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On non-intersecting arithmetic progressions

by

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1. Introduction and statement of results. Consider a set $\mathcal{Q}$ of positive integers, together with an associated family $\{a_q\}_{q\in\mathcal{Q}}$ of integers such that the arithmetic progressions (a_q mod q) are pairwise disjoint. The purpose of this paper is to provide sharper bounds for the asymptotic growth of

$$f(x) := \sup_{\mathcal{Q}\subset\mathbb{N}} |\mathcal{Q}\cap[1,x]|.$$

Erdős and Stein first conjectured that $f(x) = o(x)$ (see [6]). This was proved by Erdős and Szemerédi [6], who showed that, for a particular constant c and any $\epsilon > 0$,

$$\frac{x}{\exp\{(\log x)^{1/2+\epsilon}\}} < f(x) < \frac{x}{(\log x)^c}$$

for sufficiently large x. Erdős and Szemerédi credited Stein's help in finding this lower bound.

Croot in [2] then showed that as x tends to infinity, we have the following bounds on $f(x)$:

$$xL(-\sqrt{2}+o(1),x) \le f(x) \le xL\big(-\tfrac{1}{6}+o(1),x\big).$$

Here we use the notation $L(\alpha,x) := \exp\big\{\alpha\sqrt{\log x\log_2 x}\big\}$, $\log_2 x := \log\log x$ and $o(1)$ stands for a function that approaches 0 as $x\to\infty$. Croot further showed that

$$|\mathcal{Q}\cap[1,x]| \le xL\big(-\tfrac{1}{2}+o(1),x\big),$$

provided that $\mathcal{Q}$ contains only squarefree integers. The same estimate was later proved by Chen in [1] for arbitrary $\mathcal{Q}$.

We improve these results as follows.

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Theorem 1. *As x tends to infinity, we have*

$$xL(-1+o(1),x) \le f(x) \le xL\big(-\tfrac{1}{2}\sqrt{3}+o(1),x\big).$$

We further conjecture that $f(x) = xL(-1+o(1),x)$. Our proof of the theorem will depend on investigations of the multiplicative structure of elements of sets $\mathcal{Q} \subset [1,x]$ such that $|\mathcal{Q}| = f(x)$.

2. Proof of the lower bound. To prove the lower bound we shall construct a specific set $\mathcal{Q} \subset [1,x]$ with cardinality $xL(-1+o(1),x)$ and then show the existence of a choice of residues a_q for this $\mathcal{Q}$ that ensure all the arithmetic progressions are disjoint.

To begin, let

$$r := \left\lfloor 2\frac{\sqrt{\log x}}{\sqrt{\log_2 x}}\left(1 - \frac{3}{\sqrt{\log_2 x}}\right)\right\rfloor,$$

with $\lfloor t \rfloor$ denoting the integer part of t. Then define y_0 as the solution to the following equation:

$$\log(2y_0) = \frac{\log x}{r+1} - \frac{r}{4}\log\left(\frac{\log x}{4}\right).$$

This gives asymptotically

$$\log y_0 \sim 3\sqrt{\log x}. \tag{2.1}$$

Now we fix a prime $p_0 \in [y_0, 2y_0]$. The prime factor p_0 will divide all $q \in \mathcal{Q}$, in contrast to the construction of Erdős and Szemerédi, and similar to the construction of Croot.

Let $y_k := y_0\big(\frac{1}{4}\log x\big)^{k/2}$ for $k \in \mathbb{N}$, so that for x sufficiently large and all $k \ge 1$ we have

$$\pi(2y_{k+1}) \le y_k\frac{\sqrt{\log x}}{\log y_k} \le y_k\frac{\sqrt{\log x}}{\log y_0} < y_k. \tag{2.2}$$

Now we define our set $\mathcal{Q}$ by

$$\mathcal{Q} := \{p_0p_1\cdots p_r : \forall k \in [1,r],\ p_k \in (y_k, 2y_k]\}.$$

We have

$$\mathcal{Q} \subset \Big[\prod_{k=0}^{r} y_k,\ 2^{r+1}\prod_{k=0}^{r} y_k\Big].$$

Moreover, $\mathcal{Q} \subset [1,x]$, since, by the definition of y_0 and y_k, we have

$$2^{r+1}\prod_{k=0}^{r} y_k = (2y_0)^{r+1}\left(\frac{1}{4}\log x\right)^{r(r+1)/4} = x.$$

It remains to estimate $|\mathcal{Q}|$. By the previous line, $y_1 \cdots y_r = \mathrm{e}^{O(r)} x/y_0$, and, by (2.1),

$$2\sqrt{\log x} \le \log y_0 \le \log y_k \le \log y_0 + \frac{r}{2} \log\left(\frac{\log x}{4}\right) \le 3\sqrt{\log x \log_2 x}$$

for sufficiently large x and for $0 \le k \le r$. Therefore

$$\begin{aligned} |\mathcal{Q}| &= \prod_{k=1}^{r} (\pi(2y_k) - \pi(y_k)) = \mathrm{e}^{O(r)} \prod_{k=1}^{r} \frac{y_k}{\log y_k} \\ &= \frac{x}{(\log x)^{r(1+o(1))/2}} = xL(-1 + o(1), x). \end{aligned}$$

Now we construct the a_q with $a_q \in [1, q]$. Each $q \in \mathcal{Q}$ can be written as $q = p_0 p_1 \cdots p_r$ with $p_k \in [y_k, 2y_k]$. Using the Chinese Remainder Theorem, we may define a_q entirely by its residues modulo p_k. Let $r_k := \pi(p_k)$, and note $r_{k+1} < p_k$ by (2.2). Then define a_q by

$$a_q \equiv r_{k+1} \pmod{p_k} \quad (0 \le k \le r-1), \qquad a_q \equiv 0 \pmod{p_r}.$$

It only remains to show that the arithmetic progressions so formed are disjoint. Let $n \in \mathbb{N}$ and suppose there exists $q \in \mathcal{Q}$ such that $n \equiv a_q \pmod{q}$. We will show that q is unique. First, let m_1 be the representative of the residue class $n \pmod{p_0}$ in $[1, p_0]$. If we let $p(m)$ denote the mth prime number, then we have $p_1 = p(m_1)$. Iterating this procedure, we obtain $p_k = p(m_k)$ where m_k is the representative of the residue class $n \pmod{p_{k-1}}$ in $[1, p_{k-1}]$, and

$$q = p_0 \prod_{k=1}^{r} p(m_k).$$

This completes the proof of the lower bound. ∎

3. Preliminary lemmas for the upper bound

3.1. Some auxiliary upper bounds. We begin with three lemmas that show that moduli q with certain bad properties are so rare that they may be excluded from consideration in the upper bound without affecting the main term. The first lemma will imply that we only need to consider moduli q with a "small" number of prime factors.

LEMMA 3.1. *Let $A > 0$. As x tends to infinity, we have, uniformly in $2 \le y \le x$, $\alpha \in [0, A]$,*

$$\left|\{n \le y : \omega(n) \ge \alpha\sqrt{\log x/\log_2 x}\}\right| \le yL\left(-\tfrac{1}{2}\alpha + o(1), x\right).$$

Here, we use the usual definitions of the distinct prime divisor counting function, $\omega(n)$, and the prime divisor counting function, $\Omega(n)$, which are given by

$$\omega(n) := \sum_{p|n} 1 \quad \text{and} \quad \Omega(n) := \sum_{p^k|n,\, k\geq 1} 1.$$

Proof. The proof is a classic application of the method of parameters, also known as Rankin's method (see, for example, Section III.5 of [8]). If $z \geq 1$ then

$$\begin{aligned}\left|\{n \leq y : \omega(n) \geq \alpha\sqrt{\log x/\log_2 x}\}\right| &\leq z^{-\alpha\sqrt{\log x/\log_2 x}} \sum_{n\leq y} z^{\omega(n)} \\ &\leq \mathrm{e}yz^{-\alpha\sqrt{\log x/\log_2 x}} \sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^{1+1/\log x}} \\ &\leq \mathrm{e}yz^{-\alpha\sqrt{\log x/\log_2 x}} \zeta(1+1/\log x)^z \\ &\leq \mathrm{e}yz^{-\alpha\sqrt{\log x/\log_2 x}} (2\log x)^z.\end{aligned}$$

Choosing $z = \sqrt{\log x}/\log_2 x$, we obtain the desired upper bound. ∎

We now introduce a function h defined by

$$h(q) := \prod_{p^\nu \| q} \nu.$$

The following lemma will imply that we only need to consider moduli q with $h(q) \leq \mathrm{e}^{\sqrt{\log x}}$.

Lemma 3.2. *For x sufficiently large, we have the following bound:*

$$\left|\{n \leq x : h(n) \geq \mathrm{e}^{\sqrt{\log x}}\}\right| \leq x\mathrm{e}^{-\frac{1}{5}\sqrt{\log x}\log_2 x}.$$

Proof. Let $y := \frac{1}{5}\sqrt{\log x}$. For any integer n, write $n = n_1 n_2$ where all prime factors of n_1 are $\leq y$ and all prime factors of n_2 are $> y$. For $n = n_1 n_2 \leq x$, we have

$$h(n_1) \leq \left(\frac{\log x}{\log 2}\right)^{\pi(y)} \leq \mathrm{e}^{\frac{1}{2}\sqrt{\log x}}$$

for x sufficiently large. Therefore, integers $n \leq x$ with $h(n) \geq \mathrm{e}^{\sqrt{\log x}}$ satisfy $h(n_2) \geq \mathrm{e}^{\frac{1}{2}\sqrt{\log x}}$. The inequality $\nu \leq 2^{\nu-1}$ is valid for all $\nu \in \mathbb{N}$ and implies that

$$\mathrm{e}^{\frac{1}{2}\sqrt{\log x}} \leq h(n_2) \leq 2^{\Omega(n_2)-\omega(n_2)}.$$

Since for $p > y$,

$$\sum_{\nu\geq 2} \frac{\sqrt{2}^{(\nu-1)\log_2 x}}{p^{\nu(1+1/\log x)}} \leq \frac{(\log x)^{(\log 2)/2}}{p^2} \frac{1}{1-(\log x)^{(\log 2)/2}/p} \ll \frac{(\log x)^{(\log 2)/2}}{p^2},$$

we have

$$\left|\{n \le x : h(n) \ge \mathrm{e}^{\sqrt{\log x}}\}\right| \le \mathrm{e}x\mathrm{e}^{-\frac{1}{4}\sqrt{\log x}\log_2 x} \sum_{n=1}^{\infty} \frac{\sqrt{2}^{(\Omega(n_2)-\omega(n_2))\log_2 x}}{n^{1+1/\log x}}$$
$$\ll x(\log x)^{(\log 2)/2}\mathrm{e}^{-\frac{1}{4}\sqrt{\log x}\log_2 x},$$

with the last inequality obtained by writing the Dirichlet series in the form of an Euler product. ∎

Our final lemma of this subsection states that there cannot be too many elements of $\mathcal{Q}$ with a given squarefree part. Here we define

$$\ker(n) := \prod_{p\mid n} p$$

to be the "squarefree kernel" of n.

LEMMA 3.3. *Let $H \in \mathbb{N}$. For any squarefree q, there are at most $H^2 2^{\omega(q)}$ integers n with* $\ker(n) = q$ *and with $h(n) \le H$.*

Proof. Let $K = \omega(q)$. The number of integers in question is at most

$$\sum_{a_1,\dots,a_K \in \mathbb{N}} \left(\frac{H}{a_1 \cdots a_K}\right)^2 = H^2(\pi^2/6)^K \le H^2 2^K. \ ∎$$

3.2. Combinatorics of intersecting families. We call a family of non-empty sets $\mathcal{A}$ an *intersecting family* if $|S \cap T| \ge 1$ for all $S, T \in \mathcal{A}$. (We assume that no set is repeated in $\mathcal{A}$.) We call an intersecting family *set-minimal* if for any $S \in \mathcal{A}$ and a proper subset $S' \subset S$, there exists $T \in \mathcal{A}$ such that $|S' \cap T| = 0$. In particular, if $|\mathcal{A}| = 1$, then the set $S \in \mathcal{A}$ has one element. Let $\mathcal{A}^r$ denote $\{S \in \mathcal{A} : |S| \le r\}$.

LEMMA 3.4. *If $\mathcal{A}$ is a set-minimal intersecting family of sets, each with at most n elements, and $r \le n$, then $|\mathcal{A}^r| \le rn^{r-1}$.*

Proof. Suppose to the contrary that $|\mathcal{A}^r| > rn^{r-1}$. Select a set $S_1 \in \mathcal{A}^r$. Then there must exist an $x_1 \in S_1$ such that the number of sets in $\mathcal{A}^r$ that contain x_1 exceeds n^{r-1}; otherwise, $\mathcal{A}$ could not be an intersecting family. However, not all sets in $\mathcal{A}$ can contain x_1; otherwise, $\{x_1\}$ is a non-trivial subset of a set in $\mathcal{A}^r$ that intersects all sets in $\mathcal{A}$, contradicting set-minimality. Thus there exists $S_2 \in \mathcal{A}$ that does not contain x_1 and an $x_2 \in S_2$ such that the number of sets in $\mathcal{A}^r$ that contain x_1, x_2 exceeds n^{r-2}. If $r \ge 3$, then there must be a set S_3 which does not contain x_1 and does not contain x_2, and for some $x_3 \in S_3$, the number of sets in $\mathcal{A}^r$ that contain x_1, x_2, x_3 exceeds n^{r-3}. We can continue in this way until we find $x_1, \dots, x_r$ such that there is more than one set in $\mathcal{A}^r$ that contains these r elements, which is impossible. ∎

Remark. The proof of Lemma 3.4 follows the general steps of a proof of Erdős and Lovász ([4, p. 621]).

Consider some set $\mathcal{Q} \subset \mathbb{N}$ of moduli with associated residues $\{a_q : q \in \mathcal{Q}\}$ such that the arithmetic progressions ($a_q \bmod q$) are all disjoint. The non-intersection property is equivalent to the condition that for any $q_1, q_2 \in \mathcal{Q}$ there exists a prime p and an exponent $\nu \geq 1$ such that $p^\nu \mid (q_1, q_2)$ and $a_{q_1} \not\equiv a_{q_2} \pmod{p^\nu}$.

Each pair q_1, q_2 in $\mathcal{Q}$ must share at least one prime in common, but not all $q \in \mathcal{Q}$ must share the same prime: it could be that some q are divisible by 2 and 3, some divisible by 3 and 5, and some by 2 and 5, or something considerably more complicated. Regardless, if we consider the subset of elements in $\mathcal{Q}$ that, say, are both divisible by 2 and 3, then each pair of numbers in this subset with a_q equivalent modulo 6 must share some prime other than 2, 3.

We say a set $\{n_1, \ldots, n_k\}$ of squarefree integers > 1 is *intersecting* (respectively, *minimal*) *with size* ℓ if the corresponding collection of sets $\mathcal{A} = \{S_1, \ldots, S_k\}$ with $S_j = \{p : p \mid n_j\}$ is intersecting (respectively, set-minimal) with size ℓ.

Lemma 3.5. *For any finite, intersecting set $\mathcal{B}$ of squarefree integers > 1, there is a minimal, intersecting set $\mathcal{C}$ of squarefree integers > 1 such that for all $B \in \mathcal{B}$, there exists $C \in \mathcal{C}$ such that $C \mid B$.*

Proof. Construct $\mathcal{C}$ iteratively. Start with $\mathcal{C} = \mathcal{B}$ and repeat the following until $\mathcal{C}$ is minimal:

- If there exists $C \in \mathcal{C}$ and a prime $p \mid C$ such that $(C/p, C') > 1$ for all $C' \in \mathcal{C}$, then replace C by C/p. If C/p is duplicated in $\mathcal{C}$, then remove the duplicate.

This process must terminate, since $\mathcal{B}$ is finite. ∎

4. Proof of the upper bound

4.1. Constructing $\mathcal{Q}'$. Consider a set $\mathcal{Q} \subset [1, x]$ of moduli and disjoint progressions $\{a_q \bmod q : q \in \mathcal{Q}\}$ such that $|\mathcal{Q}| = f(x) =: S$ and suppose x is large. We first construct a subset $\mathcal{Q}' \subset \mathcal{Q}$ with cardinality S' satisfying the following conditions:

(1) $S' \geq S \cdot L(o(1), x)$;
(2) $\mathcal{Q}' \subset [xL(-2, x), x]$;
(3) for each $q \in \mathcal{Q}'$, $h(q) \leq \mathrm{e}^{\sqrt{\log x}}$;
(4) there is an integer $K \in [1, 3\sqrt{\log x / \log_2 x}]$ such that for each $q \in \mathcal{Q}'$, $\omega(q) = K$; and
(5) the numbers $\ker(q)$ for $q \in \mathcal{Q}'$ are distinct.

By Lemmas 3.1 and 3.2, together with the already proven lower bound on S, there exists a subset of $\mathcal{Q}$ with cardinality at least $S/2$ that satisfies conditions (2) and (3) and for which every element has at most $3\sqrt{\log x/\log_2 x}$ prime factors. By the pigeonhole principle, we can find a further subset of cardinality at least $S/(6\sqrt{\log x/\log_2 x})$ and an integer $K \in [1, 3\sqrt{\log x/\log_2 x}]$ such that each element has exactly K distinct prime factors. Finally, using Lemma 3.3, there is a further subset (which we call $\mathcal{Q}'$) of cardinality at least

$$\frac{S}{6(\log x/\log_2 x)^{1/2}2^K \mathrm{e}^{2\sqrt{\log x}}} = S \cdot L(o(1), x)$$

such that the numbers $\ker(q)$ for $q \in \mathcal{Q}'$ are distinct.

4.2. The descending chain. Now, as Croot did originally, we construct a descending chain of subsets

$$\mathcal{Q}' \supset \mathcal{Q}_1 \supset \mathcal{Q}'_1 \supset \cdots \supset \mathcal{Q}_R \supset \mathcal{Q}'_R$$

with corresponding cardinalities $S' \ge S_1 \ge S'_1 \ge \cdots \ge S_R \ge S'_R$ as well as a sequence of residue classes $\{m_r \bmod P_r\}_{r=1}^R$ such that

(1) $\omega(P_1 \cdots P_R) = K$ and the numbers $P_1, \ldots, P_R$ are pairwise coprime;
(2) for each $q \in \mathcal{Q}_r$, $P_1 \cdots P_r \mid q$ and $\gcd(P_r, q/P_r) = 1$;
(3) for each $q \in \mathcal{Q}'_r$, $a_q \equiv m_r \pmod{P_r}$; and
(4) we have

$$P_r S'_r \ge S_r \ge \frac{S'_{r-1}}{h(P_r)^2 7^{w_r} K^{w_r - 1}}, \qquad w_r = \omega(P_r). \tag{4.1}$$

Suppose $r \ge 1$ and $\mathcal{Q}'_0 = \mathcal{Q}', \mathcal{Q}_1, \mathcal{Q}'_1, \ldots, \mathcal{Q}_{r-1}, \mathcal{Q}'_{r-1}$ satisfy all the required conditions. Let $\mathcal{B}_r = \{q/(P_1 \cdots P_{r-1}) : q \in \mathcal{Q}'_{r-1}\}$. By Lemma 3.5 (with $\mathcal{B} = \{\ker(b) : b \in \mathcal{B}_r\}$), there is a minimal, intersecting set $\mathcal{C}_r$ of squarefree integers so that for all $B \in \mathcal{B}_r$, there is a $C \in \mathcal{C}_r$ with $C \mid B$. For each B, let $C(B)$ denote the least $C \mid B$ with $C \in \mathcal{C}_r$. There must exist some choice of $w_r \ge 1$ such that

$$\bigl|\{B \in \mathcal{B}_r : \omega(C(B)) = w_r\}\bigr| \ge \frac{S'_{r-1}}{2^{w_r}},$$

since $\sum_{w \ge 1} 1/2^w = 1$. By Lemma 3.4, the number of elements $C \in \mathcal{C}_r$ with $\omega(C) = w_r$ is at most $w_r K^{w_r - 1}$. Hence, there exists some such C so that

$$\bigl|\{B \in \mathcal{B}_r : C(B) = C\}\bigr| \ge \frac{S'_{r-1}}{2^{w_r} w_r K^{w_r - 1}} \ge \frac{S'_{r-1}}{4^{w_r} K^{w_r - 1}}.$$

Since $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, for some integer P_r, composed of prime divisors of C,

$$\begin{aligned}\big|\{B \in \mathcal{B}_r : C(B) = C,\ P_r \mid B,\ \gcd(C, B/P_r) = 1\}\big| \\ \geq \left(\frac{6}{\pi^2}\right)^{w_r} \frac{S'_{r-1}}{4^{w_r} K^{w_r-1} h(P_r)^2} \geq \frac{S'_{r-1}}{7^{w_r} K^{w_r-1} h(P_r)^2}.\end{aligned}$$

Then we define

$$\mathcal{Q}_r := \{P_1 \cdots P_{r-1} B \in \mathcal{Q}'_{r-1} : C(B) = C,\ P_r \mid B,\ \gcd(C, B/P_r) = 1\}$$

for this choice of C and P_r.

Now consider the subsets

$$\mathcal{Q}_r(a) := \{q \in \mathcal{Q}_r : a_q \equiv a \pmod{P_r}\}.$$

The union of $\mathcal{Q}_r(a)$ over all a from 1 to P_r is $\mathcal{Q}_r$, so there exists a_r such that

$$|\mathcal{Q}_r(a_r)| \geq S_r/P_r. \tag{4.2}$$

We then define $\mathcal{Q}'_r := \mathcal{Q}_r(a_r)$ for this choice of a_r. The process terminates when $\mathcal{Q}'_R$ consists of a single element $P_1 \cdots P_R$.

4.3. Completing the proof. By iterating (4.1) and letting

$$W_r := \sum_{k=1}^{r} w_r \quad \text{and} \quad V_r := h(P_1 \cdots P_r),$$

we have

$$S_r \geq \frac{S'}{7^{W_r} V_r^2 \prod_{j=1}^{r} K^{w_j-1} \prod_{j=1}^{r-1} P_j}.$$

Let $c = R/\sqrt{\log x/\log_2 x}$ and $d = K/\sqrt{\log x/\log_2 x}$. By Lemma 3.1,

$$\begin{aligned}S_r &\leq \left|\left\{m \leq \frac{x}{P_1 \cdots P_r} : \omega(m) = K - W_r\right\}\right| \\ &\leq \frac{x}{P_1 \cdots P_r} L\big(-\tfrac{1}{2}d + o(1), x\big)(\log x)^{W_r/2}.\end{aligned}$$

This estimate is uniform in W_r.

By (3), $V_r \leq \mathrm{e}^{\sqrt{\log x}}$. Comparing the upper and lower bounds for S_r, we obtain

$$\begin{aligned}P_r &\leq \frac{x}{S'} L\big(-\tfrac{1}{2}d + o(1), x\big)(\log x)^{W_r/2} V_r^2 \Big(\prod_{j=1}^{r} K^{w_j-1}\Big) 7^{W_r} \\ &= \frac{x}{S'} L\big(-\tfrac{1}{2}d + o(1), x\big)(\log x)^{W_r/2} \exp\Big\{\sum_{j=1}^{r} (w_j - 1) \log K\Big\}.\end{aligned}$$

By multiplying the upper bounds for each P_j together and using the lower bound (2), we have

$$xL(-2,x) \le \prod_{r=1}^{R} P_r \le \left(\frac{x}{S'}L\big(-\tfrac{1}{2}d+o(1),x\big)\right)^R$$
$$\times \exp\Big\{\sum_{r=1}^{R}(R-r+1)\big(\tfrac{1}{2}w_r\log_2 x+(w_r-1)\log K\big)\Big\}.$$

We claim that the sum is maximized when $w_1 = K-R+1$ and $w_r = 1$ for $r \ge 2$. It suffices to show that if $w_r > 1$, $r < R$, then replacing w_r with $w_r - 1$ and replacing w_{r+1} with $w_{r+1}+1$ always decreases the value of the sum. Note that under such an operation only the rth and $(r+1)$th terms change value: the rth term changes by an amount

$$-(R-r+1)\big(\tfrac{1}{2}\log_2 x+\log K\big),$$

while the $(r+1)$th term changes by an amount

$$(R-r)\big(\tfrac{1}{2}\log_2 x+\log K\big).$$

Therefore, noting that $\log K < \frac{1}{2}\log_2 x$ for sufficiently large x, we have

$$xL(-2,x) \le \left(\frac{x}{S'}L\big(-\tfrac{1}{2}d+o(1),x\big)\right)^R \exp\big\{\tfrac{1}{2}R(K-R+1)\log_2 x\big\}$$
$$\times \exp\Big\{R(K-R)\log K+\tfrac{1}{2}\sum_{r=2}^{R}(R-r+1)\log_2 x\Big\}$$
$$\le \left(\frac{x}{S'}L\big(-\tfrac{1}{2}d+o(1),x\big)\exp\big\{\big(K-\tfrac{3}{4}R-\tfrac{1}{4}\big)\log_2 x\big\}\right)^R.$$

So taking Rth roots and rearranging gives

$$S' \le xL\left(-\frac{1}{c}-\frac{3c}{4}+\frac{d}{2}+o(1),x\right).$$

However, by Lemma 3.1, we also have

$$S' \le xL\big(-\tfrac{1}{2}d+o(1),x\big).$$

So,

$$S \le S'\cdot L(o(1),x) \le xL\left(-\max\left\{\frac{1}{c}+\frac{3c}{4}-\frac{d}{2},\frac{d}{2}\right\}+o(1),x\right)$$
$$\le xL\left(-\min_{0\le c\le d\le 3}\max\left\{\frac{1}{c}+\frac{3c}{4}-\frac{d}{2},\frac{d}{2}\right\}+o(1),x\right)$$
$$= xL\big(-\tfrac{1}{2}\sqrt{3}+o(1),x\big).$$

This proves the upper bound. ■

5. Conditional bounds

5.1. More on intersecting families. As we remarked in the introduction, we believe that the lower bound given in Theorem 1 is closer to the truth.

CONJECTURE 1. *We have* $f(x) = xL(-1+o(1), x)$.

We believe the weakness of our method lies in the use of Lemma 3.4. The bound given in Lemma 3.4 is, however, nearly sharp by Theorem 7 of [4] (there exist sets with $|\mathcal{A}^n| > n!$). We can avoid the use of Lemma 3.4 with the following.

CONJECTURE 2. *There are constants* $t(1), t(2), \ldots$ *satisfying* $\log t(j) = o(j \log j)$ *as* $j \to \infty$ *and such that for any finite intersecting family* $\mathcal{A}$ *of finite sets, there is a non-empty set* $\mathcal{C}$ *so that*

$$\#\{S \in \mathcal{A} : \mathcal{C} \subseteq S\} \ge \frac{|\mathcal{A}|}{t(|\mathcal{C}|)}.$$

REMARK. We can show the conclusion of Conjecture 2 holds with a sequence $t(1), \ldots$ satisfying $t(j) \ll j^{j+2}$.

THEOREM 2. *Conjecture* 2 *implies Conjecture* 1.

Proof. The proof is nearly identical to the proof of Theorem 1, with the following differences. In the "descending chain" argument (Section 4.2), apply Conjecture 2 with

$$\mathcal{A} = \big\{\{p : p \mid B\} : B \in \mathcal{B}_r\big\}.$$

We can then find a set $\mathcal{C}$ of w_r primes with product C so that

$$\#\{B \in \mathcal{B}_r : C \mid B\} \ge \frac{S'_{r-1}}{t(w_r)}.$$

The remainder of the argument is as before, except that the factor $w_r K^{w_r-1}$ is replaced by $t(w_r)$ throughout. In the final Section 4.3, the sum of $(w_j - 1) \log K$ is replaced by

$$\sum_{j=1}^{R} \log t(w_j) = o(\log K) \sum_{j=1}^{R} w_j = o(K \log K) = o(\sqrt{\log x \log_2 x}).$$

This leads to the estimate

$$S' \le \max_{0 \le c \le 3} xL\left(-\frac{1}{c} - \frac{c}{4} + o(1), x\right) \le xL(-1+o(1), x). \ \blacksquare$$

5.2. Sunflowers. There is a close connection between our Conjecture 2 and the theory of so-called Δ-sets (also known as sunflowers). A Δ-*system of size* k (*sunflower of size* k) is a collection of k sets whose pairwise intersections are all identical (this common intersection may be the empty set).

A famous problem of Erdős and Rado [5] is to bound $\phi(k,n)$, the maximum cardinality of a family of n-element sets that contains no Δ-set of size k. In [5], Erdős and Rado proved that

$$(k-1)^n \le \phi(k,n) < (k-1)^n n!$$

and conjectured that for each $k \ge 3$ there is a constant C_k so that $\phi(k,n) \le C_k^n$. The conjecture remains open for all k, the best bound known today being Kostochka's estimate [7]

$$\phi(k,n) \ll_k n!\left(\frac{30k\log_3 n}{\log_2 n}\right)^n.$$

Our Conjecture 2 implies a much stronger bound.

THEOREM 3. *Assume Conjecture* 2. *Then uniformly for* $k \ge 3$,

$$\log\phi(k,n) \le n\log(k-1) + o(n\log n) \qquad (n\to\infty).$$

Proof. Let $\mathcal{A}$ be a family of n-element sets of maximum cardinality $\phi(k,n)$. In particular, $\mathcal{A}$ does not contain k mutually disjoint sets. Thus, there is an intersecting subfamily $\mathcal{A}' \subseteq \mathcal{A}$ of size $\ge \frac{1}{k-1}\phi(k,n)$. By Conjecture 2, there is a set $\mathcal{C}$ so that

$$\mathcal{A}_1 = \{S-\mathcal{C} : \mathcal{C}\subseteq S\in\mathcal{A}\}$$

has cardinality

$$|\mathcal{A}_1| \ge \frac{|\mathcal{A}'|}{t(|\mathcal{C}|)} \ge \frac{|\mathcal{A}|}{(k-1)t(|\mathcal{C}|)}.$$

The set $\mathcal{A}_1$ contains no Δ-system of size k, since if $\{S_i\}_{i=1}^k$ is such a Δ-system, then $\{S_i\cup\mathcal{C}\}_{i=1}^k$ would be a Δ-system of size k for $\mathcal{A}$, which we know does not exist. Therefore $|\mathcal{A}_1| \le \phi(k,n-|\mathcal{C}|)$. Combining these two estimates gives

$$\phi(k,n) \le \max_{1\le j\le n}(k-1)t(j)\phi(k,n-j).$$

Iterating this last inequality and using $\phi(k,0)=1$ yields

$$\phi(k,n) \le \max_{1\le i\le n}(k-1)^i \max_{j_1+\cdots+j_i=n} t(j_1)\cdots t(j_i) \le (k-1)^n\exp\{o(n\log n)\}. \ ■$$

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