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Corrigendum to Theorem 5 of the paper "Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$"

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by

OTO STRAUCH (Bratislava) and JÁNOS T. TÓTH (Ostrava)

In the proof of Theorem 5 in [2], step 3 is incorrect. We want to thank S. V. Konyagin who has pointed it out. The wrong Theorem 5 asserts that for every increasing sequence of positive integers x_n, $n = 1, 2, \ldots$, with a positive lower asymptotic density, if there exists an interval (u, v) containing no limit points of the ratio sequence x_m/x_n, $m, n=1, 2, \ldots$, where u, v are limit points, then there are infinitely many such intervals. In the new form of Theorem 5 we replace intervals (u, v) containing no limit points of x_m/x_n with intervals having some *zero asymptotic density* of x_m/x_n and we reformulate it in terms of *distribution functions* of x_m/x_n. We prove that if there exists an interval (u, v), containing no limit points of x_m/x_n, then every distribution function of x_m/x_n has infinitely many intervals with constant values, assuming positive lower asymptotic density of x_n. For an illustration, we give two examples. In Example 1, x_m/x_n has only one such interval (u, v), and in Example 2 it has infinitely many, and in both cases every distribution function of x_m/x_n has infinitely many intervals with constant values. Finally, we discuss via Examples 1 and 2 a possibility of adding a proposition contained in the incorrect step 3 as an assumption of Theorem 5.

To do this we need the following concept used in [3] (see [1] for a general account).

A function $g : [0, 1] \to [0, 1]$ will be called a *distribution function* (abbreviated d.f.) if $g(0) = 0$, $g(1) = 1$, and g is nondecreasing. We will identify any two distribution functions coinciding a.e. on $[0, 1]$. A point $\beta \in [0, 1]$

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is called a *point of increase* (or a point of the spectrum) of the d.f. $g(x)$ if either $g(x) > g(\beta)$ for every $x > \beta$ or $g(x) < g(\beta)$ for every $x < \beta$, $x \in [0,1]$. Now, for x_n we define the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

and consider a step d.f.

$$F(X_n, x) = \frac{\#\{i \le n : x_i/x_n < x\}}{n}$$

for $x \in [0,1)$ and $F(X_n, 1) = 1$. A d.f. g is a d.f. of the block sequence X_n if there exists a sequence of positive integers $n_1 < n_2 < \dots$ such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

a.e. on $[0,1]$. The set of all d.f. of the sequence of blocks X_n is denoted by $G(X_n)$. Finally, denote the counting function by $A(t) = \#\{n \in \mathbb{N} : x_n < t\}$ and define the *lower asymptotic density* $\underline{d}$ and *upper asymptotic density* $\overline{d}$ of x_n by

$$\underline{d} = \liminf_{t\to\infty} \frac{A(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n}, \qquad \overline{d} = \limsup_{t\to\infty} \frac{A(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

A corrected form of Theorem 5 of [2] is as follows:

THEOREM. *Assume that* $\underline{d} > 0$. *If there exists an interval* $(u,v) \subset [0,1]$ *such that every* $g \in G(X_n)$ *has a constant value on* (u,v) *(maybe different), then every* $g \in G(X_n)$ *has infinitely many intervals with constant values such that* g *increases at their endpoints.*

Proof. Since

$$x_i < x x_m \iff x_i < \left(x\frac{x_m}{x_n}\right)x_n,$$

we have

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right) \tag{1}$$

for every $m \le n$ and $x \in [0,1)$. Using the Helly selection principle, we can select a subsequence (m_k, n_k) of the sequence (m,n) such that $F(X_{n_k}) \to g(x)$ and $F(X_{m_k}) \to \widetilde{g}(x)$ as $k \to \infty$; furthermore $x_{m_k}/x_{n_k} \to \beta$ and $n_k/m_k \to \alpha$, but α may be infinity. Assuming $\beta > 0$ and $g(\beta - 0) > 0$, we have $\alpha < \infty$ and

$$\widetilde{g}(x) = \alpha g(x\beta) \quad \text{a.e. on } [0,1]. \tag{2}$$

Thus, if $\widetilde{g}(x)$ has a constant value on (u,v), then $g(x)$ must be constant on the interval $(u\beta, v\beta)$. Furthermore, if $\underline{d} > 0$, then for every $g \in G(X_n)$ we have

$$(\underline{d}/\overline{d})x \le g(x) \le (\overline{d}/\underline{d})x \tag{3}$$

for every $x \in [0,1]$. Thus, there exists a sequence $\beta_k \in (0,1)$ such that $\beta_k \searrow 0$ and $g(x)$ increases at β_k, $g(\beta_k) > 0$, $k = 1,2,\ldots$ For such β_k and $g(x)$, applying the Helly principle, we can find sequences α_k and $\widetilde{g}_k(x) \in G(X_n)$ such that

$$\widetilde{g}_k(x) = \alpha_k g(x\beta_k)$$

a.e. on $[0,1]$. Every $\widetilde{g}_k(x)$ has a constant value on the interval (u,v), hence, $g(x)$ must be constant on the intervals $(u\beta_k, v\beta_k)$ for $k = 1,2,\ldots$

For completeness we provide

Proof of (2). First, we prove

$$\lim_{k\to\infty} F\left(X_{n_k}, x\frac{x_{m_k}}{x_{n_k}}\right) = g(x\beta). \tag{4}$$

Setting, for abbreviation, $\beta_k = x_{m_k}/x_{n_k}$ and substituting $u = x\beta_k$ we find

$$\begin{aligned} 0 &\le \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2\,dx \\ &= \frac{1}{\beta_k}\int_0^{\beta_k} (F(X_{n_k}, u) - g(u))^2\,du \le \frac{1}{\beta_k}\int_0^1 (F(X_{n_k}, u) - g(u))^2\,du \to 0, \end{aligned}$$

which leads to $F(X_{n_k}, x\beta_k) - g(x\beta_k) \to 0$ as $k \to \infty$ (here, necessarily, $\beta > 0$). Furthermore,

$$\begin{aligned} &\int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta))^2\,dx \\ &\qquad = \int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k) + g(x\beta_k) - g(x\beta))^2\,dx \\ &\qquad \le 2\Big(\int_0^1 (F(X_{n_k}, x\beta_k) - g(x\beta_k))^2\,dx + \int_0^1 (g(x\beta_k) - g(x\beta))^2\,dx\Big). \end{aligned}$$

Since $g(x)$ is continuous a.e. on $[0,1]$, $g(x\beta_k) - g(x\beta) \to 0$ a.e. and applying the Lebesgue dominant convergence theorem, we have $\int_0^1 (g(x\beta_k) - g(x\beta))^2\,dx \to 0$, which gives (4) and implies (2). Further, $\alpha < \infty$ follows from (1) and $g(\beta - 0) > 0$.

Proof of (3). Since

$$\#\{i \le n : x_i/x_n < x\} = \#\{i = 1,2,\ldots : x_i < xx_n\},$$

we have

$$\frac{F(X_n, x)n}{xx_n} = \frac{A(xx_n)}{xx_n} \quad \text{for every } x \in [0,1].$$

Whenever $g \in G(X_n)$, there exists n_k such that $F(X_{n_k}, x) \to g(x)$ a.e. and $n_k/x_{n_k} \to d_1$. Then for some $d_2(x)$ with $\lim_{k\to\infty} A(xx_{n_k})/(xx_{n_k}) = d_2(x)$ we get

$$\frac{g(x)}{x} d_1 = d_2(x)$$

a.e. on $[0,1]$. Using the fact that $\underline{d} \le d_1 \le \overline{d}$ and $\underline{d} \le d_2 \le \overline{d}$, we have $(g(x)/x)\underline{d} \le \overline{d}$ and $(g(x)/x)\overline{d} \ge \underline{d}$ a.e. If $\underline{d} > 0$, these inequalities are valid for every $x \in (0,1]$. ■

Further properties of $G(X_n)$ can be found in [3], e.g. if $\underline{d} > 0$, then each $g \in G(X_n)$ is everywhere continuous on $[0,1]$.

The basic idea of the following type of sequences x_n is also due to Konyagin.

Example 1. Let $k_0 < k_1 < k_2 < \dots$ be an increasing sequence of positive integers, n_0 and m_0 be two integers and γ, δ and a be real numbers satisfying

(i) $k_s - k_{s-1} \to \infty$ as $s \to \infty$,
(ii) $0 < \gamma < \delta$, $a > 1$, $n_0 \le m_0$ and $1/a^{n_0} \le \gamma/\delta$.

(In what follows, we will abbreviate the interval $(\gamma\lambda, \delta\lambda)$ as $(\gamma,\delta)\lambda$.) Let x_n be an increasing sequence of all integer points lying in the intervals

$$(\gamma,\delta)a^{k_s m_0 n_0 + j n_0}, \qquad 0 \le j < (k_{s+1} - k_s)m_0,\ s = 0,2,4,\dots,$$
$$(\gamma,\delta)a^{k_s m_0 n_0 + j m_0}, \qquad 0 \le j < (k_{s+1} - k_s)n_0,\ \ s = 1,3,5,\dots,$$

i.e. we have a sequence of intervals of the form $(\gamma,\delta)(a^{n_0})^i$ and $(\gamma,\delta)(a^{m_0})^j$, where these forms alternate on common $(\gamma,\delta)(a^{n_0 m_0})^{k_s}$.

Complement of limit points. Let X be the complement in $[0,1]$ of the limit points of x_m/x_n. Define

$$I(n_0) = \left(\frac{\delta}{\gamma a^{n_0}}, \frac{\gamma}{\delta}\right), \qquad I(m_0) = \left(\frac{\delta}{\gamma a^{m_0}}, \frac{\gamma}{\delta}\right),$$

$$B(n_0,j) = I(n_0) \cup \frac{I(n_0)}{a^{n_0}} \cup \dots \cup \frac{I(n_0)}{(a^{n_0})^{j-1}}$$
$$\cup \frac{1}{(a^{n_0})^j}\left(I(m_0) \cup \frac{I(m_0)}{a^{m_0}} \cup \frac{I(m_0)}{(a^{m_0})^2} \cup \frac{I(m_0)}{(a^{m_0})^3} \cup \dots\right),$$

$$B(m_0,j) = I(m_0) \cup \frac{I(m_0)}{a^{m_0}} \cup \dots \cup \frac{I(m_0)}{(a^{m_0})^{j-1}}$$
$$\cup \frac{1}{(a^{m_0})^j}\left(I(n_0) \cup \frac{I(n_0)}{a^{n_0}} \cup \frac{I(n_0)}{(a^{n_0})^2} \cup \frac{I(n_0)}{(a^{n_0})^3} \cup \dots\right).$$

Then

$$X = \Big(\bigcap_{j=0}^{\infty} B(n_0, j)\Big) \cap \Big(\bigcap_{j=0}^{\infty} B(m_0, j)\Big). \tag{5}$$

Thus, in all cases $X \supset I(n_0)$ and assuming additionally

(iii) $1 < n_0 < m_0$, $\gcd(n_0, m_0) = 1$,

(iv) $\dfrac{1}{a^{n_0}} < \left(\dfrac{\gamma}{\delta}\right)^2$,

(v) $\left(\dfrac{\gamma}{\delta}\right)^2 \leq \dfrac{a^{n_0}}{a^{m_0}}$, $\left(\dfrac{\gamma}{\delta}\right)^2 \leq \dfrac{a^{m_0}}{a^{2n_0}}$,

(vi) $\left(\dfrac{\gamma}{\delta}\right)^2 \leq \dfrac{(a^{n_0})^{[m_0 k/n_0]+1}}{(a^{m_0})^{k+1}}$, $\left(\dfrac{\gamma}{\delta}\right)^2 \leq \dfrac{(a^{m_0})^k}{(a^{n_0})^{[m_0 k/n_0]+1}}$, $k = 1, \ldots, n_0 - 2$,

we have

$$X = I(n_0) \neq \emptyset. \tag{6}$$

The assumptions (i)–(vi) hold for $k_s = s^2$, $\gamma = 1$, $\delta = 2$, $a = 2$, $n_0 = 3$ and $m_0 = 4$. Here $X = (1/2^2, 1/2)$.

Proof of (5) and (6). We briefly mention the following steps.

1. For terms $x_n \in (\gamma, \delta)a^{k_s m_0 n_0 + j n_0}$, $n \to \infty$, we have two possibilities:

   (a) s even $\to \infty$, j fixed;
   (b) s even $\to \infty$, $j \to \infty$.

Similarly, for $x_n \in (\gamma, \delta)a^{k_s m_0 n_0 + j m_0}$ we have

   (c) s odd $\to \infty$, j fixed;
   (d) s odd $\to \infty$, $j \to \infty$.

By direct computation we find that $B(n_0, j)$ is the complement of the limit points of x_m/x_n having x_n of type (a), $B(m_0, j)$ of type (c), $B(m_0, 0)$ of type (b) and $B(n_0, 0)$ of type (d).

2. Define

$$A(n_0) = I(n_0) \cup \frac{I(n_0)}{(a^{n_0})^1} \cup \ldots \cup \frac{I(n_0)}{(a^{n_0})^{m_0-2}} \cup \frac{I(n_0)}{(a^{n_0})^{m_0-1}},$$

$$A(m_0) = I(m_0) \cup \frac{I(m_0)}{(a^{m_0})^1} \cup \ldots \cup \frac{I(m_0)}{(a^{m_0})^{n_0-2}} \cup \frac{I(m_0)}{(a^{m_0})^{n_0-1}}.$$

Since $A(n_0)$ and $A(m_0)$ lie in $I = (\delta/(\gamma a^{m_0 n_0}), \gamma/\delta)$ we have

$$B(n_0, 0) \cap B(m_0, 0) = (A(n_0) \cap A(m_0)) \cup \frac{A(n_0) \cap A(m_0)}{a^{m_0 n_0}} \cup \frac{A(n_0) \cap A(m_0)}{a^{2m_0 n_0}} \cup \frac{A(n_0) \cap A(m_0)}{a^{3m_0 n_0}} \cup \ldots$$

3. Assumptions (iii) and (vi) imply

$$A(n_0)\cap A(m_0)=I(n_0)\cup\frac{I(n_0)}{(a^{n_0})^{m_0-1}}.$$

4. Applying (v) we have

$$a^{m_0}\,\frac{A(n_0)}{a^{sm_0n_0}}\cap\left(\frac{A(n_0)\cap A(m_0)}{a^{sm_0n_0}}\cup\frac{A(n_0)\cap A(m_0)}{a^{(s-1)m_0n_0}}\right)=\frac{I(n_0)}{(a^{n_0})^{m_0-1}a^{sm_0n_0}}\cup\frac{I(n_0)}{a^{(s-1)m_0n_0}},$$

which gives

$$B(n_0,0)\cap B(m_0,0)\cap B(m_0,n_0-1)=I(n_0).\ \blacksquare$$

Distribution functions. Here we assume only (i) and (ii). Define

$$I(n_0,t)=\frac{1}{t\gamma+(1-t)\delta}\left(\frac{\delta}{a^{n_0}},\gamma\right),\qquad I(m_0,t)=\frac{1}{t\gamma+(1-t)\delta}\left(\frac{\delta}{a^{m_0}},\gamma\right),$$
$$I(t)=\frac{1}{t\gamma+(1-t)\delta}(\gamma,\delta).$$

The set $G(X_n)$ of all d.f. of X_n has the structure

$$G(X_n)=\{g_{n_0,j,t}(x):j=0,1,\dots,\ t\in[0,1]\}\cup\{g_{m_0,j,t}(x):j=0,1,\dots,\ t\in[0,1]\},$$

where the d.f. $g_{n_0,j,t}(x)$ has constant values on the intervals

$$I(n_0,t),\frac{I(n_0,t)}{a^{n_0}},\dots,\frac{I(n_0,t)}{(a^{n_0})^{j-1}},\frac{I(m_0,t)}{(a^{n_0})^{j}},\frac{I(m_0,t)}{(a^{n_0})^{j}(a^{m_0})},\frac{I(m_0,t)}{(a^{n_0})^{j}(a^{m_0})^2},\dots,$$

while on the complement intervals in $[0,1]$

$$\left(\frac{\gamma}{t\gamma+(1-t)\delta},1\right),\frac{I(t)}{a^{n_0}},\frac{I(t)}{(a^{n_0})^2},\dots,\frac{I(t)}{(a^{n_0})^j},\frac{I(t)}{(a^{n_0})^j(a_{m_0})},\frac{I(t)}{(a^{n_0})^j(a_{m_0})^2},\dots\tag{7}$$

it has a constant derivative

$$g'_{n_0,j,t}(x)=1/d,\tag{8}$$

where $\underline{d}\le d\le\overline{d}$ and

$$d=\frac{\delta-\gamma}{t\gamma+(1-t)\delta}\left(1-t+\frac{1}{a^{n_0}-1}-\frac{1}{(a^{n_0})^j}\left(\frac{1}{a^{n_0}-1}-\frac{1}{a^{m_0}-1}\right)\right).$$

Here

$$\underline{d}=\frac{\delta-\gamma}{\gamma}\cdot\frac{1}{a^{m_0}-1},\qquad\overline{d}=\frac{\delta-\gamma}{\delta}\cdot\frac{a^{n_0}}{a^{n_0}-1}.$$

These assertions characterize the d.f. $g_{n_0,j,t}(x)$. Similarly we define d.f. $g_{m_0,j,t}(x)$, exchanging n_0 with m_0 in the intervals and derivatives defined above.

Proof of (8). 1. If $F(X_n, x) \to g(x)$ for some $n \to \infty$, then we can select a subsequence of n such that $n/x_n \to d$ and, for some $t \in [0,1]$,

$$x_n = (t\gamma + (1-t)\delta)a^{k_s m_0 n_0 + j n_0} + o(a^{k_s m_0 n_0 + j n_0}), \qquad s \text{ even} \to \infty,$$
$$x_n = (t\gamma + (1-t)\delta)a^{k_s m_0 n_0 + j m_0} + o(a^{k_s m_0 n_0 + j m_0}), \qquad s \text{ odd} \to \infty,$$

and vice versa for any $t \in [0,1]$ and any x_n of these forms we have $n/x_n \to d > 0$, which implies $F(X_n, x) \to g(x)$ for some d.f. $g(x)$, since we have

$$\frac{\Delta F(X_n, x)}{\Delta x} = \frac{1/n}{(i+1)/x_n - i/x_n} = \frac{x_n}{n}$$

on intervals (7). For such x_n, the complement of (7) contains no x_m/x_n.

2. We directly compute the limit d for cases (a)–(d) specified in step 1 of the above proof. ■

EXAMPLE 2. In Example 1 we put $k_s = s$ for $s = 0, 1, 2, \ldots$, i.e. x_n is a sequence of all integer points lying in the intervals

$$\begin{aligned}
&(\gamma,\delta)(a^{n_0})^0, (\gamma,\delta)(a^{n_0})^1, \ldots, (\gamma,\delta)(a^{n_0})^{m_0-1},\\
&(\gamma,\delta)(a^{m_0})^{n_0}, (\gamma,\delta)(a^{m_0})^{n_0+1}, \ldots, (\gamma,\delta)(a^{m_0})^{2n_0-1},\\
&(\gamma,\delta)(a^{n_0})^{2m_0}, (\gamma,\delta)(a^{n_0})^{2m_0+1}, \ldots, (\gamma,\delta)(a^{n_0})^{3m_0-1},\\
&(\gamma,\delta)(a^{m_0})^{3n_0}, (\gamma,\delta)(a^{m_0})^{3n_0+1}, \ldots
\end{aligned}$$

Complement of limit points. Define

$$\begin{aligned}
B(n_0, j) &= I(n_0) \cup \frac{I(n_0)}{a^{n_0}} \cup \ldots \cup \frac{I(n_0)}{(a^{n_0})^{j-1}}\\
&\qquad \cup \frac{1}{(a^{n_0})^j}\left(A(m_0) \cup \frac{A(n_0)}{a^{m_0 n_0}} \cup \frac{A(m_0)}{a^{2m_0 n_0}} \cup \frac{A(n_0)}{a^{3m_0 n_0}} \cup \ldots\right),\\
B(m_0, j) &= I(m_0) \cup \frac{I(m_0)}{a^{m_0}} \cup \ldots \cup \frac{I(m_0)}{(a^{m_0})^{j-1}}\\
&\qquad \cup \frac{1}{(a^{m_0})^j}\left(A(n_0) \cup \frac{A(m_0)}{a^{m_0 n_0}} \cup \frac{A(n_0)}{a^{2m_0 n_0}} \cup \frac{A(m_0)}{a^{3m_0 n_0}} \cup \ldots\right).
\end{aligned}$$

Then

$$X = \Big(\bigcap_{j=0}^{m_0-1} B(n_0, j)\Big) \cap \Big(\bigcap_{j=0}^{n_0-1} B(m_0, j)\Big). \tag{9}$$

For $n_0 = m_0$ this gives (cf. [2, Ex. 1])

$$X = \bigcup_{i=0}^{\infty} \frac{I(n_0)}{(a^{n_0})^i}.$$

Assuming (i)–(vi) we have

$$\begin{aligned}(10)\qquad X = I(n_0) &\cup \frac{I(n_0)}{a^{2m_0n_0}} \cup \frac{I(n_0)}{a^{4m_0n_0}} \cup \frac{I(n_0)}{a^{6m_0n_0}} \cup \dots \\ &\cup a^{n_0}\left(\frac{I(n_0)}{a^{2m_0n_0}} \cup \frac{I(n_0)}{a^{4m_0n_0}} \cup \frac{I(n_0)}{a^{6m_0n_0}} \cup \dots\right).\end{aligned}$$

Proof of (9) and (10). Similarly to proof of (5) and (6) in Example 1, but the step 4 can only be used for odd s, since here $B(m_0, n_0-1)$ contains only $a^{m_0}A(n_0)/a^{(2i+1)m_0n_0}$. ■

Distribution functions. As in Example 1,

$$I(n_0,t) = \frac{1}{t\gamma + (1-t)\delta}\left(\frac{\delta}{a^{n_0}}, \gamma\right), \qquad I(m_0,t) = \frac{1}{t\gamma + (1-t)\delta}\left(\frac{\delta}{a^{m_0}}, \gamma\right),$$

$$I(t) = \frac{1}{t\gamma + (1-t)\delta}(\gamma, \delta).$$

The set $G(X_n)$ of all d.f. of X_n has the structure

$$\begin{aligned}G(X_n) = \{&g_{n_0,j,t}(x) : j = 0,1,\dots,m_0-1,\ t\in[0,1]\} \\ &\cup \{g_{m_0,j,t}(x) : j = 0,1,\dots,n_0-1,\ t\in[0,1]\},\end{aligned}$$

where the d.f. $g_{n_0,j,t}(x)$ has constant values on the intervals

$$I(n_0,t), \frac{I(n_0,t)}{a^{n_0}}, \dots, \frac{I(n_0,t)}{(a^{n_0})^{j-1}},$$

$$\frac{I(m_0,t)}{(a^{n_0})^j}, \frac{I(m_0,t)}{(a^{n_0})^j a^{m_0}}, \dots, \frac{I(m_0,t)}{(a^{n_0})^j (a^{m_0})^{n_0-1}}, \frac{I(n_0,t)}{(a^{n_0})^j (a^{m_0n_0})},$$

$$\frac{I(n_0,t)}{(a^{n_0})^j (a^{m_0n_0}) a^{n_0}}, \dots, \frac{I(n_0,t)}{(a^{n_0})^j (a^{m_0n_0})(a^{n_0})^{m_0-1}}, \frac{I(m_0,t)}{(a^{n_0})^j (a^{2m_0n_0})}, \dots,$$

while on the complement intervals in $[0,1]$

$$\left(\frac{\gamma}{t\gamma+(1-t)\delta}, 1\right), \frac{I(t)}{a^{n_0}}, \frac{I(t)}{(a^{n_0})^2}, \dots, \frac{I(t)}{(a^{n_0})^j},$$

$$\frac{I(t)}{(a^{n_0})^j (a^{m_0})}, \frac{I(t)}{(a^{n_0})^j (a^{m_0})^2}, \dots, \frac{I(t)}{(a^{n_0})^j a^{m_0n_0}},$$

$$\frac{I(t)}{(a^{n_0})^j a^{m_0n_0}(a^{n_0})}, \frac{I(t)}{(a^{n_0})^j a^{m_0n_0}(a^{n_0})^2}, \dots, \frac{I(t)}{(a^{n_0})^j a^{2m_0n_0}}, \dots$$

it has a constant derivative

$$(11)\qquad g'_{n_0,j,t}(x) = 1/d,$$

where $\underline{d} \le d \le \overline{d}$ and

$$d = \frac{\delta - \gamma}{t\gamma + (1-t)\delta} \times \left(1 - t + \frac{1}{a^{n_0} - 1} - \frac{1}{(a^{n_0})^j} \cdot \frac{a^{m_0 n_0}}{a^{m_0 n_0} + 1}\left(\frac{1}{a^{n_0} - 1} - \frac{1}{a^{m_0} - 1}\right)\right).$$

Here

$$\underline{d} = \frac{\delta - \gamma}{\gamma}\left(\frac{1}{a^{n_0} - 1} - \frac{a^{m_0 n_0}}{a^{m_0 n_0} + 1}\left(\frac{1}{a^{n_0} - 1} - \frac{1}{a^{m_0} - 1}\right)\right),$$
$$\overline{d} = \frac{\delta - \gamma}{\delta}\left(1 + \frac{1}{a^{m_0} - 1} + \frac{a^{m_0 n_0}}{a^{m_0 n_0} + 1}\left(\frac{1}{a^{n_0} - 1} - \frac{1}{a^{m_0} - 1}\right)\right).$$

These assertions characterize d.f. $g_{n_0,j,t}(x)$. Similarly we define d.f. $g_{m_0,j,t}(x)$, exchanging n_0 with m_0 in the intervals and derivatives defined above.

Proof of (11). As the proof of (8) in Example 1. ■

Concluding remarks. Theorem 5 in [2] can also be amended by adding the assertion of the incorrect step 3 to the assumptions of this theorem. This gives the following second correct form: Assume that there exists a sequence of positive integers $g(n)$ such that $\lim_{n\to\infty} x_{g(n)}/x_n = \lambda$ and $0 < \lambda < 1$ and let $\underline{d} > 0$. If there exists an interval (u, v) containing no limit points of x_m/x_n, then there are infinitely many such intervals, e.g. $(u, v)\lambda^j$, $j = 0, 1, 2, \ldots$ All possible limits λ form a cyclic group.

By this theorem, for x_n in Example 1, there exists no such λ. We can see this directly, since such λ must be a common term of the following sequences:

$$\frac{1}{a^{n_0}}, \frac{1}{(a^{n_0})^2}, \ldots, \frac{1}{(a^{n_0})^j}, \frac{1}{(a^{n_0})^j (a^{m_0})}, \frac{1}{(a^{n_0})^j (a^{m_0})^2}, \frac{1}{(a^{n_0})^j (a^{m_0})^3}, \ldots,$$
$$j = 0, 1, 2, \ldots$$

$$\frac{1}{a^{m_0}}, \frac{1}{(a^{m_0})^2}, \ldots, \frac{1}{(a^{m_0})^j}, \frac{1}{(a^{m_0})^j (a^{n_0})}, \frac{1}{(a^{m_0})^j (a^{n_0})^2}, \frac{1}{(a^{m_0})^j (a^{n_0})^3}, \ldots,$$
$$j = 0, 1, 2, \ldots$$

For $j = 0$ we see that λ must have a form $1/a^{k m_0 n_0}$, but for $j = 1$ there exists no i such that $1/a^{k m_0 n_0} = 1/a^{n_0}(a^{m_0})^i$. Here we use only (i)–(iv).

In Example 2 we can construct λ directly: For

$$x_n = [(t\gamma + (1-t)\delta)a^{2s m_0 n_0 + j n_0}], \qquad s = 0, 1, \ldots,\ j = 0, 1, \ldots, m_0 - 1,$$

we take $x_{g(n)} = [(t\gamma + (1-t)\delta)a^{(2s-2)m_0 n_0 + j n_0}]$ and similarly for $2s + 1$. Thus $\lambda = 1/a^{2 m_0 n_0}$.

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Mathematical Institute
of the Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava, Slovakia
E-mail: strauch@mat.savba.sk

Department of Mathematics
University of Ostrava
ul. 30. Dubna 22
CZ-701 03 Ostrava 1, Czech Republic
E-mail: toth@albert.osu.cz

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