cases} m^\varepsilon & \text{for } \chi = \chi_1, \\ (\log m)^2 & \text{for } \chi \neq \chi_1, \end{cases}$$

where χ_1 denotes the unique quadratic character in X_m.

(ii) *Let $x \geq m^{12}$. Then*

$$\sum_{\substack{\chi \in X_m \\ \chi \neq \chi_1}} \left|\frac{L'(\chi,1)}{L(\chi,1)} + \Phi_\chi(x)\right| \ll (\log x)^{15}.$$

5.3. *Reducing Theorem 5 to Lemma 2.* We shall use the following elementary inequality ([8, §6.8]):

$$(5.3.1) \quad |P^{(a,b)}(z+w) - P^{(a,b)}(z)| \leq (a+b)|w|(|z|+|w|)^{a+b-1} \quad (z, w \in \mathbb{C}).$$

(This is a simple explicit version of the mean value theorem for polynomials in two variables.)

Take $z = -L'(\chi,1)/L(\chi,1)$ and $w = L'(\chi,1)/L(\chi,1) + \Phi_\chi(x)$. Then (5.3.1), together with Lemma 2(i) (for a, b fixed and $x \geq m$) gives

$$|P^{(a,b)}(\Phi_\chi(x)) - P^{(a,b)}(-L'(\chi,1)/L(\chi,1))| \\ \ll \begin{cases} m^{\varepsilon(a+b)} & \text{for } \chi = \chi_1, \\ (\log m)^{2(a+b-1)}|L'(\chi,1)/L(\chi,1) + \Phi_\chi(x)| & \text{for } \chi \neq \chi_1. \end{cases}$$

Now put $x = m^{12}$. Then the above and Lemma 2(ii) give

$$\sum_{\chi \in X_m} |P^{(a,b)}(\Phi_\chi(x)) - P^{(a,b)}(-L'(\chi,1)/L(\chi,1))| \\ \ll m^{\varepsilon(a+b)} + (\log m)^{2(a+b-1)+15} \ll m^{\varepsilon'}.$$

Therefore,

$$\frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(-L'(\chi,1)/L(\chi,1)) = \frac{1}{|X_m|} \sum_{\chi \in X_m} P^{(a,b)}(\Phi_\chi(x)) + \mathbf{O}(m^{\varepsilon'-1}) \\ = \mu^{(a,b)} + \mathbf{O}(m^{\varepsilon'-1})$$

by virtue of Theorem 4. Thus, Theorem 5 is reduced to Lemma 2. (Since Theorem 4 remains valid with X_m replaced by $X_m^\pm$, so does Theorem 5.)

5.4. *Reducing Lemma 2 to L-zero sum estimates.* We are going to prove Lemma 2 by using the explicit formula (Theorem 1 of §2). Together with the estimate (3.2.1) of the archimedean part, this gives

$$\frac{L'(\chi,1)}{L(\chi,1)}+\Phi_\chi(x)=\frac{1}{x-1}\sum_{\varrho\in Z_\chi}\frac{x^\varrho-1}{\varrho(1-\varrho)}+\mathbf{O}\left(\frac{\log x}{x}\right), \tag{5.4.1}$$

where Z_χ denotes the set of all non-trivial zeros of $L(\chi,s)$. This formula reduces Lemma 2 to appropriate estimates of the quantities

$$\sum_{\varrho\in Z_\chi}\left|\frac{x^\varrho-1}{\varrho(1-\varrho)}\right|,\qquad \sum_{\substack{\chi\in X_m\\ \chi\neq\chi_1}}\sum_{\varrho\in Z_\chi}\left|\frac{x^\varrho-1}{\varrho(1-\varrho)}\right|.$$

These will be collected in the sublemma below, of which (ii) may be of independent interest. But first we recall a classical result on zero-free regions.

It is well-known (cf., e.g., [3]) that there is an effective and absolute constant $c>0$, and (possibly) a real simple zero $\beta_1>1/2$ of $L(\chi_1,s)$, such that if $\varrho=\beta+i\gamma$ is any non-trivial zero of $L(\chi,s)$ ($\chi\in X_m$) with $|\gamma|\le T$, $T\ge 1$, then either $\chi=\chi_1$ and ϱ is one of β_1, $1-\beta_1$, or

$$\min(1-\beta,\beta)>\frac{c}{\log(mT)}. \tag{5.4.2}$$

Sublemma 5.4.3.

(i) *Let $\chi\in X_m$ be fixed. Then*

$$\sum_{\varrho\in Z_\chi}{}'\left|\frac{1}{\varrho(1-\varrho)}\right|\ll(\log m)^2, \tag{5.4.4}$$

$$\frac{1}{\beta_1(1-\beta_1)}\ll m^\varepsilon, \tag{5.4.5}$$

$$\sum_{\substack{\varrho\in Z_\chi\\ |\gamma|>T}}\left|\frac{x^\varrho}{\varrho(1-\varrho)}\right|\ll\frac{x\log(mT)}{T}\qquad (x,T>1). \tag{5.4.6}$$

(ii) *For $x\ge(mT)^6$ and $T>1$,*

$$\sum_{\chi\in X_m}\sum_{\substack{\varrho\in Z_\chi\\ |\gamma|\le T}}{}'\left|\frac{x^\varrho}{\varrho(1-\varrho)}\right|\ll x(\log x)^{15}.$$

Here, $\sum'$ denotes the sum over $\varrho\neq\beta_1$.

Reducing Lemma 2 to the sublemma. (i) By (5.4.1), we have, for $x\ge m$ (≥ 2),

$$\left|\frac{L'(\chi,1)}{L(\chi,1)}+\Phi_\chi(x)\right| \ll \sum_{\varrho\in Z_\chi}\left|\frac{1}{\varrho(1-\varrho)}\right| + \frac{\log m}{m}$$
$$\ll \begin{cases} m^\varepsilon & (\chi=\chi_1),\\ (\log m)^2 & (\chi\neq\chi_1),\end{cases}$$

by the first two inequalities in the sublemma. Since $|\Phi_\chi(m)| \le |\Phi_{\chi_0}(m)| \ll \log m$, the same holds for $|L'(\chi,1)/L(\chi,1)|$; whence Lemma 2(i).

(ii) By (5.4.1) for $\chi\neq\chi_1$,

$$\left|\frac{L'(\chi,1)}{L(\chi,1)}+\Phi_\chi(x)\right| \ll \frac{1}{x-1}\sum_{|\gamma|\le T}{}'\left|\frac{x^\varrho}{\varrho(1-\varrho)}\right| + \frac{1}{x-1}\sum_{|\gamma|>T}\left|\frac{x^\varrho}{\varrho(1-\varrho)}\right|$$
$$+\frac{1}{x-1}\sum{}'\left|\frac{1}{\varrho(1-\varrho)}\right| + \mathbf{O}\Big(\frac{\log x}{x}\Big),$$

where the indication $\varrho\in Z_\chi$ is suppressed. Hence, by Sublemma 5.4.3(i), the right hand side above is

$$\ll \frac{1}{x}\sum_{|\gamma|\le T}{}'\left|\frac{x^\varrho}{\varrho(1-\varrho)}\right| + \frac{\log(mT)}{T} + \frac{(\log m)^2}{x} + \frac{\log x}{x}.$$

Now let $T=m$, $x\ge m^{12}$. Then by this and Sublemma 5.4.3(ii),

$$\sum_{\chi\neq\chi_1}\left|\frac{L'(\chi,1)}{L(\chi,1)}+\Phi_\chi(x)\right| \ll \frac{1}{x}\sum_{\chi\in X_m}\sum_{|\gamma|\le m}{}'\left|\frac{x^\varrho}{\varrho(1-\varrho)}\right| + \log m$$
$$\ll (\log x)^{15} + \log m \ll (\log x)^{15},$$

as desired.

5.5. *Proof of Sublemma 5.4.3(i).* First, since $\beta_1>1/2$ and $1-\beta_1\gg m^{-\varepsilon}$ (e.g., [3, §21]), (5.4.5) is obvious. Secondly, (5.4.4) and (5.4.6) follow directly by using the well-known inequality (e.g. [3])

$$\sharp\{\varrho \mid |\gamma-T|<1\} \ll \log(m(T+2))$$

to estimate the number of terms in a given interval, and

$$\left|\frac{1}{\varrho(1-\varrho)}\right| \ll \begin{cases} \log m & (|\gamma|\le 1,\ \varrho\neq\beta_1),\\ \gamma^{-2} & (|\gamma|>1),\end{cases}$$

to estimate each summand. (The first case of the above inequality follows from (5.4.2) for $T=1$.)

5.6. *Proof of Sublemma 5.4.3(ii).* For each $0\le\sigma\le 1$ and $T\ge 2$, set

$$\begin{cases} N(\sigma,T,\chi) = \sharp\{\varrho=\beta+i\gamma\in Z_\chi \mid \beta\ge\sigma,\ |\gamma|\le T\},\\ N(\sigma,T,m) = \displaystyle\sum_{\chi\in X_m} N(\sigma,T,\chi).\end{cases}$$

Then, firstly, as is well-known,

$$N(0,T,m) \ll mT\log(mT). \tag{5.6.1}$$

Secondly, by Montgomery [13, Theorem 12.1] (cf. also a sharper result due to Huxley–Jutila [5]),

$$N(\sigma,T,m) \ll (mT)^{5(1-\sigma)/2}(\log(mT))^{14} \quad \text{for } \sigma \geq 4/5. \tag{5.6.2}$$

Now let $T \geq 2, x > 1$, and consider the sum

$$\tilde{S}(x,m,T) = \sum_{\chi\in X_m} \sideset{}{'}\sum_{\substack{\varrho\in Z_\chi \\ |\gamma|\leq T}} x^\beta. \tag{5.6.3}$$

We claim that

$$\tilde{S}(x,m,T) \ll x(\log x)^{14} \quad \text{for } x \geq (mT)^6. \tag{5.6.4}$$

To prove this claim, first note that the sum in (5.6.3) over those ϱ with $\beta \leq 4/5$ is

$$\ll x^{4/5}N(0,T,m) \ll x^{4/5}(mT)\log(mT) \ll x^{4/5+1/6}\log x \ll x,$$

by (5.6.1). The remaining sum is given by

$$-\int_{4/5}^{1} x^\sigma d_\sigma N(\sigma,T,m) \leq x^{4/5}N(4/5,T,m) + \int_{4/5}^{1} (x^\sigma \log x)N(\sigma,T,m)\,d\sigma.$$

The first term of the right hand side is, again, $\ll x$, while by (5.6.2), the second term is

$$\ll (\log x)(mT)^{5/2}(\log(mT))^{14} \int_{4/5}^{1} (x/(mT)^{5/2})^\sigma\,d\sigma. \tag{5.6.5}$$

But since

$$\int_{4/5}^{1} (x/(mT)^{5/2})^\sigma\,d\sigma \leq (x/(mT)^{5/2})/\log(x/(mT)^{5/2}) \ll (x/(mT)^{5/2})/\log x$$

by our assumption on x, (5.6.5) is

$$\ll x(\log(mT))^{14} \ll x(\log x)^{14}.$$

This proves the claim (5.6.4).

Now, let us finally estimate the sum

$$S(x,m,T) = \sum_{\chi\in X_m} \sideset{}{'}\sum_{\substack{\varrho\in Z_\chi \\ |\gamma|\leq T}} \left|\frac{x^\varrho}{\varrho(1-\varrho)}\right|$$

in question. First, the sum over those ϱ with $\beta \leq 4/5$ is $\ll x^{4/5}m(\log m)^2$ by Sublemma 5.4.3(i); hence it is $\ll x$. So, let us restrict ourselves to those

ϱ with $\beta \geq 4/5$. Then since

$$|\varrho(1-\varrho)| \geq \mathrm{Re}(\varrho(1-\varrho)) = \beta(1-\beta) + \gamma^2,$$

and $\beta > 1/2$, $1-\beta > c/\log(mT)$, we may use $|\varrho(1-\varrho)| > c/(2\log(mT))$ when $|\gamma| \leq 2$, and $|\varrho(1-\varrho)| > \gamma^2$ when $|\gamma| > 2$, to estimate $S(x,m,T)$. This gives

$$\begin{aligned} S(x,m,T) &\ll (2/c)\log(mT)\tilde{S}(x,m,2) + \sum_{\substack{j\geq 0\\ 2^{j+1}\leq T}} \sum_{\chi\in X_m} \sum_{2^j\leq|\gamma|\leq 2^{j+1}} \frac{1}{\gamma^2}x^\beta \\ &\ll \log(mT)\tilde{S}(x,m,2) + \sum_{j\geq 0} 2^{-2j}\tilde{S}(x,m,2^{j+1}). \end{aligned}$$

But the condition $x \geq (mT)^6$ in (5.6.4) remains valid if T is replaced by anything smaller; hence

$$S(x,m,T) \ll (\log(mT))x(\log x)^{14} \ll x(\log x)^{15},$$

as desired.

This settles the proof of the sublemma, hence of Lemma 2, and hence that of Theorem 5.

6. Applications to distributions of $L'(\chi,1)/L(\chi,1)$

6.1. In this section, we shall give several remarks on the distribution of $L'(\chi,1)/L(\chi,1)$, which can be deduced from Corollary 4.1.2 and Theorem 5.

The case $a+b=1$. The decomposition of $\zeta_{\mathbb{Q}(\mu_m)}(s)/\zeta(s)$ into the product of $L(\chi,s)$ ($\chi \in X_m$), and the similar one for the real cyclotomic field, give rise to the additive decompositions

$$\begin{aligned} \gamma_{\mathbb{Q}(\mu_m)} &= \gamma_{\mathbb{Q}} + \sum_{\chi\neq\chi_0} L'(\chi,1)/L(\chi,1), \\ \gamma_{\mathbb{Q}(\mu_m)^+} &= \gamma_{\mathbb{Q}} + \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} L'(\chi,1)/L(\chi,1) \end{aligned}$$

(see §2.1 for the definition of γ_K). So, the case $(a,b)=(1,0)$ of Corollary 4.1.2 and Theorem 5 give the following estimates.

Corollary 6.1.1.

$$|\gamma_{\mathbb{Q}(\mu_m)}|, |\gamma_{\mathbb{Q}(\mu_m)^+}| = \begin{cases} \mathbf{O}((\log m)^2) & (\textit{under GRH}), \\ \mathbf{O}(m^\varepsilon) & (\textit{unconditionally}). \end{cases}$$

As is pointed out in [6], [7], when we study the range of values of the Euler–Kronecker invariants γ_K for a given family of global fields K (where for each family we impose the additional condition $|d_K| \leq x$, d_K the discriminant, x grows), it is important to study the upper and the lower bound carefully and separately, because each reflects the arithmetic nature of the family in a different way. The upper bound is usually small (under GRH, $\ll \log\log|d_K|$) and the arithmetic implication is not well understood. The lower bound for the most general family is negative and has larger absolute values ($\gg -\log|d_K|$), and when the family has increasingly many primes with small norms, the negative range is in fact much wider than the positive range. Even for the family of number fields with a given degree $N > 2$, the negative range is wider. Now, in the case of cyclotomic fields $\mathbb{Q}(\mu_m)$ or $\mathbb{Q}(\mu_m)^+$, each of which has only few primes with small norms, the situation is very much different. In this case, we have a GRH upper bound $(2+\varepsilon)\log m$ [6, Theorem 1], and as for the lower bound, numerical evidence and some L-zero interpretations suggest that their Euler–Kronecker invariants are always *positive* [7]. However, the best result for the lower bound we have obtained so far is the above corollary for their absolute values.

More recently (after the preliminary version of this article containing the above conditional estimate had been circulated), we were informed that Andrey Badzyan has proved (among other things in [1]) the following stronger estimate by using (our explicit formula for $\Phi_K(x)$ and) sieve methods

$$|\gamma_{\mathbb{Q}(\mu_m)}| = \mathbf{O}((\log m)(\log\log m)) \quad \text{(under GRH).}$$

The case $a+b=2$. Write $(\alpha,\beta) = \mathrm{Re}(\alpha\bar{\beta})$ for any $\alpha,\beta \in \mathbb{C}$. Then Theorem 5 for $(a,b) = (2,0),(1,1),(0,2)$ gives:

Corollary 6.1.2. *Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Then*

$$\lim_{m\to\infty} \frac{1}{|X_m|} \sum_{\chi\in X_m} \left(\alpha, \frac{L'(\chi,1)}{L(\chi,1)}\right)^2 = \frac{1}{2} \lim_{m\to\infty} \frac{1}{|X_m|} \sum_{\chi\in X_m} \left|\frac{L'(\chi,1)}{L(\chi,1)}\right|^2 = \frac{1}{2}\,\mu^{(1,1)}.$$

This remains valid if X_m is replaced by $X_m^{\pm}$.

The case $a+b=3$. In this case, $\mu^{(2,1)} = 0.0705\ldots$. This shows that the average of $(\mathrm{Re}(L'(\chi,1)/L(\chi,1)))^3$ for large m is $-\frac{3}{4}\mu^{(2,1)}$, which is negative! The same argument shows that the average of $(\mathrm{Re}(L'(\chi,1)/L(\chi,1)))^k$ is negative for large m, for any odd integer $k \geq 3$. (Recall, from the case $a+b=1$, that when $k=1$ this average tends to 0 as $m\to\infty$ and that conjecturally it is positive for each m.) What this might indicate is the possibility that the distribution of $L'(\chi,1)/L(\chi,1)$ on the left half plane is more spread out while that on the right is more numerous and accumulated near the y-axis. Such a tendency becomes evident only when m is large (see Figures 4, 5 below).

The case $(a,b) = (2,2)$. We have $\mu^{(2,2)} = 1.25\ldots$, which is considerably larger than the square of $\mu^{(1,1)}$. The result for $a+b=2$ alone does not exclude the possibility that the points $L'(\chi,1)/L(\chi,1)$ are distributed near the circle with center O and radius $\sqrt{\mu^{(1,1)}}$ with their arguments nearly uniformly distributed. But this result for $(2,2)$ shows that this cannot be the case.

We have computed other higher powers; for example, we can show that the average of $(\mathrm{Re}(L'(\chi,1)/L(\chi,1)))^8$ for large m exceeds 2.09.

Histograms for some m. Figures 4 resp. 5 are histograms for the distribution of $L'(\chi,1)/L(\chi,1)$ on the domain $\{z = x+iy \in \mathbb{C}\}$, for $\chi \in X_m^+$ resp. X_m^-, when $m = 104849$. (Incidentally, $\log\log m = 2.4475\ldots$.) The difference of slopes on the left and the right sides seems to be in accordance with the last remark in the above description for the case $a+b=3$.

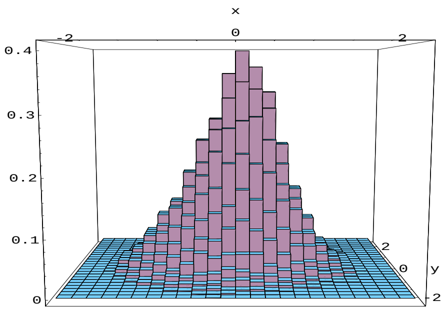

Fig. 4. $m = 104849$, even characters

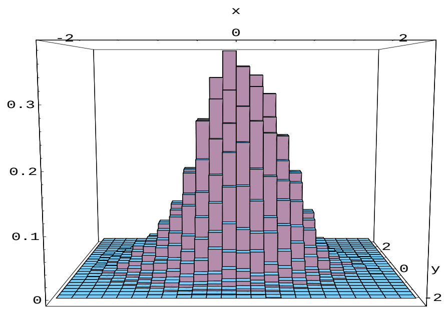

Fig. 5. $m = 104849$, odd characters

A final remark. One can show the following, at least under GRH ([8, especially Remark (ii) in §6.9]). If we average, with weight $|X_m|^{-1}$, the distribution over all primes $m \le N$, then the joint histogram converges to a limit as $N \to \infty$, and the height of the limit histogram at each $z \in \mathbb{C}$ is given by $M_1(z)/(2\pi)$, where $M_\sigma(z)$ is the "M-function" constructed and studied there. In particular, the limit height at $(0,0)$ can be computed as the integral of its Fourier dual $\tilde{M}_1(z)$ over $\mathbb{C}$. Approximately, $M_1(0)/(2\pi) = 2.41\ldots/(2\pi) = 0.38\ldots$ (loc.cit., Remark 3.11.17).

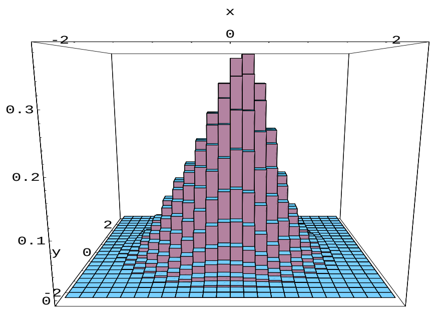

Fig. 6. $m < 21799$

Figure 6 gives the joint histogram for $N = 21799$. The height at its origin looks close to 0.38, but we must add that the value computed by using the smaller square region $|x|, |y| < 0.05$ is less close and is about $0.364\ldots$. We need better approximations from both sides to be able to determine the value more accurately.

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