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On the irreducibility of the generalized Laguerre polynomials

by

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1. Introduction. The *generalized Laguerre polynomials* are defined by

$$L_m^{(\alpha)}(x) = \sum_{j=0}^{m} \frac{(m+\alpha)(m-1+\alpha)\ldots(j+1+\alpha)(-x)^j}{(m-j)!j!},$$

where m is a positive integer and α is an arbitrary complex number. In 1929, I. Schur [4] established the irreducibility over the rationals of $L_m^{(0)}(x)$, the classical Laguerre polynomials, for every m. In 1931, I. Schur [5] considered $L_m^{(\alpha)}(x)$ in general and showed that $L_m^{(1)}(x)$ is irreducible over the rationals for every m. The case $\alpha \notin \{0,1\}$ remained open. The purpose of this paper is to establish the following:

Theorem 1. *Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers m, the polynomial $L_m^{(\alpha)}(x)$ is irreducible over the rationals.*

Before going to the proof, it is worth noting that reducible $L_m^{(\alpha)}(x)$ do exist even with $\alpha = 2$. In particular, we give the following examples:

$$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6),$$

$$L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30),$$

$$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040),$$

$$L_4^{(12/5)}(x) = \frac{1}{15000}(25x^2-420x+1224)(25x^2-220x+264),$$

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$$L_5^{(39/5)}(x) = \frac{-1}{375000}(5x-84)(625x^4 - 29500x^3 + 448400x^2 - 2662080x + 5233536).$$

It is not difficult to show that in fact there are infinitely many positive integers α for which $L_2^{(\alpha)}(x)$ is reducible (a product of two linear polynomials).

Theorem 1 is a direct consequence of the following more general result:

THEOREM 2. *Let α be a rational number which is not a negative integer. Then for all but finitely many positive integers m, the polynomial*

$$\sum_{j=0}^{m} a_j \frac{(m+\alpha)(m-1+\alpha)\dots(j+1+\alpha)x^j}{(m-j)!j!}$$

is irreducible over the rationals provided only that $a_j \in \mathbb{Z}$ for $0 \le j \le m$ and $|a_0| = |a_m| = 1$.

I. Schur obtained his irreducibility results for $L_m^{(0)}(x)$ and $L_m^{(1)}(x)$ through general results similar to the above. Recent work of a similar nature has been done by Filaseta [1, 2] and by Filaseta and Trifonov [3]. We note also that the above results can be made effective so that for any fixed $\alpha \in \mathbb{Q}$, α not a negative integer, it is possible to determine a finite set $S = S(\alpha)$ of m such that the polynomial in Theorem 2 is irreducible (for a_j as stated there) provided $m \notin S$.

2. A proof of Theorem 2. For a prime p and a non-zero integer a, we define $\nu(a) = \nu_p(a) = e$ where $p^e \parallel a$. We set $\nu(0) = \infty$. We begin with the following preliminary results.

LEMMA 1. *Let k be a positive integer. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$, $p \mid b_j$ for all $j \in \{0, 1, \dots, n-k\}$, and $\nu(b_j) > \nu(b_0) - j/k$ for $1 \le j \le n$. Then for any integers $a_0, a_1, \dots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j$ cannot have a factor of degree k in $\mathbb{Z}[x]$.*

LEMMA 2. *Let a, b, c and d be integers with $bc - ad \neq 0$. Then the largest prime factor of $(am+b)(cm+d)$ tends to infinity as the integer m tends to infinity.*

Lemma 1 is a consequence of Lemma 2 in [1]. Observe that $f(x)$ satisfies the same conditions as $g(x)$ in the lemma so that the lemma can be established by simply showing the conditions on $g(x)$ imply $g(x)$ cannot have a factor of degree k (see [1] for details). Lemma 2 above is a fairly easy consequence of the fact that the Thue equation $ux^3 - vy^3 = w$ has finitely many solutions in integers x and y where u, v, and w are fixed integers with $w \neq 0$. It also immediately follows from Corollary 1.2 of [6]. We omit the proofs.

Fix α now as in Theorem 2. Throughout the argument we suppose as we may that m is large. Define

$$c_j = \binom{m}{j}(m+\alpha)(m-1+\alpha)\dots(j+1+\alpha) \quad \text{for } 0 \le j \le m.$$

We want to show that for all but finitely many positive integers m, the polynomial $f(x) = \sum_{j=0}^m a_j c_j x^j$ is irreducible over the rationals, where a_j are arbitrary integers with $|a_0| = |a_m| = 1$. Motivated by Lemma 1, we consider instead $g(x) = \sum_{j=0}^m c_j x^j$. Let u and v be relatively prime integers with $v > 0$ such that $\alpha = u/v$. The condition that α is not a negative integer implies that for each $j \in \{0, 1, \dots, m-1\}$, $m - j + \alpha$ and, hence, $v(m-j) + u$ cannot be zero. We assume that $g(x)$ has a factor in $\mathbb{Z}[x]$ of degree $k \in [1, m/2]$ and establish the theorem by obtaining a contradiction to Lemma 1. We divide the argument into cases depending on the size of k.

Case 1: $k > m/\log^2 m$. For a and b integers with $b > 0$, let $\pi(x; b, a)$ denote the number of primes $\le x$ which are $\equiv a \pmod{b}$. Then the Prime Number Theorem for Arithmetic Progressions implies that if $\gcd(a, b) = 1$, then

$$\begin{aligned}\pi(x; b, a) &= \frac{1}{\phi(b)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x}{\log^4 x}\right) \\ &= \frac{1}{\phi(b)}\left(\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\left(\frac{x}{\log^4 x}\right)\right).\end{aligned}$$

By considering $\pi(x; b, a) - \pi(x-h; b, a)$, it follows that for a and b fixed, the interval $(x-h, x]$ contains a prime $\equiv a \pmod{b}$ if $h = x/(2\log^2 x)$ and if x is sufficiently large. Taking $a = u$, $b = v$, and $x = vm + u$, we deduce that for some integer $j \in [0, k)$, the number $v(m-j) + u$ is prime. Call such a prime p, and observe that $p \ge 2vm/3$ (since v is a positive integer and m is large). We deduce that p does not divide v. Observe that

$$c_l = \binom{m}{l} \frac{(vm+u)(v(m-1)+u)\dots(v(l+1)+u)}{v^{m-l}} \quad \text{for } 0 \le l \le m.$$

For $j \in \{0, 1, \dots, k-1\}$, the numbers $v(m-j)+u$ appear in the numerator of the fraction on the right-hand side above whenever $0 \le l \le m-k$. Therefore,

$$\nu_p(c_l) \ge 1 \quad \text{for } 0 \le l \le m-k. \tag{1}$$

Since $c_m = 1$, $\nu_p(c_m) = 0$. To obtain a contradiction to Lemma 1 for the case under consideration, we show that $\nu_p(c_0) = 1$; the contradiction will be achieved since (1) and $k \le m-k$ imply $\nu(c_l) \ge 1 > 1 - l/k$ for $1 \le l \le k$ and since the inequality $\nu(c_l) > 1 - l/k$ is clear for $k < l \le m$. Recall that $p \nmid v$ and that $p \ge 2vm/3$. For $j \in \{0, 1, \dots, m-1\}$, we deduce the inequality

$$2p > vm + u \ge v(m-j) + u \ge v + u > -p.$$

As α is not a negative integer, none of $v(m-j)+u$ can be zero. Hence, p itself is the only multiple of p among the numbers $v(m-j)+u$ with $0 \le j \le m-1$. Since $c_0 = (vm+u)(v(m-1)+u)\dots(v+u)/v^m$, we obtain $\nu_p(c_0) = 1$.

CASE 2: $k_0 \le k \le m/\log^2 m$ *with* $k_0 = k_0(u,v)$ *a sufficiently large integer.* Let $z = k(\log\log k)^{1/2}$. We first show that there is a prime $p > z$ that divides $v(m-j)+u$ for some $j \in \{0,1,\dots,k-1\}$. Then (1) follows as before, and we will obtain a contradiction to Lemma 1 by showing $\nu(c_j) > \nu(c_0) - j/k$ for $1 \le j \le m$.

Let
$$T = \{v(m-j)+u : 0 \le j \le k-1\}.$$
Since m is large, we deduce that the elements of T are each $\ge m/2$. Also, observe that $\gcd(u,v) = 1$ implies that each element of T is relatively prime to v. For each prime $p \le z$, we consider an element $a_p = v(m-j)+u \in T$ with $\nu_p(a_p)$ as large as possible. We let
$$S = T - \{a_p : p \nmid v,\, p \le z\}.$$
By the Prime Number Theorem,
$$\pi(z) \le \frac{2k(\log\log k)^{1/2}}{\log k}.$$
We combine this momentarily with $|S| \ge k - \pi(z)$. Since $k \le m/\log^2 m$, we obtain $m \ge k\log^2 k$. Consider a prime $p \le z$ with p not dividing v, and let $r = \nu_p(a_p)$. By the definition of a_p, if $j > r$, then there are no multiples of p^j in T (and, hence, in S). For $1 \le j \le r$, there are $\le [k/p^j]+1$ multiples of p^j in T and, hence, at most $[k/p^j]$ multiples of p^j in S. Therefore,
$$\nu_p\Big(\prod_{s\in S} s\Big) \le \sum_{j=1}^{r}\left[\frac{k}{p^j}\right] \le \nu_p(k!) \quad \text{and} \quad \prod_{s\in S}\prod_{p\le z} p^{\nu_p(s)} \le k! \le k^k.$$
On the other hand,
$$\prod_{s\in S} s \ge \left(\frac{m}{2}\right)^{|S|} \ge \left(\frac{k\log^2 k}{2}\right)^{k-\pi(z)}.$$
Recalling our bound on $\pi(z)$, we obtain
$$\begin{aligned}\log\Big(\prod_{s\in S} s\Big) &\ge (k-\pi(z))(\log k + 2\log\log k - \log 2)\\ &\ge \left(k - \frac{2k\sqrt{\log\log k}}{\log k}\right)(\log k + 2\log\log k - \log 2)\\ &\ge k\log k + 2k\log\log k + O(k\sqrt{\log\log k}).\end{aligned}$$

Since $k \ge k_0$ where k_0 is sufficiently large,

$$\log\Big(\prod_{s\in S} s\Big) > k\log k \ge \log\Big(\prod_{s\in S}\prod_{p\le z} p^{\nu_p(s)}\Big).$$

It follows that there is a prime $p > z$ that divides some element of S and, hence, divides some element of T.

Fix a prime $p > z$ that divides an element in T, and let $\nu = \nu_p$. Fix $j \in \{1,\dots,m\}$. We show $\nu(c_j) > \nu(c_0) - j/k$. Observe that

$$\begin{aligned}\nu(c_0) - \nu(c_j) &\le \nu((vj+u)(v(j-1)+u)\dots(v+u))\\ &\le \nu((vj+|u|)!) = \sum_{i=1}^{\infty}\left[\frac{vj+|u|}{p^i}\right] < \sum_{i=1}^{\infty}\frac{vj+|u|}{p^i} = \frac{vj+|u|}{p-1}.\end{aligned}$$

Since $p > z = k(\log\log k)^{1/2}$ and $k \ge k_0$, we deduce that $(vj+|u|)/(p-1) < j/k$ and the inequality $\nu(c_j) > \nu(c_0) - j/k$ follows. Hence, as indicated at the beginning of this case, we obtain a contradiction to Lemma 1.

Case 3: $2 \le k < k_0$. By Lemma 2 (with $a = v$, $b = u$, $c = v$, and $d = u - v$), the largest prime factor of the product $(vm+u)(v(m-1)+u)$ tends to infinity. Since m is large, we deduce that there is a prime $p > (v+|u|)k_0$ that divides $(vm+u)(v(m-1)+u)$. The argument now follows as in the previous case. In particular,

$$\frac{\nu(c_0)-\nu(c_j)}{j} < \frac{vj+|u|}{j(p-1)} \le \frac{v+|u|}{p-1} \le \frac{1}{k_0} < \frac{1}{k} \quad \text{for } 1 \le j \le m.$$

Hence, in this case, we also obtain a contradiction.

Case 4: $k = 1$. From Lemma 2, if $u \ne 0$, then the largest prime factor of $m(vm+u)$ tends to infinity with m. We consider a large prime factor p of this product. In particular, we suppose that $p > v + |u|$. Note this implies $p \nmid v$. As in the previous case, we are through if $p \mid (vm+u)$. So suppose $p \mid m$. The binomial coefficient $\binom{m}{j}$ appears in the definition of c_j, and this is sufficient to guarantee that $\nu(c_j) \ge 1$ and $\nu(c_{m-j}) \ge 1$ for $1 \le j \le p-1$. On the other hand,

$$c_j = \binom{m}{j}\frac{(vm+u)(v(m-1)+u)\dots(v(j+1)+u)}{v^{m-j}}.$$

For $j \le m-p$, the numerator of the fraction on the right is a product of $\ge p$ consecutive terms in the arithmetic progression $vt+u$ with $\gcd(p,v) = 1$; thus, $\nu(c_{m-j}) \ge 1$ for $j \ge p$. This implies that (1) holds with $k = 1$. It follows, along the lines of the previous two cases, that $\nu(c_j) > \nu(c_0) - j/k$ for $1 \le j \le m$. A contradiction to Lemma 1 is again obtained (and the proof of the theorem is complete).

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