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Nonreciprocal algebraic numbers of small Mahler's measure

by

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1. Introduction. Let d be a positive integer and let α be an algebraic number of degree d with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ over $\mathbb{Q}$. The *Mahler measure* of α with minimal polynomial $P(x) := a_d(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{Z}[x]$, $a_d > 0$, is given by

$$M(\alpha) = M(P) := a_d \prod_{j=1}^{d} \max(1, |\alpha_j|).$$

Then $M(\alpha) \geq 1$ and, by Kronecker's theorem, $M(\alpha) = 1$ if and only if α is either zero or a root of unity.

Let $T \geq 1$ be a fixed real number. *How many irreducible polynomials in $\mathbb{Z}[x]$ of degree d (or at most d) have their Mahler measures in the interval $[1, T)$?* This question was first raised by Mignotte [12] (see also [13]) who gave the first upper bound $2(8d)^{2d+1}$ on the number of irreducible polynomials of degree d whose Mahler measures are smaller than 2. The problem was further studied in [2], [3] and [4]. In particular, an asymptotical formula for the number of integer polynomials of degree at most d and of Mahler's measure at most T when d is fixed and $T \to \infty$ was established by Chern and Vaaler in [2]. However, the problem is much more difficult when T is small, say, fixed and $d \to \infty$. Although Kronecker's theorem gives the answer when the interval is a singleton (the number of integer irreducible polynomials of degree at most d with Mahler's measure 1 is equal to the number of solutions of $\varphi(n) \leq d$, where φ is Euler's totient function), Lehmer's question if for each $T > 1$ there is an irreducible polynomial $P \in \mathbb{Z}[x]$ whose Mahler measure satisfies $1 < M(P) < T$ remains open.

Currently, the best upper bound for the number of irreducible polynomials of degree at most d having Mahler's measures in $[1, T)$ follows from [4]. There exist at most $T^{d+16\log d/\log\log d}$ integer polynomials of degree at most d

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whose Mahler measures belong to the interval $[1, T)$. However, when T is fixed and $d \to \infty$ this bound seems to be very far from the true bound.

The first nontrivial lower bound was obtained in [3]: for each $d \geq 2$ there are at least $(d-3)^2/2$ irreducible integer polynomials (in fact, nonreciprocal polynomials) of degree d with Mahler measures smaller than 2. Recall that a polynomial P is called *reciprocal* if it satisfies $P(x) = \pm P^*(x)$, where $P^*(x) = x^{\deg P} P(1/x)$, and *nonreciprocal* otherwise. The algebraic number is reciprocal iff its minimal polynomial in $\mathbb{Z}[x]$ is reciprocal. Of course, 'most' of the algebraic numbers are nonreciprocal, so it is natural to expect that 'most' of the irreducible polynomials in $\mathbb{Z}[x]$ whose Mahler measures are small, say less than 2, are nonreciprocal too. However, this is not the case for Mahler measures smaller than 1.32.

Let

$$\theta := 1.32471\ldots \quad \text{and} \quad \theta_1 := 1.32479\ldots$$

be the roots of the polynomials

$$x^3 - x - 1 \quad \text{and} \quad 4x^8 - 5x^6 - 2x^4 - 5x^2 + 4,$$

respectively. In [17] Smyth showed that the Mahler measure of a nonreciprocal algebraic number α is at least θ. Moreover, in [18] it is shown that if α is a nonreciprocal algebraic number satisfying $1 \leq M(\alpha) \leq \theta_1$ then $\alpha = \pm\theta^{\pm 1/n}$ with $n \in \mathbb{N}$ and so $M(\alpha) = \theta$. In particular, this implies that the interval $[1, \theta)$ contains no nonreciprocal Mahler measures at all and that the number of irreducible nonreciprocal polynomials of degree at most d with Mahler measures in the interval $[1, \theta_1]$ is between $c_1 d$ and $c_2 d$. The above mentioned result of [3] implies that the number of nonreciprocal irreducible polynomials in $\mathbb{Z}[x]$ of degree at most d with Mahler measures in $[1, 2)$ is at least $\sum_{k=2}^{d} (k-3)^2/2 = d^3/6 + O(d^2)$.

In this paper, we improve this bound:

Theorem 1.1. *There is an absolute constant $c > 0$ such that for each $d \geq 2$ there exist at least cd^5 monic irreducible nonreciprocal polynomials $P \in \mathbb{Z}[x]$ satisfying $\deg P \leq d$ and $1 \leq M(P) < 2$.*

In fact, we prove the following more general result:

Theorem 1.2. *For each $\varepsilon > 0$ and each integer $k \geq 2$ there exist two positive numbers $c_0 := c(\varepsilon, k)$ and $d(\varepsilon, k)$ such that for every integer $d \geq d(\varepsilon, k)$ there exist at least $c_0 d^k$ monic irreducible nonreciprocal polynomials in $\mathbb{Z}[x]$ of degree at most d whose Mahler measures belong to the interval $[\theta, \lambda_k + \varepsilon)$, where $\lambda_k := M(1 + x_1 + \cdots + x_k)$.*

Recall that

$$\log M(P(x_1, \ldots, x_k)) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})| \, dt_1 \cdots dt_k$$

for $P \in \mathbb{C}[x_1, \ldots, x_k]$. In the table below we give the first five values of λ_k (starting with $k = 2$) with three correct decimal digits. Since $\lambda_5 < 2$, the table shows that Theorem 1.1 is a special case of Theorem 1.2 with $k = 5$ and, for instance, $\varepsilon = 1/8 = 0.125$. (The interval for the degree $2 \le d \le d(1/8, 5)$ is covered by reducing the constant $c_0 = c(1/8, 5)$ to c, if necessary.)

k	2	3	4	5	6
$\lambda_k = M(1 + x_1 + \cdots + x_k)$	1.381	1.531	1.723	1.872	2.019

The values

$$\lambda_2 = \exp(\log M(1 + x_1 + x_2)) = \exp\left(\frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^2}\right),$$

$$\lambda_3 = \exp(\log M(1 + x_1 + x_2 + x_3)) = \exp(7\zeta(3)/2\pi^2)$$

have been evaluated by Smyth (see [19] and Appendix 1 in [1]). In the next section we shall explain how the numerical values given in the above table have been found.

For the proof of Theorem 1.2 we will construct monic irreducible nonreciprocal polynomials as divisors of some Newman hexanomials $1 + x^{r_1} + \cdots + x^{r_k}$, where the integers $1 \le r_1 < \cdots < r_k \le d$ satisfy some additional restrictions including $2r_j < r_{j+1}$ for $j = 1, \ldots, k-1$.

2. Computation of Mahler measures. The Mahler measures in the above table have been calculated by evaluating the integral

$$\log M(1 + x_1 + \cdots + x_k) = \int_{I_k} \log|F_k(t_1, \ldots, t_k)|\, dt_1 \cdots dt_k$$

of the function

$$F_k(t_1, \ldots, t_k) := 1 + e^{2\pi i t_1} + \cdots + e^{2\pi i t_k}$$

over the k-dimensional hypercube $I_k := [0, 1]^k$.

Firstly, Jensen's formula was applied to the integral

$$\int_{I_k} \log|F_k(t_1, \ldots, t_k)|\, dt_1 \cdots dt_k = \int_{I_{k-1}} \log^+|F_{k-1}(t_1, \ldots, t_{k-1})|\, dt_1 \cdots dt_{k-1},$$

where $\log^+$ denotes the positive part of the logarithmic function, given by the identity

$$\log^+|z| := \log\max\{1, |z|\}, \qquad z \in \mathbb{C}.$$

This transformation resolves the problem of singularities at points where the function F_k vanishes. In addition, it reduces the dimension of the integration domain. Secondly, calculations with complex numbers have been replaced by calculations with real numbers using the identities

$$\log^+ |F_{k-1}(t_1,\dots,t_{k-1})| = \frac{1}{2}\log^+ |F_{k-1}(t_1,\dots,t_{k-1})|^2,$$
$$|F_{k-1}(t_1,\dots,t_{k-1})|^2 = (1+\cos 2\pi t_1+\dots+\cos 2\pi t_{k-1})^2 + (\sin 2\pi t_1+\dots+\sin 2\pi t_{k-1})^2.$$

Finally, the resulting integral was evaluated numerically using the `Cuba` library [8] for the multidimensional integration through `Mathematica` interface.

The integration was performed using the global adaptive subdivision algorithm `Cuhre` for dimensions $k \le 5$. For $k = 6$ the rate of convergence was quite slow and the reported error was considerable, hence we applied the stratified sampling algorithm `Divonne` in nondeterministic quasi-random mode; the resulting value 2.019 was subsequently also tested in `Divonne` in deterministic mode. Other algorithms (such as `Suave` or `NIntegrate`, available in `Mathematica`) were used to cross-check the results.

3. Auxiliary lemmas. The next result was conjectured by Boyd [1] and proved by Lawton [9]. One can also find its proof in Schinzel's book [16, pp. 237–243].

LEMMA 3.1. *Let* $\mathbf{r}$ *be a vector in* $\mathbb{Z}^k$, $P \in \mathbb{C}[x_1,\dots,x_k]$, *and let*

$$\mu(\mathbf{r}) := \min\{\|\mathbf{s}\| : \mathbf{s} \in \mathbb{Z}^k \text{ and } \mathbf{r}\cdot\mathbf{s} = 0\},$$

where $\|\mathbf{s}\| = \|(s_1,\dots,s_k)\| = \max_{1\le i\le k}|s_i|$. *Then*

$$\lim_{\mu(\mathbf{r})\to\infty} M(P(x^{r_1},\dots,x^{r_k})) = M(P(x_1,\dots,x_k)).$$

In order to apply Lemma 3.1 we shall need the following observation.

LEMMA 3.2. *Let* $\mathbf{s}$ *be a nonzero vector in* $\mathbb{Z}^k$. *Then the number of vectors* $\mathbf{r} = (r_1,\dots,r_k) \in \mathbb{Z}^k$ *satisfying* $1 \le r_1 < \dots < r_k \le d$ *and* $\mathbf{r}\cdot\mathbf{s} = 0$ *is less than* $\binom{d}{k-1}$.

Proof. The result is trivial for $k = 1$. Assume that $k \ge 2$. Since $\mathbf{s} = (s_1,\dots,s_k)$ is nonzero, we must have $s_i \ne 0$ for some index i. Denote by $\mathbf{r}'$ an arbitrary vector $\mathbf{r}' := (r_1,\dots,r_{i-1},r_{i+1},\dots,r_k)$ with strictly increasing positive integer coordinates, the largest of which does not exceed d. Clearly, there are at most $\binom{d}{k-1}$ such vectors $\mathbf{r}'$. Consider any vector $\mathbf{r} = (r_1,\dots,r_k) \in \mathbb{Z}^k$ satisfying $1 \le r_1 < \dots < r_k \le d$ and $\mathbf{r}\cdot\mathbf{s} = 0$ corresponding to the vector $\mathbf{r}'$. Note that such a vector $\mathbf{r}$ does not exist if $r_{i+1} - r_{i-1} = 1$ for $1 < i < k$ (resp. $r_2 = 1$ for $i = 1$ and $r_{k-1} = d$ for $i = k$). Since $s_i r_i = -\sum_{j\ne i} s_j r_j$, to each such vector $\mathbf{r}'$ corresponds at most one value for the remaining component r_i of the vector $\mathbf{r}$. Therefore, there are less than $\binom{d}{k-1}$ such vectors $\mathbf{r}$. ∎

Every monic polynomial $P \in \mathbb{Z}[x]$ can be written in the form $P(x) = Q(x)R(x)$, where $Q(x)$ is the product of all nonreciprocal monic polynomials dividing $P(x)$ (with respective multiplicities) and, similarly, $R(x)$ is the product of all reciprocal polynomials dividing $P(x)$. (The polynomials Q and R can be equal to 1.) We refer to the polynomial $Q(x)$ as the *nonreciprocal part* of $P(x)$. Results on reducibility of Newman polynomials (those with coefficients $0, 1$) are given in [6], [7], [10], [11], [14], [15]. Some of these results are more precise for small k than the lemma given below. However, we prefer to give this result of Filaseta [5], since it can be used for every $k \geq 2$.

LEMMA 3.3. *Let $P(x) = 1 + x^{r_1} + \cdots + x^{r_k} \in \mathbb{Z}[x]$, where $1 \leq r_1 < \cdots < r_k$ and $r_{j+1} > \frac{1+\sqrt{5}}{2} r_j$ for each $j = 1, \ldots, k-1$. Then the nonreciprocal part of $P(x)$ is either irreducible or identically* 1.

4. Putting things together: proof of Theorem 1.2. Let $k \geq 2$ and $d \geq 2^k - 1$ be two integers, and let S_k be the set of k-nomials of the form

$$1 + x^{r_1} + \cdots + x^{r_{k-1}} + x^{r_k},$$

where r_j are positive integers lying in the intervals

$$(2^j - 2)M + 1 \leq r_j \leq (2^j - 1)M$$

for $j = 1, \ldots, k$ with $M := [d/(2^k - 1)]$. Then $1 \leq r_1 < \cdots < r_k \leq d$ and $2r_j < r_{j+1}$ for each $j = 1, \ldots, k-1$. Clearly, $|S_k| = M^k$.

Observe that each polynomial in S_k is nonreciprocal in view of $r_{k-1} + r_1 \leq 2r_{k-1} < r_k$. By Lemma 3.3, each $P \in S_k$ has a unique monic irreducible nonreciprocal factor $Q \in \mathbb{Z}[x]$ of degree at least 2 and at most d. We claim that all these Q are distinct.

Indeed, for a contradiction assume that there are $P_1(x) := 1 + x^{r_1} + \cdots \cdots + x^{r_n}$ and $P_2(x) := 1 + x^{u_1} + \cdots + x^{u_n}$ in S_k whose nonreciprocal parts are the same. Then $P_1(x) = Q(x)R_1(x)$ and $P_2(x) = Q(x)R_2(x)$ with some nonreciprocal polynomial Q and some two distinct reciprocal polynomials $R_1, R_2 \in \mathbb{Z}[x]$. Notice that the polynomial

$$P_1(x)P_2^*(x) = Q(x)R_1(x)Q^*(x)R_2^*(x) = \pm Q(x)Q^*(x)R_1(x)R_2(x)$$

is reciprocal, since so are R_1, R_2 and QQ^*. Since

$$P_2^*(x) = 1 + x^{u_n - u_{n-1}} + \cdots + x^{u_n - u_1} + x^{u_n},$$

using

$$u_n - u_{n-1} \geq (2^n - 2)M + 1 - (2^{n-1} - 1)M = (2^{n-1} - 1)M + 1 > r_{n-1},$$

we see that the first (lowest) n terms of the polynomial $P_1(x)P_2^*(x)$ are

$$1 + x^{r_1} + \cdots + x^{r_{n-1}}.$$

Analogously, by the inequality $r_{n-1} + u_n < r_n + u_n - u_{n-1}$, we see that the last (highest) n terms of this polynomial are

$$x^{r_n+u_n-u_{n-1}} + \cdots + x^{r_n+u_n-u_1} + x^{r_n+u_n}.$$

Since the degree of the reciprocal polynomial $P_1(x)P_2^*(x)$ is $r_n + u_n$, by considering the first n and the last n terms, we must have

$$r_i = (r_n + u_n) - (r_n + u_n - u_i) = u_i$$

for $i = 1, \ldots, n-1$. Hence the nonzero difference

$$\begin{aligned}&Q(x)(R_1(x) - R_2(x))\\ &\quad = P_1(x) - P_2(x) = 1 + x^{r_1} + \cdots + x^{r_n} - 1 - x^{u_1} - \cdots - x^{u_n} = x^{r_n} - x^{u_n}\end{aligned}$$

is the product of a power of x and some cyclotomic polynomials, so it cannot be divisible by $Q(x)$, a contradiction. This proves our claim.

The claim implies that there exist $L := M^k$ distinct monic irreducible nonreciprocal polynomials $Q_i(x)$, $i = 1, \ldots, L$, which divide L distinct polynomials of the set S_k. It remains to show that 'most' of them have small Mahler's measure.

Fix $\varepsilon > 0$ and fix an integer $k \geq 2$. By Lemma 3.1 applied to the polynomial in k variables $P(x_1, \ldots, x_k) := 1 + x_1 + \cdots + x_k$, for each $\varepsilon > 0$ there is a constant $C(\varepsilon, k)$ such that

$$|M(1 + x_1 + \cdots + x_k) - M(1 + x^{r_1} + \cdots + x^{r_k})| < \varepsilon$$

whenever $\mu(\mathbf{r}) > C(\varepsilon, k)$. Obviously, there only finitely many, say $B := B(\varepsilon, k)$, vectors $\mathbf{s} \in \mathbb{Z}^k$ satisfying $\|\mathbf{s}\| \leq C(\varepsilon, k)$. To each of those B vectors we may apply Lemma 3.2. This gives at most $B\binom{d}{k-1} \leq Bd^{k-1}$ vectors $\mathbf{r} = (r_1, \ldots, r_k)$, $1 \leq r_1 < \cdots < r_k \leq d$ with $\mu(\mathbf{r}) \leq C(\varepsilon, k)$ for which the modulus of the difference between Mahler measures of the polynomials $1 + x_1 + \cdots + x_k$ and $P(x) = 1 + x^{r_1} + \cdots + x^{r_k}$ can be greater than or equal to ε. Therefore, the inequality

$$\lambda_k - \varepsilon < M(P) < \lambda_k + \varepsilon$$

holds for each $P \in S_k$ with at most Bd^{k-1} exceptions.

It follows that there is a subset S_k^* of S_k with cardinality

$$L - Bd^{k-1} = M^k - Bd^{k-1} = [d/(2^k - 1)]^k - Bd^{k-1} \gg d^k$$

(where the last inequality holds for d large enough, say $d \geq d(\varepsilon, k)$) such that $M(P_i) < \lambda_k + \varepsilon$ for each $P_i \in S_k^*$. Since the nonreciprocal parts Q_i of those $P_i(x) = Q_i(x)R_i(x)$ are all distinct and

$$\lambda_k + \varepsilon > M(P_i) = M(Q_i)M(R_i) \geq M(Q_i) \geq \theta,$$

where the last inequality holds by Smyth's result [17], there exist at least $|S_k^*| \gg d^k$ distinct monic irreducible nonreciprocal polynomials Q_i of degree at most d with Mahler measures lying in the interval $[\theta, \lambda_k + \varepsilon)$. This completes the proof of Theorem 1.2.

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