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Sums of one prime and two prime squares

by

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1. Introduction. Let

$$\mathcal{A} = \{n : n \in \mathbb{N},\ n \equiv 1 \pmod 2,\ n \not\equiv 2 \pmod 3\},$$
$$\mathcal{C} = \{n : n \in \mathbb{N},\ n \equiv 3 \pmod{24},\ n \not\equiv 0 \pmod 5\}.$$

In 1938 Hua [3] proved that almost all $n \in \mathcal{A}$ are representable as sums of two squares of primes and a kth power of a prime for odd k, and almost all $n \in \mathcal{C}$ are representable as sums of two squares of primes and a kth power of a prime for even k. The natural question then becomes: how good a bound can we get on the possible exceptional sets? Let $E_k(N)$ denote the number of exceptions up to N for the problem with kth power of a prime. Hua's result actually shows that $E_k(N) \ll N(\log N)^{-A}$ for some positive constant A. Later Schwarz [6] refined Hua's result to show that

$$E_k(N) \ll N(\log N)^{-A} \quad \text{for any } A > 0.$$

In 1993 Leung and Liu [4] improved this to $E_k(N) \ll N^{1-\delta}$ for some fixed $\delta > 0$.

For the special case $k = 1$,

$$n = p_1 + p_2^2 + p_3^2. \tag{1.1}$$

In 2004 Wang [7] proved that $E_1(N) \ll N^{13/30+\varepsilon}$. In 2006 Wang and Meng [8] improved it to $E_1(N) \ll N^{5/12+\varepsilon}$. In this note we shall prove the following result.

Theorem. *Let $\varepsilon > 0$ be given. Then for all large N we have*

$$E_1(N) \ll N^{5/14+\varepsilon}.$$

The improvement is due to the application of a sieve method. The basic idea is to show that the argument of [2] used for four squares of primes can

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be adapted to work for a prime and two squares of primes to give the same size exceptional set. We can therefore quote much from the proof in [2], sketching the necessary changes.

2. Outline and preliminary results. To prove the Theorem, it suffices to estimate the number of exceptional integers in the set $\mathcal{B} := \mathcal{A} \cap (N/2, N]$. Here N is our main parameter, which we assume to be "sufficiently large". We write

$$P = N^{1/7-\varepsilon}, \quad Q = NP^{-1}\mathcal{L}^{-100}, \quad M = N\mathcal{L}^{-9}, \quad \mathcal{L} = \log N. \tag{2.1}$$

We use c and ε to denote an absolute constant and a sufficiently small positive number, not necessarily the same at each occurrence.

Let

$$\mathfrak{M} = \bigcup_{1\le q\le P} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right]. \tag{2.2}$$

These are the major arcs, and so the minor arcs $\mathfrak{m}$ are given by

$$\mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}. \tag{2.3}$$

Let us begin with

$$\sum_{\substack{p_1+p_2^2+m^2=n\\ M<p_1,p_2^2,m^2\le N}} (\log p_1)(\log p_2)\rho(m) = \int_0^1 f(\alpha)g(\alpha)h(\alpha)e(-\alpha n)\,d\alpha, \tag{2.4}$$

in which $e(x) = \exp(2\pi i x)$ and

$$\begin{aligned} f(\alpha) &= \sum_{M<p\le N} (\log p)\, e(\alpha p), \qquad g(\alpha) = \sum_{M<p^2\le N} (\log p)\, e(\alpha p^2),\\ h(\alpha) &= \sum_{M<m^2\le N} \rho(m) e(\alpha m^2). \end{aligned} \tag{2.5}$$

Here $\rho(m)$ satisfy

$$\rho(m) \le \begin{cases} 1 & \text{if } m \text{ is prime},\\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \sum_{m\le X} \rho(m) \gg X\mathcal{L}^{-1} \tag{2.6}$$

for $N^{1/4} \le X \le N^{1/2}$. This means that ρ is a non-trivial lower bound for the characteristic function of the set of primes in $[M^{1/2}, N^{1/2}]$.

The new idea introduced in Section 3 of [2], and which we use here, is as follows. The maximum saving we can make for $g(\alpha)$ on the minor arcs with our current knowledge is $N^{1/16}$, but this can be increased to $N^{1/14}$ for $h(\alpha)$. The final exponent for the exceptional set is then $\frac{1}{2} - 2\cdot\frac{1}{14} = \frac{5}{14}$ using an argument of Wooley that motivates (4.3)–(4.6) below.

Let $\theta(m,\alpha)$ be the function which is 1 except when there exist integers a and q such that

$$|q\alpha - a| < Q^{-1},\ (a,q)=1,\ q\le P, \qquad (m,q) \text{ is divisible by a prime } p \ge N^{1/14},$$

in which case $\theta(m,\alpha)=0$. Define

$$k(\alpha) = \sum_{M<m^2\le N} \rho(m)\theta(m,\alpha)e(\alpha m^2), \qquad l(\alpha) = h(\alpha) - k(\alpha). \tag{2.7}$$

It is easy to see that, for $\alpha\in\mathfrak{m}$, $h(\alpha)=k(\alpha)$ and

$$l(\alpha) \ll N^{3/7} \tag{2.8}$$

for all α.

For a positive integer k and $\chi \bmod q$, define

$$C_k(\chi,a) = \sum_{h=1}^{q} \overline{\chi}(h)e(ah^k/q), \qquad C_k(q,a) = C_k(\chi_0,a). \tag{2.9}$$

Here χ_0 is the principal character modulo q.

If χ_1,χ_2,χ_3 are characters modulo q, then let

$$\begin{aligned} B(n,q;\chi_1,\chi_2,\chi_3) \\ = \frac{1}{\phi^3(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C_1(\chi_1,a)C_2(\chi_2,a)C_2(\chi_3,a)e(-an/q), \end{aligned} \tag{2.10}$$

and

$$A(q) = B(n,q;\chi_0,\chi_0,\chi_0), \qquad \mathfrak{S}(n,X) = \sum_{q\le X} A(q). \tag{2.11}$$

Lemma 1 (Lemma 7.1 of [8]). *For $n\in\mathcal{A}$, we have*

$$\mathfrak{S}(n,X) \gg 1.$$

Lemma 2 (Lemma 3.1 of [8]). *Let $\chi_j \pmod{r_j}$ with $j=1,2,3$ be primitive characters, $r_0=[r_1,r_2,r_3]$, and χ_0 the principal character modulo q. Then*

$$\sum_{\substack{q\le x\\ r_0|q}} |B(n,q;\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)| \ll r_0^{-1/2+\varepsilon}(\log x)^{10}.$$

Lemma 3 (Theorem 1.1 of [1]). *Let $\ell\in\mathbb{N}$, $R,T,X\ge 1$ and $\kappa := 1/\log X$. Then there is an absolute positive constant c such that*

$$\sum_{\substack{r\sim R\\ \ell|r}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \int_{-T}^{T} \Big| \sum_{X\le n\le 2X} \frac{\Lambda(n)\chi(n)}{n^{\kappa+i\tau}} \Big| d\tau \ll (\ell^{-1}R^2TX^{11/20} + X)(\log RTX)^c,$$

where $\sum^*_{\chi\,(\mathrm{mod}\, r)}$ means summation over the primitive characters modulo r. The implied constant is absolute.

3. The major arcs. Let

$$f^*(\alpha) = \frac{C_1(q,a)}{\phi(q)} \sum_{M<m\le N} e(\beta m), \tag{3.1}$$

$$g^*(\alpha) = \frac{C_2(q,a)}{\phi(q)} \sum_{M<m^2\le N} e(\beta m^2), \tag{3.2}$$

$$k^*(\alpha) = \frac{C_2(q,a)}{\phi(q)} \sum_{M<m^2\le N} \varrho(m) e(\beta m^2), \tag{3.3}$$

where $\varrho(m)$ is defined in (4.3) of [2]. We now consider

$$\int_{\mathfrak{M}} f(\alpha)g(\alpha)k(\alpha)e(-\alpha n)\,d\alpha - \int_{\mathfrak{M}} f^*(\alpha)g^*(\alpha)k^*(\alpha)e(-\alpha n)\,d\alpha, \tag{3.4}$$

which we think of as the error term over $\mathfrak{M}$.

Define

$$W_1(\chi,\beta) = \sum_{M<p\le N} (\log p)\,\chi(p)e(\beta p) - D(\chi) \sum_{M<m\le N} e(\beta m),$$

$$W_2(\chi,\beta) = \sum_{M<p^2\le N} (\log p)\,\chi(p)e(\beta p^2) - D(\chi) \sum_{M<m^2\le N} e(\beta m^2),$$

$$W^\sharp(\chi,\beta) = \sum_{M<m^2\le N} \rho(m)\chi(m)e(\beta m^2) - D(\chi) \sum_{M<m^2\le N} \varrho(m)e(\beta m^2),$$

where $D(\chi)$ is 1 or 0 according as χ is principal or not.

Similar to (4.1) of [2], we can write the $f(\alpha)$, $g(\alpha)$ and $k(\alpha)$ as

$$f\left(\frac{a}{q}+\beta\right) = \frac{C_1(q,a)}{\phi(q)} \sum_{M<m\le N} e(\beta m) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C_1(\chi,a)W_1(\chi,\beta), \tag{3.5}$$

$$g\left(\frac{a}{q}+\beta\right) = \frac{C_2(q,a)}{\phi(q)} \sum_{M<m^2\le N} e(\beta m^2) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C_2(\chi,a)W_2(\chi,\beta), \tag{3.6}$$

$$k\left(\frac{a}{q}+\beta\right) = \frac{C_2(q,a)}{\phi(q)} \sum_{M<m^2\le N} \varrho(m)e(\beta m^2) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C_2(\chi,a)W^\sharp(\chi,\beta). \tag{3.7}$$

So we can use (3.5)–(3.7) to express the difference in (3.4) as a linear combination of error terms involving $f^*(\alpha)$, $g^*(\alpha)$ and $k^*(\alpha)$, and $W_1(\chi,\beta)$,

$W_2(\chi,\beta)$ and $W^\sharp(\chi,\beta)$. In these error terms, the most troublesome is

$$\sum_{q\le P}\sum_{\chi_1 \bmod q}\sum_{\chi_2 \bmod q}\sum_{\chi_3 \bmod q} B(n,q;\chi_1,\chi_2,\chi_3)J(n,q,\chi_1,\chi_2,\chi_3). \tag{3.8}$$

Here $B(n,q;\chi_1,\chi_2,\chi_3)$ is defined in (2.10), and

$$J(n,q,\chi_1,\chi_2,\chi_3) = \int_{-1/qQ}^{1/qQ} W^\sharp(\chi_3,\beta)W_1(\chi_1,\beta)W_2(\chi_2,\beta)e(-\beta n)\,d\beta.$$

Suppose χ_j^* mod r_j, $r_j \mid q$, is the primitive character inducing χ_j. If χ mod q, $q\le P$, is induced by a primitive character χ^* mod r, $r\mid q$, we have

$$W_j(\chi,\beta) = W_j(\chi^*,\beta), \quad j=1,2, \tag{3.9}$$

$$W^\sharp(\chi,\beta) = W^\sharp(\chi^*,\beta) + O(r^{-2}N^{13/28}), \tag{3.10}$$

where the error term comes from the integers in the set

$$\{m^2\in[M,N] : (m,q)>1,\ (m,r)=1,\ \rho(m)\ne 0\}.$$

When $r\le PN^{-3/28} < N^{1/28}$, this set contains $\ll N^{1/2-3/28} \ll r^{-2}N^{13/28}$ integers; when $r > PN^{-3/28}$, it is empty.

By Cauchy's inequality,

$$J(n,q,\chi_1,\chi_2,\chi_3) \ll (W^\sharp(\chi_3^*) + r_3^{-2}N^{13/28})W_1(\chi_1^*)W_2(\chi_2^*), \tag{3.11}$$

where for a character χ mod r,

$$W^\sharp(\chi) = \max_{|\beta|\le 1/rQ}|W^\sharp(\chi,\beta)|,$$

$$W_j(\chi) = \Big(\int_{-1/rQ}^{1/rQ}|W_j(\chi,\beta)|^2\,d\beta\Big)^{1/2}, \quad j=1,2. \tag{3.12}$$

By (3.11), the quantity (3.8) is

$$\ll \sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1}\sum_{r_2\le P}\sideset{}{^*}\sum_{\chi_2}\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3}(W^\sharp(\chi_3)+r_3^{-2}N^{13/28}) \times W_1(\chi_1)W_2(\chi_2)B(n,\chi_1,\chi_2,\chi_3). \tag{3.13}$$

Here $\sum_{r_j}\sum^*_{\chi_j}$ denotes summation over the primitive characters to moduli $r_j\le P$, and

$$B(n,\chi_1,\chi_2,\chi_3) = \sum_{\substack{q\le P\\ r_0\mid q}} |B(n,q;\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)|,$$

where $r_0=[r_1,r_2,r_3]$ and χ_0 is the principal character modulo q.

By Lemma 2 we have

$$B(n,\chi_1,\chi_2,\chi_3) \ll r_0^{-1/2+\varepsilon}\mathcal{L}^{10},$$

and by Lemma 2.4 of [5],

$$\sum_{r\le R}\sideset{}{^*}\sum_{\chi}[r,d]^{-1/2+\varepsilon}W_2(\chi)\ll d^{-1/2+\varepsilon}\mathcal{L}^c$$

whenever $R\le N^{1/6-\varepsilon}$. Thus the sum in (3.13) does not exceed

$$\mathcal{L}^c\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1}\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3}[r_1,r_3]^{-1/2+\varepsilon}(W^\sharp(\chi_3)+r_3^{-2}N^{13/28})W_1(\chi_1). \tag{3.14}$$

Following Section 6 of [5], but using Lemma 3 instead of Theorem 4.1 of [5], with a few changes, we get

$$\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1}[r_1,r_3]^{-1/2+\varepsilon}W_1(\chi_1)\ll r_3^{-1/2+\varepsilon}N^{1/2}\mathcal{L}^c, \tag{3.15}$$

so (3.14) does not exceed

$$\begin{aligned}\mathcal{L}^cN^{1/2}&\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3}(r_3^{-1/2+\varepsilon}W^\sharp(\chi_3)+r_3^{-5/2+\varepsilon}N^{13/28})\\ &\ll\mathcal{L}^cN^{1/2}\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3}r_3^{-1/2+\varepsilon}W^\sharp(\chi_3)+N^{27/28+\varepsilon}.\end{aligned} \tag{3.16}$$

By the argument of page 8 of [2], if $\rho(m)$ satisfies conditions (i), (ii), (iv) and (v) in [2], then for any fixed $A>0$ we have

$$\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3}r_3^{-1/2+\varepsilon}W^\sharp(\chi_3)\ll N^{1/2}\mathcal{L}^{-A-c}. \tag{3.17}$$

Therefore, by (3.9)–(3.17) we have

$$\sum_{q\le P}\sum_{\chi_1 \bmod q}\sum_{\chi_2 \bmod q}\sum_{\chi_3 \bmod q}B(n,q;\chi_1,\chi_2,\chi_3)J(n,q,\chi_1,\chi_2,\chi_3)\ll N\mathcal{L}^{-A} \tag{3.18}$$

for any fixed $A>0$.

Hence the sum in (3.8) is $O(N\mathcal{L}^{-A})$ for any fixed $A>0$. Similarly, the other error terms in (3.4) can be estimated in the same way, so the difference in (3.4) is $O(N\mathcal{L}^{-A})$.

By the standard major arcs techniques we have

$$\int_{\mathfrak{M}}f^*(\alpha)g^*(\alpha)k^*(\alpha)e(-\alpha n)\,d\alpha=P_0\mathfrak{S}(n,P)(1+o(1)), \tag{3.19}$$

where

$$N\mathcal{L}^{-1}\ll P_0=\sum_{\substack{m_1+m_2^2+m_3^2=n\\ M<m_1,m_2^2,m_3^2\le N}}\varrho(m_3)\ll N\mathcal{L}^{-1}, \tag{3.20}$$

by (4.4) of [2], and $\mathfrak{S}(n,P)$ is defined by (2.11).

By Lemma 1, (3.4) and (3.19)–(3.20) we obtain the following result:

LEMMA 4. *Suppose that* $\rho(m)$ *satisfies conditions* (i), (ii), (iv) *and* (v) *in* [2]. *Then for sufficiently large* $n \in \mathcal{A}$, *we have*

$$\int_{\mathfrak{M}} f(\alpha)g(\alpha)k(\alpha)e(-\alpha n)\,d\alpha \gg N\mathcal{L}^{-1}. \tag{3.21}$$

4. Proof of Theorem. Let $\mathcal{E}(N)$ be the set of integers $n \in \mathcal{B}$ such that

$$n \neq p_1 + p_2^2 + p_3^2. \tag{4.1}$$

It is sufficient to prove that

$$\mathcal{E}(N) \ll N^{5/14+\varepsilon}. \tag{4.2}$$

Let $|\mathcal{E}(N)|$ denote the cardinality of $\mathcal{E}(N)$ and $Z(\alpha)$ be its generating function:

$$Z(\alpha) = \sum_{n \in \mathcal{E}(N)} e(-\alpha n).$$

Then by (2.2)–(2.6) we have

$$\int_0^1 f(\alpha)g(\alpha)h(\alpha)Z(\alpha)\,d\alpha \le 0.$$

By Lemma 4, it follows that

$$\int_{\mathfrak{M}} f(\alpha)g(\alpha)k(\alpha)Z(\alpha)\,d\alpha \gg |\mathcal{E}(N)|N\mathcal{L}^{-1}.$$

Thus

$$\begin{aligned}\Big|\int_{\mathfrak{M}} f(\alpha)g(\alpha)k(\alpha)Z(\alpha)\,d\alpha - \int_0^1 f(\alpha)g(\alpha)h(\alpha)Z(\alpha)\,d\alpha\Big| \\ = \Big|\int_{\mathfrak{M}} f(\alpha)g(\alpha)(k(\alpha)-h(\alpha))Z(\alpha)\,d\alpha - \int_{\mathfrak{m}} f(\alpha)g(\alpha)h(\alpha)Z(\alpha)\,d\alpha\Big| \\ \gg |\mathcal{E}(N)|N\mathcal{L}^{-1}.\end{aligned}$$

By Lemma 1 of [2] and (2.8) we have

$$\begin{aligned}|\mathcal{E}(N)| &\ll \mathcal{L}N^{-1}\Big(\int_{\mathfrak{M}} |f(\alpha)g(\alpha)(k(\alpha)-h(\alpha))Z(\alpha)|\,d\alpha \\ &\qquad + \int_{\mathfrak{m}} |f(\alpha)g(\alpha)h(\alpha)Z(\alpha)|\,d\alpha\Big) \\ &\ll \mathcal{L}N^{-1}N^{3/7+\varepsilon/5}\int_0^1 |f(\alpha)g(\alpha)Z(\alpha)|\,d\alpha.\end{aligned} \tag{4.3}$$

Now we apply the device introduced by Wooley [9] and used by Harman and Kumchev [2], namely by Cauchy's inequality and Parseval's identity we have

$$\int_0^1 |f(\alpha)g(\alpha)Z(\alpha)|\,d\alpha \ll \Big(\int_0^1 |f(\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int_0^1 |g(\alpha)Z(\alpha)|^2\,d\alpha\Big)^{1/2}. \tag{4.4}$$

It is easy to see that

$$\int_0^1 |f(\alpha)|^2\,d\alpha = \sum_{M<p\le N} (\log p)^2 \ll N\mathcal{L},$$

$$\begin{aligned}\int_0^1 |g(\alpha)Z(\alpha)|^2\,d\alpha &= \sum_{\substack{p_1^2+n_1=p_2^2+n_2\\ M<p_i^2\le N,\, n_i\in\mathcal{E}(N)}} (\log p_1)(\log p_2)\\ &\ll N^{1/2+\varepsilon/4}|\mathcal{E}(N)| + |\mathcal{E}(N)|^2N^{\varepsilon/4}.\end{aligned} \tag{4.5}$$

Therefore

$$\int_0^1 |f(\alpha)g(\alpha)Z(\alpha)|\,d\alpha \ll N^{3/4+\varepsilon/4}|\mathcal{E}(N)|^{1/2} + N^{1/2+\varepsilon/2}|\mathcal{E}(N)|. \tag{4.6}$$

So by (4.3)–(4.6) we have

$$|\mathcal{E}(N)| \ll N^{5/28+\varepsilon/2}|\mathcal{E}(N)|^{1/2}.$$

From this we get

$$|\mathcal{E}(N)| \ll N^{5/14+\varepsilon}.$$

This completes the proof of the Theorem.

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