ACTA ARITHMETICA
126.2 (2007)

Small solutions of cubic equations with prime variables in arithmetic progressions

by

HAIGANG ZHOU (Shanghai) and TIANZE WANG (Kaifeng)

1. Introduction. A well known work of A. Baker [1] on the solvability of certain Diophantine inequalities involving primes first raised the problem on bounds for small solutions of some equations with prime variables. This problem is now called the Baker problem. As for the linear equation, the Baker problem was settled qualitatively in [6] by M. C. Liu and K. M. Tsang, and was generalized to the case of prime variables in arithmetic progressions in [8] by M. C. Liu and T. Z. Wang. Similar investigations have been applied to quadratic equations with five prime variables in [2], [7], [10] and [11]; and some different types of qualitative and quantitative results have been given.

In this paper, we are going to consider the cubic equation

$$a_1p_1^3 + \cdots + a_9p_9^3 = b \tag{1.1}$$

with prime variables in arithmetic progressions modulo large integer $k \geq 1$. One novelty of our investigations is that we will overcome the difficulties coming from the twisting of the nonlinearity and the rareness of primes in arithmetic progressions of large modulus as in [10]. The other novelty is that we can transform the congruent solvability condition similar to that in [7] to an easy to check form, by giving an elementary necessary and sufficient solvability condition using cubic residue characters. This needs some delicate analysis of the singular series, and forms one of the main themes of the present paper. Further, the best qualitative bound for small solutions is given.

Throughout this paper, we always use $a_1, \ldots, a_9$; $c_1, \ldots, c_9$; b and k to stand for integers satisfying

$$a_1 \cdots a_9 c_1 \cdots c_9 \neq 0, \quad k > 0, \tag{1.2}$$

2000 *Mathematics Subject Classification*: 11P32, 11D25, 11N13, 11L20.

Key words and phrases: cubic equation, prime variable, prime solution.

The second author is supported partially by the National Natural Science Foundation of China (Grant No. 10671056).

$$(c_j, k) := \gcd(c_j, k) = 1, \quad 1 \le j \le 9. \tag{1.3}$$

We define k^* to be $3k$ or k according as k is divisible by 3 or not, and assume

$$a_1 + \cdots + a_9 \equiv b \pmod 2, \tag{1.4}$$

$$a_1 c_1^3 + \cdots + a_9 c_9^3 \equiv b \pmod{k^*}. \tag{1.5}$$

We sometimes use a_{10} to denote $-b$, and for any subset $\{i_1, \dots, i_9\}$ of $\{1, \dots, 10\}$ suppose

$$(a_{i_1}, \dots, a_{i_9}) := \gcd(a_{i_1}, \dots, a_{i_9}) = 1. \tag{1.6}$$

Put $\omega = (-1 + \sqrt{3}\, i)/2$, and let $\mathbb{Z}[\omega]$ denote the ring of algebraic integers in the quadratic field $\mathbb{Q}(\omega)$ as in [5]. For any rational prime p with $p \equiv 1 \pmod 3$ we let π stand for a fixed primary prime divisor of p in $\mathbb{Z}[\omega]$, and $\chi_\pi(\cdot)$ denote the cubic residue character modulo π. If a rational prime $p \ge 7$ with $p \equiv 1 \pmod 3$ divides exactly eight of the ten numbers $a_1, \dots, a_{10}$, and if $(a_i, p) = (a_j, p) = 1$, then we suppose

$$\chi_\pi(a_i) = \chi_\pi(a_j). \tag{1.7}$$

Moreover, for any rational prime $p \le 96$ with $p \equiv 1 \pmod 3$, i.e. $p = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79$, we assume that the congruence

$$a_1 n_1^3 + \cdots + a_9 n_9^3 \equiv b \pmod p \tag{1.8}$$

is solvable in $\mathbb{F}_p^\times$, the multiplicative group of the finite field $\mathbb{F}_p$. Throughout this paper, we put

$$A := \max\{3, k, |a_1|, \dots, |a_9|\}.$$

We use C and c to denote positive effective absolute constants, not necessarily the same at different occurrences.

Our main results are as follows.

THEOREM 1. *Assume* (1.2)–(1.8). *If* $a_1, \dots, a_9$ *are all positive*, *then there exists an effective absolute constant* $C > 0$ *such that the equation*

$$\begin{cases} a_1 p_1^3 + \cdots + a_9 p_9^3 = b, \\ p_j \equiv c_j \pmod k, \quad 1 \le j \le 9, \end{cases} \tag{1.9}$$

is solvable whenever $b \ge A^C$.

THEOREM 2. *Assume* (1.2)–(1.8). *If* $a_1, \dots, a_9$ *are not of the same sign*, *then there exists an effective absolute constant* $C > 0$ *such that equation* (1.9) *has solutions in primes* p_j *satisfying*

$$\max\{p_1, \dots, p_9\} \le 3|b|^{1/3} + A^C.$$

PROPOSITION 1. *Conditions* (1.2)–(1.8) *are either natural or necessary for the solvability of equation* (1.9), *so in view of Theorems* 1 *and* 2 *they form a necessary and sufficient condition for the solvability of* (1.9).

It is trivial to see that (1.2) and (1.3) are natural for the study of equation (1.9). Now we assume that (1.9) is solvable in odd primes. Since every odd solution satisfies $p_j^3 \equiv 1 \pmod 2$ for $1 \le j \le 9$, (1.9) implies $a_1 + \cdots + a_9 \equiv b \pmod 2$, which is (1.4). If $3 \nmid k$, then $k^* = k$ by definition; so (1.5) is clearly necessary for the solvability of (1.9). If $3 \mid k$, then $p_j \equiv c_j \pmod k$ clearly gives $p_j^3 \equiv c_j^3 \pmod{3k}$; so the solvability of (1.9) also implies (1.5). Condition (1.6) is natural, since otherwise, the remaining a_j must be divisible by $(a_{i_1}, \dots, a_{i_9})$, and then we may divide both sides of the first equality of (1.9) by $(a_{i_1}, \dots, a_{i_9})$. To see (1.7), we set $p_{10} = 1$ (similar usage may occur below); then the solvability of (1.9) implies

$$a_i p_i^3 + a_j p_j^3 \equiv 0 \pmod p.$$

This clearly implies (1.7). Finally, the necessity of (1.8) is trivial, and the proof of Proposition 1 is complete.

Remark. The bound A^C in Theorems 1 and 2 is best possible if we are not concerned with the exact value of C.

2. Outline of the proofs of Theorems 1 and 2. We shall use the circle method, so we introduce a large parameter N which is fixed throughout this paper. Put

$$P := N^{\delta}, \quad L := \log N, \quad Q := NP^{-20}L^{-100}; \tag{2.1}$$

here and throughout, δ is a fixed sufficiently small constant which may depend on some fixed small positive absolute constant $\varepsilon > 0$. We always assume

$$P^{\delta} \ge A. \tag{2.2}$$

By Dirichlet's lemma on rational approximations, each α in $[1/Q, 1 + 1/Q]$ may be written as

$$\alpha = a/q + \eta, \quad |\eta| \le 1/(qQ), \tag{2.3}$$

for some integers a and q with $(a, q) = 1$ and $1 \le a \le q \le Q$. We denote by $m(a, q)$ the set of α satisfying (2.3), and define the *major arcs* $\mathfrak{M}$ and *minor arcs* $C(\mathfrak{M})$ as follows:

$$\mathfrak{M} := \bigcup_{1 \le q \le P} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} m(a, q), \quad C(\mathfrak{M}) := [1/Q, 1 + 1/Q] \setminus \mathfrak{M}. \tag{2.4}$$

It is clear that all the $m(a, q)$'s are mutually disjoint for $q \le P$ since $2P < Q$. As usual write $e(x) = \exp(2\pi i x)$ for any real x, and let $\Lambda(n)$ denote the von

Mangoldt function. For $1 \le j \le 9$ put

$$S_j(\alpha) := \sum_{\substack{N/100<|a_j|n^3\le N\\ n\equiv c_j \ (\mathrm{mod}\, k)}} \Lambda(n)e(a_j\alpha n^3), \tag{2.5}$$

and define

$$r(b) := \sum_{(p_1,\dots,p_9)} (\log p_1)\cdots(\log p_9), \tag{2.6}$$

where the summation is over all prime 9-tuples $(p_1,\dots,p_9)$ satisfying $a_1p_1^3 + \cdots + a_9p_9^3 = b$, $N/100 < |a_j|p_j^3 \le N$ and $p_j \equiv c_j \ (\mathrm{mod}\, k)$ with $1 \le j \le 9$. Then using Hölder's inequality and Hua's lemma (Theorem 4 in [4]) to treat the error term we have

$$r(b) = \int_{\mathfrak{M}} + \int_{C(\mathfrak{M})} + O(N^{11/6}L^c). \tag{2.7}$$

To prove Theorems 1 and 2, we only need to prove that for some N satisfying (2.2), i.e. $N \ge A^{\delta^{-2}}$, $r(b)$ has a positive lower bound if

(i) $b = N$ when all the a_j with $1 \le j \le 9$ are positive;
(ii) $N \ge 20|b|$ when a_j with $1 \le j \le 9$ are not of the same sign.

So by (2.7) we need a lower bound for $\int_{\mathfrak{M}}$ and an upper bound for $\int_{C(\mathfrak{M})}$. The former will be given in Lemma 6.1, and the latter in Lemma 7.2. Then the combination of Lemmas 6.1, 7.2 and the definition of $r(b)$ in (2.6) proves Theorems 1 and 2.

3. Simplification for $\int_{\mathfrak{M}}$. In the following, we always abbreviate

$$d := (k,q), \quad D := [k,q]. \tag{3.1}$$

When $(\ell,q) = 1$ and $\ell \equiv c_j \ (\mathrm{mod}\, d)$, we let s_j be the unique solution modulo D to the pair of the congruences $n \equiv c_j \ (\mathrm{mod}\, k)$, $n \equiv \ell \ (\mathrm{mod}\, q)$. Note that $(s_j,k) = (s_j,q) = (s_j,D) = 1$. Introduce the Dirichlet character χ modulo any $q \ge 1$ and let $\chi_0 \ (\mathrm{mod}\, q)$ be the principal character. For $1 \le j \le 9$, $\chi \ (\mathrm{mod}\, D)$, and any integer a with $(a,q) = 1$, define

$$\begin{aligned} G_j(\chi,a) &:= \sum_{\substack{\ell=1\\ \ell\equiv c_j \ (\mathrm{mod}\, d)}}^{q} \chi(s_j)e(a_ja\ell^3/q), \\ G_j(q,a) &:= G_j(\chi_0 \ (\mathrm{mod}\, D), a). \end{aligned} \tag{3.2}$$

Define a large parameter T by

$$T := N^{\sqrt{\delta}} = P^{1/\sqrt{\delta}}. \tag{3.3}$$

It is well known (see, e.g., [3, §14]) that there exists a small constant $c > 0$ such that the function

$$\prod_{q\le kP}\ \prod_{\chi\,(\mathrm{mod}\,q)}^{*} L(s,\chi)$$

has at most one zero $\widetilde{\beta}$ in the region

$$\sigma\ge 1-\frac{c}{\log P},\quad |t|\le T, \tag{3.4}$$

where the $*$ indicates that the product over χ runs through all primitive characters, s is any complex variable, $\sigma=\operatorname{Re}s$, $t=\operatorname{Im}s$; such a zero $\widetilde{\beta}$, if it exists, is real, simple and unique, and corresponds to a nonprincipal primitive character $\widetilde{\chi}$ to a modulus $\widetilde{r}$ with $3\le\widetilde{r}\le kP$. We call $\widetilde{\chi}$ and $\widetilde{\beta}$ the *exceptional character* and *exceptional zero* respectively. From [3, §14] we have

$$1-\frac{c}{\log P}\le\widetilde{\beta}\le 1-\frac{c}{\widetilde{r}^{1/2}\log^2\widetilde{r}}. \tag{3.5}$$

Define $\widetilde{E}=1$ or 0 according as $\widetilde{r}\,|\,D$ or not. For $1\le j\le 9$, put

$$N_j:=(N/|a_j|)^{1/3},\quad N_j':=(N/(100|a_j|))^{1/3}. \tag{3.6}$$

Let

$$\begin{gathered} I_j(\eta):=\int_{N_j'}^{N_j} e(a_j\eta x^3)\,dx,\quad \widetilde{I}_j(\eta):=\int_{N_j'}^{N_j} x^{\widetilde{\beta}-1}e(a_j\eta x^3)\,dx,\\ I_j(\chi,\eta):=\sum_{|\gamma|\le T}{}' \int_{N_j'}^{N_j} x^{\varrho-1}e(a_j\eta x^3)\,dx. \end{gathered} \tag{3.7}$$

Put

$$C_j(a,q,\eta):=\sum_{\chi\,(\mathrm{mod}\,D)} G_j(\overline{\chi},a)I_j(\chi,\eta), \tag{3.8}$$

$$H_j(a,q,\eta):=G_j(q,a)I_j(\eta)-\widetilde{E}G_j(\widetilde{\chi}\chi_0,a)\widetilde{I}_j(\eta)-C_j(a,q,\eta). \tag{3.9}$$

When we multiply out the product $\prod_{j=1}^{9}H_j(a,q,\eta)$ using (3.9), we get a sum of 3^9 terms which can be classified into the following three categories:

- $\mathcal{J}_1$: the term $\prod_{j=1}^{9}G_j(q,a)I_j(\eta)$,
- $\mathcal{J}_2$: the $3^9-2^9=19171$ terms each of which has at least one $C_j(a,q,\eta)$ as factor,
- $\mathcal{J}_3$: the remaining $2^9-1=511$ terms.

For $1\le v\le 3$ define

$$\begin{aligned} M_v:=\sum_{1\le q\le P}\frac{1}{\varphi(D)^9}&\sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(-\frac{ab}{q}\Big)\\ &\times\int_{-\infty}^{\infty} e(-b\eta)\{\text{sum of the terms in } \mathcal{J}_v\}\,d\eta. \end{aligned} \tag{3.10}$$

Then, with the help of [9, Lemmas 4.3 and 4.5], we can conclude, using similar arguments to those in [10],

$$\int_{\mathfrak{M}} = M_1 + M_2 + M_3 + O(N^2|a_2\cdots a_9|^{-1/3}P^{-27}). \tag{3.11}$$

4. Estimation of M_1. We first give a lemma which can be proved by the method of [6, Lemma 4.7] and will be used to treat the singular integral.

LEMMA 4.1. *For any complex number ϱ_j with $0 < \operatorname{Re}\varrho_j \le 1$, we have*

$$\begin{aligned}\int_{-\infty}^{\infty} e(-b\eta)\Big(\prod_{j=1}^{9}\int_{N_j'}^{N_j} x^{\varrho_j-1}e(a_j\eta x^3)\,dx\Big)\,d\eta \\ = N^2(3^9|a_9|)^{-1}\int_{\mathcal{D}}\prod_{j=1}^{9}((Nx_j)^{(\varrho_j-1)/3}x_j^{-2/3})\,dx_1\cdots dx_8,\end{aligned}\tag{4.1}$$

where

$$x_9 := (bN^{-1} - a_1x_1 - \cdots - a_8x_8)/a_9 \tag{4.2}$$

and

$$\mathcal{D} := \{(x_1,\ldots,x_8) : 1/(100|a_j|) \le x_j \le 1/|a_j|,\ 1 \le j \le 9\}. \tag{4.3}$$

Furthermore, if either (i) *not all the a_j's are of the same sign and $N \ge 20|b|$, or* (ii) *all the a_j's are positive and $N = b$, then*

$$\int_{\mathcal{D}}\prod_{j=1}^{9}x_j^{-2/3}\,dx_1\cdots dx_8 \asymp |a_1\cdots a_8|^{-1/3}|a_9|^{2/3}. \tag{4.4}$$

For any character χ modulo $q\ge 1$ and any integers a and c with $(c,k)=1$, let $d=(k,q)$ and put

$$G^*(\chi,a) := \sum_{\substack{\ell=1\\ \ell\equiv c\,(\mathrm{mod}\,d)}}^{q} \chi(\ell)e(a\ell^3/q). \tag{4.5}$$

LEMMA 4.2. *Let $\chi \pmod{p^\alpha}$ be any character and $\alpha \ge 0$. Then*

(a) $G^*(\chi,a)=0$ *if χ is primitive*, $\mathrm{ord}_p(k) \le \alpha-1$, *and $p\,|\,a$;*
(b) $G^*(\chi\chi_0,a) = 0$ *if χ_0 is modulo p^t, $p\nmid a$ and* $\mathrm{ord}_p(k) \le \max\{1,\alpha\}$, $t \ge \theta + \max\{1,\alpha\}$, *where* $\theta = 1 + [3/p] - [2/p]$;
(c) $|G^*(\chi,a)| \le 2(2,p)(a,p^\alpha)^{1/2}p^{\alpha/2}$.

Proof. (a) In view of $p\,|\,a$, we can write $a = a'p$. Writing $\ell = vp^{\alpha-1}+u$ with $1 \le u \le p^{\alpha-1}$ and $0 \le v \le p-1$, and noting $(k,p^\alpha) = p^{\mathrm{ord}_p(k)}$ since

$\mathrm{ord}_p(k) \le \alpha - 1$, we get

$$G^*(\chi, a) = \sum_{\substack{1 \le u \le p^{\alpha-1} \\ u \equiv c \, (\mathrm{mod}\, p^{\mathrm{ord}_p(k)})}} e(a'u^3/p^{\alpha-1}) F(u), \tag{4.6}$$

where $F(u) = \sum_{0 \le v \le p-1} \chi(vp^{\alpha-1} + u)$. Clearly, $F(u)$ is a periodic function with period $p^{\alpha-1}$. Since $\chi \pmod{p^\alpha}$ is primitive, there exists an integer $1 < m < p^\alpha$ such that $m \equiv 1 \pmod{p^{\alpha-1}}$ and $\chi(m) \ne 1$. Thus

$$\chi(m)F(u) = \sum_{0 \le v \le p-1} \chi(mvp^{\alpha-1} + mu) = \sum_{0 \le v \le p-1} \chi(vp^{\alpha-1} + u) = F(u),$$

which implies $F(u) = 0$, and part (a) follows from (4.6).

(b) For $1 \le \ell \le p^t$, write $\ell = vp^{t-\theta} + u$ with $1 \le u \le p^{t-\theta}, 0 \le v \le p^\theta - 1$. Then since $\mathrm{ord}_p(k) \le \max\{1, \alpha\}$ and $t \ge \theta + \max\{1, \alpha\}$ we have

$$\begin{aligned} e(a\ell^3/p^t) &= e(a(v^3 p^{3t-3\theta} + 3u^2 v p^{t-\theta} + 3uv^2 p^{2t-2\theta} + u^3)/p^t) \\ &= e(au^3/p^t) e(3au^2 v/p^\theta), \end{aligned}$$

$$\chi\chi_0(\ell) = \chi\chi_0(vp^{t-\theta} + u) = \chi(u)\chi_0(u), \quad (k, p^t) = p^{\mathrm{ord}_p(k)} \le p^{t-\theta}.$$

So by definition and in view of $p \nmid a$, $\theta = 1 + [3/p] - [2/p]$, we get

$$G^*(\chi\chi_0, a) = \sum_{\substack{1 \le u \le p^{t-\theta} \\ u \equiv c \, (\mathrm{mod}\, p^{\mathrm{ord}_p(k)})}} \chi\chi_0(u) e(au^3/p^t) \sum_{0 \le v \le p^\theta - 1} e(3au^2 v/p^\theta) = 0.$$

This proves part (b).

(c) By definition and the orthogonality of characters, we have

$$|G^*(\chi, a)| \le \frac{1}{\varphi(d)} \sum_{\chi_1 \, (\mathrm{mod}\, d)} \Big| \sum_{\ell=1}^{p^\alpha} \chi_1\chi(\ell) e(a\ell^3/p^\alpha) \Big|.$$

Note that $d = (k, p^\alpha)$. So $\chi_1\chi$ is a character modulo p^α. Thus the last sum over ℓ can be bounded by $2(2,p)(a, p^\alpha)^{1/2} p^{\alpha/2}$ by [7, Lemma 3.1(c)]. This proves part (c). The proof of Lemma 4.2 is complete.

Now we turn to the investigation of the singular series. Let

$$A(q) := \frac{\varphi(d)^9}{\varphi(q)^9} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(-\frac{ab}{q}\Big) \prod_{j=1}^{9} G_j(q, a). \tag{4.7}$$

Then $A(q)$ is a multiplicative function of q. So we are led to evaluate $A(q)$ when q is a prime power p^m. Firstly, by (1.3) and (1.4), direct computations yield $A(1) = A(2) = 1$. For any integer $m \ge 2$, we can compute $A(2^m)$ as follows. If $2^m \mid k$ (so $(k, 2^m) = 2^m$), then in view of (1.5),

$$A(2^m) = \sum_{\substack{a=1\\(a,2)=1}}^{2^m} e\Big(-\frac{ab}{2^m}\Big) \prod_{j=1}^{9} e(a_j a c_j^3/2^m) = \varphi(2^m).$$

If $2^m \nmid k$, we suppose $2^v \parallel k$, i.e. $2^v \mid k$ but $2^{v+1} \nmid k$. Then $0 \le v \le m-1$ and $(k, 2^m) = 2^v$. Also, in view of (1.6), there exists a $1 \le j_0 \le 9$ such that $(a_{j_0}, 2) = 1$. Introducing Dirichlet characters $\chi \pmod{2^v}$, we get

$$\sum_{\substack{\ell=1\\(\ell,2)=1\\ \ell\equiv c_{j_0} \,(\mathrm{mod} 2^v)}}^{2^m} e(a_{j_0} a \ell^3/2^m) = \frac{1}{\varphi(2^v)} \sum_{\chi \,(\mathrm{mod}\, 2^v)} \overline{\chi}(c_{j_0}) \sum_{\substack{\ell=1\\(\ell,2^m)=1}}^{2^m} \chi(\ell) e(a_{j_0} a \ell^3/2^m). \tag{4.8}$$

For any $\chi \pmod{2^v}$, in view of $m \ge 1 + \max\{1, v\}$, by Lemma 4.2(b) we see that the last sum over ℓ in (4.8) vanishes. Thus by (4.7) we obtain $A(2^m) = 0$.

Gathering together the above, we obtain

Lemma 4.3. *Under the assumptions* (1.3)–(1.5), *we have* $A(1) = A(2) = 1$, *and for any integer* $m \ge 2$,

$$A(2^m) = \begin{cases} \varphi(2^m) & \text{if } 2^m \mid k, \\ 0 & \text{if } 2^m \nmid k. \end{cases}$$

Now we begin to compute $A(3^m)$ for $m \ge 1$. If $m = 1$, we consider two cases according as $3 \mid k$ or not. When $3 \mid k$, by (1.5) we have $A(3) = \varphi(3)$. When $3 \nmid k$, so $(3, k) = 1$, we have

$$A(3) = \frac{1}{\varphi(3)^9} \sum_{\substack{a=1\\(a,3)=1}}^{3} e\Big(-\frac{b}{3}\Big) \prod_{j=1}^{9} (e(a_j/3) + e(-a_j/3)) = (-1)^{10-n} \varphi(3)^{n-9},$$

where n is the number of integers among $a_1, \ldots, a_{10}$ divisible by 3. To compute $A(9)$ we consider three cases according as $(k, 9) = 1, 3$ or 9. If $(k, 9) = 1$, (4.7) gives

$$\begin{aligned} A(9) &= \Big(\frac{1}{\varphi(9)}\Big)^9 \sum_{\substack{a=1\\(a,3)=1}}^{9} e\Big(-\frac{ab}{9}\Big) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\(\ell,3)=1}}^{9} e(a_j a \ell^3/9) \\ &= \sum_{\substack{a=1\\(a,3)=1}}^{9} e\Big(-\frac{ab}{9}\Big) \prod_{j=1}^{9} \cos\Big(\frac{2\pi a a_j}{9}\Big). \end{aligned}$$

If $(k,9)=3$ or 9, by (4.7) and (1.5), direct computation yields $A(9)=\varphi(9)$. For any integer $m\geq 3$, we can compute $A(3^m)$ as follows. If $3^m\mid k$ (so $(k,3^m)=3^m$), then in view of (1.5) we have $A(3^m)=\varphi(3^m)$. If $3^{m-1}\mid k$ but $3^m\nmid k$ (so $(k,3^m)=3^{m-1}$), then

$$A(3^m)=\left(\frac{1}{3}\right)^9\sum_{\substack{a=1\\(a,3)=1}}^{3^m}e\left(-\frac{ab}{3^m}\right)\prod_{j=1}^{9}\sum_{\substack{\ell=1\\(\ell,3)=1\\\ell\equiv c_j\ (\mathrm{mod}\ 3^{m-1})}}^{3^m}e(a_ja\ell^3/3^m). \tag{4.9}$$

Note that $\ell\equiv c_j\pmod{3^{m-1}}$ must imply $(\ell,3)=1$ since $(c_j,k)=1$. So the last summation variable ℓ in (4.9) can be written as $\ell=3^{m-1}t+c_j$ with $0\leq t\leq 2$, and the sum over ℓ is

$$\sum_{0\leq t\leq 2}e(a_ja(3^{m-1}t+c_j)^3/3^m)=3e(a_jac_j^3/3^m).$$

Thus by (4.9) we get

$$A(3^m)=\sum_{\substack{a=1\\(a,3)=1}}^{3^m}e\left(-\frac{ab}{3^m}\right)\prod_{j=1}^{9}e(a_jac_j^3/3^m)=\varphi(3^m).$$

If $3^{m-1}\nmid k$, we suppose $3^v\parallel k$. Then $0\leq v\leq m-2$, $(k,3^m)=3^v$. Also, in view of (1.6), there exists a $1\leq j_0\leq 9$ such that $(a_{j_0},3)=1$. Introducing the Dirichlet character $\chi\pmod{3^v}$, we get

$$\begin{aligned}\sum_{\substack{\ell=1\\(\ell,3)=1\\\ell\equiv c_{j_0}\ (\mathrm{mod}\ 3^v)}}^{3^m}&e(a_ja\ell^3/3^m)\\&=\frac{1}{\varphi(3^v)}\sum_{\chi\ (\mathrm{mod}\ 3^v)}\overline{\chi}(c_{j_0})\sum_{\substack{\ell=1\\(\ell,3^m)=1}}^{3^m}\chi(\ell)e(a_{j_0}a\ell^3/3^m).\end{aligned} \tag{4.10}$$

For any $\chi\pmod{3^v}$, in view of $m\geq 2+\max\{1,v\}$, by Lemma 4.2(b) we see that the last sum over ℓ in (4.10) vanishes, and so does (4.10). Thus by (4.7) we obtain $A(3^m)=0$.

Gathering together the above, we obtain

LEMMA 4.4. *Under the assumptions* (1.3), (1.5) *and* (1.6), *let n denote the number of a_j's* $(1\leq j\leq 10)$ *divisible by* 3. *Then*

$$A(3)=\begin{cases}\varphi(3) & \textit{if } 3\mid k,\\(-1)^{10-n}\varphi(3)^{n-9} & \textit{if } 3\nmid k,\end{cases}$$

$$A(9)=\begin{cases}\varphi(9) & \text{if } 3\,|\,k,\\ \displaystyle\sum_{\substack{a=1\\(a,3)=1}}^{9} e\Big(-\frac{ab}{9}\Big)\prod_{j=1}^{9}\cos\Big(\frac{2\pi a a_j}{9}\Big) & \text{if } 3\nmid k,\end{cases}$$

and for $m\geq 3$,

$$A(3^m)=\begin{cases}\varphi(3^m) & \text{if } 3^{m-1}\,|\,k,\\ 0 & \text{if } 3^{m-1}\nmid k.\end{cases}$$

Now, we compute $A(p^m)$ for $p\geq 5$ and $m\geq 1$. If $p\nmid k$, then $(k,p^m)=1$; so by (4.7),

$$A(p^m)=\varphi(p^m)^{-9}\sum_{\substack{a=1\\(a,p)=1}}^{p^m} e\Big(-\frac{ab}{p^m}\Big)\prod_{j=1}^{9}\sum_{\substack{\ell=1\\(\ell,p)=1}}^{p^m} e(a_j a\ell^3/p^m).$$

In view of (1.6), there exists a $1\leq j_0\leq 9$ such that $(a_{j_0},p)=1$. So if $m\geq 2$, then by Lemma 4.2(b) the last sum over ℓ with $j=j_0$ vanishes for any a with $(a,p)=1$; and this leads to $A(p^m)=0$ for $p\geq 5$, $m\geq 2$ and $p\nmid k$.

Next, we consider the case $p\,|\,k$. For any $m\geq 1$, if $p^m\,|\,k$, then $(k,p^m)=p^m$. Thus by (4.7) and (1.5) we get $A(p^m)=\varphi(p^m)$. If $p^m\nmid k$, we suppose $p^v\,\|\,k$; then $1\leq v\leq m-1$, and $(k,p^m)=p^v$. By (4.7) we get

$$A(p^m) = \Big(\frac{\varphi(p^v)}{\varphi(p^m)}\Big)^9\sum_{\substack{a=1\\(a,p^m)=1}}^{p^m} e\Big(-\frac{ab}{p^m}\Big)\prod_{j=1}^{9}\sum_{\substack{\ell=1\\(\ell,p^m)=1\\ \ell\equiv c_j \,(\mathrm{mod}\, p^v)}}^{p^m} e(a_j a\ell^3/p^m). \tag{4.11}$$

If we introduce Dirichlet characters $\chi \ (\mathrm{mod}\, p^v)$, the last sum over ℓ in (4.11) with $j=j_0$ is

$$\frac{1}{\varphi(p^v)}\sum_{\chi\,(\mathrm{mod}\,p^v)}\overline{\chi}(c_{j_0})\sum_{\substack{\ell=1\\(\ell,p)=1}}^{p^m}\chi(\ell)e(a_{j_0}a\ell^3/p^m),$$

where $p\nmid a_{j_0}$. Since $m\geq 1+v$, by Lemma 4.2(b) the last sum over ℓ vanishes for any $\chi \ (\mathrm{mod}\, p^v)$, and this leads to the vanishing of (4.11).

Now we are in a position to consider $A(p)$ for $5\leq p\nmid k$. For $p\equiv 2 \ (\mathrm{mod}\, 3)$ (so $(p-1,3)=1$), it is known that for any $1\leq a\leq p-1$, the equation $x^3=a$ has exactly one solution in the multiplicative group $\mathbb{F}_p^\times=\{1,\dots,p-1\}$ of the finite field $\mathbb{F}_p$. So when ℓ runs over $\mathbb{F}_p^\times$, ℓ^3 will run over $\mathbb{F}_p^\times$ as well. Thus (4.7) yields

$$(4.12)\qquad A(p) = \varphi(p)^{-9} \sum_{\substack{a=1\\(a,p)=1}}^{p} e\Big(-\frac{ab}{p}\Big) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\(\ell,p)=1}}^{p} e(a_j a\ell/p)$$
$$= (-1)^{10-n}\varphi(p)^{n-9}.$$

Here, and in what follows, n is always used to denote the number of integers divisible by p among $a_1, \ldots, a_{10}$ with $a_{10} = -b$. For $p \equiv 1 \pmod 3$, let g_1 and g_2 be two fixed cubic non-residues from $1, \ldots, p-1$ whose indices relative to a given primitive root modulo p are congruent to 1 and 2 respectively modulo 3. Then a^3, g_1a^3 and g_2a^3 will run through respectively the cubic residues, the cubic non-residues whose indices are $\equiv 1 \pmod 3$, and the cubic non-residues whose indices are $\equiv 2 \pmod 3$ three times as a assumes $1, \ldots, p-1$. Hence by (4.7) we have

$$(4.13)\qquad A(p) = \frac{1}{3}\,\varphi(p)^{-9} \sum_{a=1}^{p-1} \bigg(e\Big(-\frac{a^3b}{p}\Big) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\(\ell,p)=1}}^{p} e(a_j a^3\ell^3/p)$$
$$+ e\Big(-\frac{g_1a^3b}{p}\Big) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\(\ell,p)=1}}^{p} e(a_j g_1 a^3\ell^3/p)$$
$$+ e\Big(-\frac{g_2a^3b}{p}\Big) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\(\ell,p)=1}}^{p} e(a_j g_2 a^3\ell^3/p)\bigg).$$

Again, for $p \equiv 1 \pmod 3$, we can write $4p = a^2 + 27b^2$ with $a \equiv 1 \pmod 3$ uniquely determined, and we can define a unique $\theta = \theta(p)$ up to sign as in [3, §3]. Put $\lambda_1 = 2\sqrt{p}\cos\theta$, $\lambda_2 = 2\sqrt{p}\cos(\theta - 2\pi/3)$ and $\lambda_3 = 2\sqrt{p}\cos(\theta + 2\pi/3)$. Then from [3, §3] we have $\sum_{\ell=1}^{p-1} e(\ell^3/p) = \lambda_1 - 1$, $\sum_{\ell=1}^{p-1} e(g_1\ell^3/p) = \lambda_2 - 1$ and $\sum_{\ell=1}^{p-1} e(g_2\ell^3/p) = \lambda_3 - 1$. Further, let u, v and w denote respectively the number of cubic residues, of cubic non-residues whose indices are $\equiv 1 \pmod 3$, and of cubic non-residues whose indices are $\equiv 2 \pmod 3$ among $a_1, \ldots, a_{10}$. Then we have $n + u + v + w = 10$. It then follows from (4.13) that

$$(4.14)\qquad A(p) = \frac{1}{3}\,\varphi(p)^{n-9}\{(\lambda_1 - 1)^u(\lambda_2 - 1)^v(\lambda_3 - 1)^w$$
$$+ (\lambda_1 - 1)^v(\lambda_2 - 1)^w(\lambda_3 - 1)^u$$
$$+ (\lambda_1 - 1)^w(\lambda_2 - 1)^u(\lambda_3 - 1)^v\}.$$

Now we obtain the following

LEMMA 4.5. *Under the assumptions* (1.3), (1.5) *and* (1.6), *for* $p \ge 5$ *and* $m \ge 1$ *we have*

$$A(p^m)=\begin{cases}\varphi(p^m) & \textit{if } p^m\,|\,k,\\ 0 & \textit{if } p^m\nmid k,\ m\ge 2;\end{cases}$$

and for $5\le p\nmid k$ *we have* (4.12) *or* (4.14) *according as* $p\equiv 2 \pmod 3$ *or* $p\equiv 1 \pmod 3$.

By Lemmas 4.3–4.5, and the multiplicativity of $A(q)$, for any real $y\ge 1$ we have

$$\sum_{q\le y}|A(q)|\ll k\prod_{\substack{3\ne p\nmid k\\ p\equiv 1 \,(\mathrm{mod}\, 3)}}(1+|A(p)|)\prod_{\substack{3\ne p\nmid k\\ p\equiv 2 \,(\mathrm{mod}\, 3)}}(1+|A(p)|). \tag{4.15}$$

For $p\equiv 2 \pmod 3$ with $p\,|\,a_1\cdots a_9$, but $p\nmid k$, in view of (4.12), it is easy to see that $|A(p)|<1$. For $p\equiv 2 \pmod 3$ with $p\nmid a_1\cdots a_9 k$, also from (4.12) we get $|A(p)|\le\varphi(p)^{-8}$. For $p\equiv 1 \pmod 3$ and $p\nmid k$, by (4.14),

$$|A(p)|\le\varphi(p)^{n-9}(2\sqrt{p}+1)^{u+v+w}=(p-1)^{n-9}(2\sqrt{p}+1)^{10-n}. \tag{4.16}$$

Gathering these, direct computations yield $|A(p)|\le 10$ for $3\ne p\nmid k$, and for all $p\nmid ka_1\cdots a_9$ (so $n=0$ or 1), $|A(p)|\le 500p^{-2}$. So (4.15) can be estimated further as

$$\ll k\prod_{\substack{3\ne p\nmid k\\ p|a_1\cdots a_9}}11\prod_{3\ne p\nmid ka_1\cdots a_9}(1+500p^{-2})\ll 11^{\omega(a_1\cdots a_9)}k, \tag{4.17}$$

where $\omega(m)$ denotes the number of distinct prime factors of the integer m. This shows that $\sum_{q\le y}|A(q)|$ can be bounded by a constant independent of y, so the series

$$\mathfrak{S}(b):=\sum_{q=1}^{\infty}A(q) \tag{4.18}$$

is absolutely convergent. In view of Lemmas 4.3–4.5, we can define

$$s(3):=\begin{cases}1+\varphi(3)+\varphi(3^2)+\cdots+\varphi(3^{1+\mathrm{ord}_3(k)})=3^{1+\mathrm{ord}_3(k)} & \\ & \text{if } 3\,|\,k,\\ 1+A(3)+A(3^2) & \text{if } 3\nmid k;\end{cases} \tag{4.19}$$

$$s(p):=\begin{cases}1+\varphi(p)+\varphi(p^2)+\cdots+\varphi(p^{\mathrm{ord}_p(k)})=p^{\mathrm{ord}_p(k)} & \\ & \text{if } 3\ne p\,|\,k,\\ 1+A(p) & \text{if } 3\ne p\nmid k.\end{cases} \tag{4.20}$$

Then for any integer m it is clear that

$$\sum_{\substack{q=1\\(q,m)=1}}^{\infty}A(q)=\prod_{p\nmid m}(1+A(p)+A(p^3)+\cdots)=\prod_{p\nmid m}s(p). \tag{4.21}$$

Next, we prove that the series $\mathfrak{S}(b)$ defined by (4.18) has a positive lower bound. By (4.19), (4.20) and (4.21) with $m = 1$ we have

$$\mathfrak{S}(b) = \prod_{p|k} s(p) \prod_{p\nmid k} s(p). \tag{4.22}$$

Note that for $3\nmid k$, by Lemma 4.4,

$$A(3^2) = \sum_{a=1,2,4} \prod_{j=1}^{9} \cos^2\left(\frac{2\pi a a_j}{9}\right) + i \sum_{\substack{a=1\\(a,3)=1}}^{9} \sin\left(\frac{2\pi a a_{10}}{9}\right) \prod_{j=1}^{9} \cos\left(\frac{2\pi a a_j}{9}\right).$$

This shows that $A(3^2)$ has positive real part when $3\nmid k$. Hence, for $3\nmid k$, $s(3) = 1+A(3)+A(3^2)$ has real part $\geq 1/2$ since by Lemma 4.4, $|A(3)| \leq 1/2$. Thus $s(3) \geq 1/2$, and this together with (4.22), (4.19) and (4.20) yields

$$\mathfrak{S}(b) \gg k \prod_{3\neq p\nmid k} (1 + A(p)). \tag{4.23}$$

For convenience we introduce, for any integer $q \geq 1$,

$$\mathcal{N}(q) := \mathrm{card}\Big\{(n_1, \ldots, n_9) : 1 \leq n_j \leq q, \quad (n_j, q) = 1, \sum_{j=1}^{9} a_j n_j^3 \equiv b \pmod q\Big\}. \tag{4.24}$$

Similar to [7, (3.8)], for $3 \neq p\nmid k$ we have

$$\varphi(p)^{-9} p\mathcal{N}(p) = 1 + A(p). \tag{4.25}$$

When $p \equiv 2 \pmod 3$ and $p\nmid k$, in view of (4.12) and $n \leq 8$, we get $A(p) \neq -1$. When $p \equiv 1 \pmod 3$ and $p\nmid k$, we separate our discussion into three cases as follows: (i) for $p \leq 96$, condition (1.8) clearly implies $\mathcal{N}(p) \geq 1$, or $A(p) \neq -1$ by (4.25); (ii) for $p \geq 97$ and $n \leq 7$, in view of (4.16), direct computation shows that $|A(p)| < 1$, so $A(p) \neq -1$; (iii) for $p \geq 97$ and $n = 8$, in view of $u + v + w + n = 10$, we have $u + v + w = 2$, so by condition (1.7) we see that the possible triplets (u, v, w) are $(2, 0, 0)$, $(0, 2, 0)$ or $(0, 0, 2)$; and by (4.14) we get

$$\begin{aligned} A(p) = \frac{1}{3}\,\varphi(p)^{n-9}\{&(\lambda_1 - 1)^2(\lambda_2 - 1)^0(\lambda_3 - 1)^0 \\ &+ (\lambda_1 - 1)^0(\lambda_2 - 1)^0(\lambda_3 - 1)^2 + (\lambda_1 - 1)^0(\lambda_2 - 1)^2(\lambda_3 - 1)^0\} \\ = \frac{3(p+1)}{3(p-1)} &= \frac{p+1}{p-1} \neq -1. \end{aligned}$$

Therefore we can conclude for any prime p with $3 \neq p\nmid k$ that $A(p) \neq -1$, and this in combination with (4.25) implies $\mathcal{N}(p) \geq 1$, thus $1 + A(p) \geq$

$\varphi(p)^{-9}p$. This in combination with (4.23) yields

$$\mathfrak{S}(b) \gg k \prod_{p \mid a_1 \cdots a_9} p\varphi(p)^{-9} \prod_{\substack{3 \neq p \nmid k \\ p \nmid a_1 \cdots a_9}} (1 + A(p)).$$

In view of $|A(p)| \le 500p^{-2}$ for $p \nmid ka_1 \cdots a_9$, the last product is $\gg 1$. So we arrive at

$$\mathfrak{S}(b) \gg k \prod_{p \mid a_1 \cdots a_9} p^{-8} \gg k|a_1 \cdots a_9|^{-8}. \tag{4.26}$$

Let $\sigma = (\log P)^{-1}$. By Lemmas 4.3–4.5, the multiplicativity of $A(q)$, and $|A(p)| \le 500p^{-2}$ for any $3 \neq p \nmid ka_1 \cdots a_9$, and $|A(p)| \le 10$ for any $3 \neq p \nmid k$, we have

$$\begin{aligned} \sum_{q \ge P} |A(q)| &\ll P^{-1}k^2 \prod_{p \mid a_1 \cdots a_9} (11p) \prod_{p \nmid a_1 \cdots a_9} (1 + 500p^{-1-\sigma}) \\ &\ll P^{-1}k^2 L^{500} |a_1 \cdots a_9|^2. \end{aligned} \tag{4.27}$$

Finally, we complete the estimate for M_1. By definition we have

$$M_1 = \sum_{1 \le q \le P} \frac{1}{\varphi(D)^9} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{ab}{q}\right) \int_{-\infty}^{\infty} e(-b\eta) \prod_{j=1}^{9} G_j(q,a) I_j(\eta)\, d\eta.$$

Note that $\varphi(d)\varphi(D) = \varphi(k)\varphi(q)$. So by (4.7) and Lemma 4.1 with $\varrho_1 = \varrho_2 = \varrho_3 = 1$ we get

$$M_1 = N^2(3^9|a_9|)^{-1}\varphi(k)^{-9}\Big(\sum_{1 \le q \le P} A(q)\Big) \int_{\mathcal{D}} \prod_{j=1}^{9} x_j^{-2/3}\, dx_1 \cdots dx_8.$$

Now let

$$M_0 := N^2(3^9|a_9|)^{-1}\varphi(k)^{-9}\mathfrak{S}(b) \int_{\mathcal{D}} \prod_{j=1}^{9} x_j^{-2/3}\, dx_1 \cdots dx_8. \tag{4.28}$$

Then by (4.18) and (4.27) we get

$$M_1 = M_0 + R,$$

where by (4.4) we have $R \ll N^2\varphi(k)^{-9}P^{-1}k^2L^{500}|a_1 \cdots a_9|^{5/3}$. Therefore we can conclude that

$$M_1 = M_0 + O(N^2\varphi(k)^{-9}P^{-1}k^2L^{500}|a_1 \cdots a_9|^{5/3}). \tag{4.29}$$

5. General singular series. Throughout this section, we let $r_1, \ldots, r_9$ be any positive integers, and let $\chi_j \pmod{r_j}$ be primitive characters. Put

$r := [r_1, \ldots, r_9]$. The purpose of this section is to estimate the sum

$$\sum := \sum_{\substack{1\le q\le P\\ r\mid D}} \left(\frac{1}{\varphi(D)}\right)^9 \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{ab}{q}\right) \prod_{j=1}^{9} G_j(\chi_j\chi_0, a), \tag{5.1}$$

where $d = (k,q)$ and $D = [k,q]$ are as in (3.1), $G_j(\chi,a)$ is defined as in (3.2) with χ modulo D, and χ_0 is the principal character modulo D. For $1 \le j \le 9$, let

$$r_j' = \prod_{\mathrm{ord}_p(r_j)>\mathrm{ord}_p(k)} p^{\mathrm{ord}_p(r_j)}, \quad r_j'' = r_j/r_j' = \prod_{\mathrm{ord}_p(r_j)\le\mathrm{ord}_p(k)} p^{\mathrm{ord}_p(r_j)}. \tag{5.2}$$

Put $r' = [r_1', \ldots, r_9']$. Then it is easy to see that

$$r \mid D \iff r_j = r_j' r_j'' \mid [k,q],\ 1\le j\le 9 \iff r_j' \mid q,\ 1\le j\le 9 \iff r'\mid q. \tag{5.3}$$

Next, (5.2) yields $(r_j', r_j'') = 1$ and $r_j = r_j' r_j''$. So one can split $\chi_j \pmod{r_j}$, $1\le j\le 9$ as $\chi_j \pmod{r_j} = \chi_j' \pmod{r_j'}\chi_j'' \pmod{r_j''}$ with both χ_j' and χ_j'' primitive since χ_j is primitive. Here we temporarily regard $\chi \pmod 1$ as primitive, and similar usage may occur below. Note that for $1\le j\le 9$, by (5.2) we have $r_j'' \mid k$, and by (5.3), if $r \mid D$ then $r_j' \mid q$. Thus we can write

$$\sum = \left(\frac{1}{\varphi(k)}\right)^9 \Big(\prod_{j=1}^{9} \chi_j''(c_j)\Big) \sum\nolimits_1, \tag{5.4}$$

where

$$\sum\nolimits_1 := \sum_{\substack{1\le q\le P\\ r'\mid q}} \left(\frac{\varphi(d)}{\varphi(q)}\right)^9 \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{ab}{q}\right) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\(\ell,q)=1\\ \ell\equiv c_j \,(\mathrm{mod}\, d)}}^{q} \chi_j'(\ell) e(a_j a\ell^3/q). \tag{5.5}$$

Now the estimation for $\sum$ is reduced to the estimation for $\sum_1$. To proceed further, we introduce the following notation similar to that in [7, (3.1) and (3.2)]:

$$Z(q;\chi_1,\ldots,\chi_9) := \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{ab}{q}\right) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\ \ell\equiv c_j \,(\mathrm{mod}\, d)}}^{q} \chi_j(\ell) e(a_j a\ell^3/q), \tag{5.6}$$

$$Y(q;\chi_1,\ldots,\chi_9) := \sum_{a=1}^{q} e\left(-\frac{ab}{q}\right) \prod_{j=1}^{9} \sum_{\substack{\ell=1\\ \ell\equiv c_j \,(\mathrm{mod}\, d)}}^{q} \chi_j(\ell) e(a_j a\ell^3/q), \tag{5.7}$$

where q is any positive integer, $\chi_1, \ldots, \chi_9$ are characters modulo q, and $d = (k,q)$. When there is no possible confusion about the character χ_j, we shall abbreviate these to $Z(q)$ and $Y(q)$ respectively. Similar to [7, Lemma 3.2] it can be easily proved that both $Z(q)$ and $Y(q)$ are multiplicative.

LEMMA 5.1. *For* $j = 1, \ldots, 9$, *let* $\chi_j \pmod{p^{\alpha_j}}$ *be primitive characters and* $\alpha = \max\{\alpha_1, \ldots, \alpha_9\} > \mathrm{ord}_p(k)$. *For any* $t \geq \alpha$, *let* $Z(p^t) = Z(p^t; \chi_1\chi_0, \ldots, \chi_9\chi_0)$ *where* χ_0 *is modulo* p^t. *Then*

(a) $Z(p^\alpha) = Y(p^\alpha)$,

(b) $Z(p^t) = 0$ *if* $t \geq \theta + \alpha$, *where* $\theta = 1 + [3/p] - [2/p]$,

(c) $\displaystyle\sum_{v=\alpha}^{\beta} \left(\frac{\varphi((k, p^v))}{\varphi(p^v)}\right)^9 Z(p^v) = \left(\frac{\varphi((k, p^\beta))}{\varphi(p^\beta)}\right)^9 Y(p^\beta)$ *for any* $\beta \geq \alpha$.

Proof. This lemma can be proved in precisely the same way as [10, Lemma 5.2] using Lemma 4.2.

Now we come to the estimate for $\sum_1$. For any integers m and n, we now use the notation $m \,\|\, n$ to denote that $m \mid n$ and every prime factor of n divides m. For the integer q in (5.5) we write

$$q = q_1 q_2, \quad r' \,\|\, q_1, \quad (r', q_2) = 1.$$

It is clear that $(q_1, q_2) = 1$. So by (5.5), (5.6) and (4.7) we get

$$\sum\nolimits_1 = \sum_{\substack{1 \leq q_1 \leq P \\ r' \| q_1}} \left(\frac{\varphi((k, q_1))}{\varphi(q_1)}\right)^9 Z(q_1; \chi_1'\chi_0, \ldots, \chi_9'\chi_0) \sum_{\substack{1 \leq q_2 \leq P/q_1 \\ (q_2, r') = 1}} A(q_2). \tag{5.8}$$

From (4.21) and (4.27), the last sum over q_2 is

$$\prod_{p \nmid r'} s(p) + O(P^{-1} q_1 k^2 L^{500} |a_1 \cdots a_9|^2). \tag{5.9}$$

By the multiplicativity of $Z(q_1; \chi_1'\chi_0, \ldots, \chi_9'\chi_0)$ and Lemma 5.1(b) we see that for $r' \,\|\, q_1$,

$$\begin{aligned} &\text{if} \quad 3 \nmid r' \quad \text{then} \quad Z(q_1; \chi_1'\chi_0, \ldots, \chi_9'\chi_0) = 0 \quad \text{except for } q_1 = r', \\ &\text{if} \quad 3 \mid r' \quad \text{then} \quad Z(q_1; \chi_1'\chi_0, \ldots, \chi_9'\chi_0) = 0 \quad \text{except for } q_1 = r' \text{ and } 3r'. \end{aligned}$$

Now define

$$\sigma := 1 \text{ or } 3 \quad \text{according as } 3 \nmid r' \text{ or } 3 \mid r'. \tag{5.10}$$

Then by (5.8) and (5.9) the main term of $\sum_1$ is

$$\Big(\prod_{p \nmid r'} s(p)\Big) \sum_{u \mid \sigma} \left(\frac{\varphi((k, ur'))}{\varphi(ur')}\right)^9 Z(ur'; \chi_1'\chi_0, \ldots, \chi_9'\chi_0)$$

when $\sigma r' \le P$. As in [7, (3.14)], by Lemma 5.1(a),(c), the above last sum over u is

$$\left(\frac{\varphi((k,\sigma r'))}{\varphi(\sigma r')}\right)^9 Y(\sigma r'; \chi_1'\chi_0, \ldots, \chi_9'\chi_0).$$

Thus, when $\sigma r' \le P$, the main term of $\sum_1$ can be written as

$$\left(\frac{\varphi((k,\sigma r'))}{\varphi(\sigma r')}\right)^9 Y(\sigma r'; \chi_1'\chi_0, \ldots, \chi_9'\chi_0) \prod_{p\nmid r'} s(p). \tag{5.11}$$

Note that, by (5.6) and Lemma 4.2(c), for any $q \ge 1$ we have

$$Z(q) \ll q^{5.5}|a_1\cdots a_9|^{1/2} 2^{9\omega(q)}. \tag{5.12}$$

So by (5.8) and (5.9), and noting $q_1 = ur', u \mid \sigma$, the error term of $\sum_1$ can be estimated as

$$\begin{aligned} &\ll P^{-1} r' k^2 L^{500} |a_1\cdots a_9|^2 \left(\frac{\varphi(k)}{\varphi(r')}\right)^9 r'^{5.5} |a_1\cdots a_9|^{1/2} 2^{9\omega(r')} \\ &\ll P^{-1} k^{11} L^{500} |a_1\cdots a_9|^{2.5}. \end{aligned}$$

This together with (5.11) gives, when $\sigma r' \le P$,

$$\begin{aligned} \textstyle\sum_1 = \left(\frac{\varphi((k,\sigma r'))}{\varphi(\sigma r')}\right)^9 Y(\sigma r'; \chi_1'\chi_0, \ldots, \chi_9'\chi_0) \prod_{p\nmid r'} s(p) \\ + O(P^{-1} k^{11} L^{500} |a_1\cdots a_9|^{2.5}). \end{aligned} \tag{5.13}$$

When $\sigma r' > P$ (so $r' \gg P$), the validity of (5.13) can be seen as follows: firstly, from (5.5) and (5.12) we have

$$\begin{aligned} \textstyle\sum_1 &\ll \sum_{\substack{1\le q\le P \\ r'\mid q}} \left(\frac{k}{\varphi(q)}\right)^9 q^{5.5} |a_1\cdots a_9|^{1/2} 2^{9\omega(q)} \\ &\ll r'^{-1} k^9 |a_1\cdots a_9|^{1/2}, \end{aligned} \tag{5.14}$$

which is $\ll P^{-1}k^9|a_1\cdots a_9|^{1/2}$ if $\sigma r' > P$. Secondly, (4.17) and (4.21) imply, for any integer n,

$$\prod_{p\nmid n} s(p) \ll \sum_{q=1}^{\infty} |A(q)| \ll 11^{\omega(a_1\cdots a_9)} k. \tag{5.15}$$

So in view of $|Y(q)|$ having the same upper bound as $|Z(q)|$ in (5.12), the main term in (5.13) is

$$\ll k^9 \varphi(r')^{-9} r'^{5.5} |a_1\cdots a_9|^{1/2} 2^{9\omega(r')} 11^{\omega(a_1\cdots a_9)} k \ll P^{-1} k^{10} |a_1\cdots a_9|$$

if $\sigma r' > P$. From (5.13) and (5.14) we infer the following

LEMMA 5.2. *Let* $\sum_1$ *be as in* (5.5), *and* σ *be as in* (5.10). *Then*

$$\sum\nolimits_1 = \left(\frac{\varphi((k,\sigma r'))}{\varphi(\sigma r')}\right)^9 Y(\sigma r';\chi_1'\chi_0,\ldots,\chi_9'\chi_0)\prod_{p\nmid r'} s(p) + O(P^{-1}k^{11}L^{500}|a_1\cdots a_9|^{2.5}),$$

and

$$\sum\nolimits_1 \ll r'^{-1}k^9|a_1\cdots a_9|^{1/2}.$$

6. The major arc integrals. In this section, we complete the estimation for the major arc integrals $\int_{\mathfrak{M}}$. We first estimate M_3, defined in (3.10). Note that if $\widetilde{\beta}$ does not exist then $M_3=0$. So in the following we assume $\widetilde{\beta}$ does indeed exist, hence $\widetilde{E}=1$. We decompose $\widetilde{r}$ as follows:

$$\widetilde{r}=\widetilde{r}'\widetilde{r}'',\quad \widetilde{r}'=\prod_{\mathrm{ord}_p(\widetilde{r})>\mathrm{ord}_p(k)} p^{\mathrm{ord}_p(\widetilde{r})},\quad \widetilde{r}''=\prod_{\mathrm{ord}_p(\widetilde{r})\le\mathrm{ord}_p(k)} p^{\mathrm{ord}_p(\widetilde{r})}. \tag{6.1}$$

Then $(\widetilde{r}',\widetilde{r}'')=1$. So we can split $\widetilde{\chi}\ (\mathrm{mod}\,\widetilde{r})$ as

$$\widetilde{\chi}\ (\mathrm{mod}\,\widetilde{r})=\widetilde{\chi}'\ (\mathrm{mod}\,\widetilde{r}')\widetilde{\chi}''\ (\mathrm{mod}\,\widetilde{r}''), \tag{6.2}$$

where $\widetilde{\chi}'$ and $\widetilde{\chi}''$ are primitive characters. Here we have regarded $\chi_0\ (\mathrm{mod}\,1)$ as primitive character. We define

$$\widetilde{\sigma}:=3 \text{ or } 1 \quad \text{according as } 3\,|\,\widetilde{r}' \text{ or not.} \tag{6.3}$$

For distinct integers $m_1,m_2,\ldots,$ taken from the set $\{1,\ldots,9\}$, let

$$\mathcal{L}(m_1,m_2,\ldots):=(\widetilde{\chi}''(c_{m_1})\widetilde{\chi}''(c_{m_2})\cdots)(\widetilde{\sigma}\widetilde{r}')^{-1}Y(\widetilde{\sigma}\widetilde{r}';\chi_1,\ldots,\chi_9), \tag{6.4}$$

where for $1\le j\le 9$,

$$\chi_j=\begin{cases}\widetilde{\chi}'\chi_0\ (\mathrm{mod}\,\widetilde{\sigma}\widetilde{r}') & \text{for } j\in\{m_1,m_2,\ldots\},\\ \chi_0\ (\mathrm{mod}\,\widetilde{\sigma}\widetilde{r}') & \text{otherwise,}\end{cases}$$

and let

$$\mathcal{P}(m_1,m_2,\ldots) := N^2(3^9|a_9|)^{-1}\int_{\mathcal{D}}\Big(\prod_{j=1}^9 x_j^{-2/3}\Big)(Nx_{m_1}Nx_{m_2}\cdots)^{(\widetilde{\beta}-1)/3}\,dx_1\cdots dx_8, \tag{6.5}$$

where $\mathcal{D}$ is defined as in (4.3). Then by (5.7) we get

$$\mathcal{L}(m_1,m_2,\ldots)=\sum_{(\widetilde{\sigma}\widetilde{r}')}(\widetilde{\chi}'(\ell_{m_1})\widetilde{\chi}''(c_{m_1})\widetilde{\chi}'(\ell_{m_2})\widetilde{\chi}''(c_{m_2})\cdots) \tag{6.6}$$

where for any $q\ge1$, $\sum_{(q)}$ denotes the sum over $\ell_1,\ldots,\ell_9$ with $1\le\ell_j\le q$, $\ell_j\equiv c_j\ (\mathrm{mod}\,(k,q))$ for $1\le j\le 9$ and $a_1\ell_1^3+\cdots+a_9\ell_9^3\equiv b\ (\mathrm{mod}\,q)$.

Now we can estimate M_3. By the definition of $H_j(a,q,\eta)$ in (3.9), the 511 terms in $\mathcal{J}_3$ can be classified into nine types with the vth $(1\le v\le 9)$

type consisting of $\binom{9}{v}$ terms, each of which is a product of v pieces of $-G_j(\widetilde{\chi}\chi_0, a)\widetilde{I}_j(\eta)$ and $9-v$ pieces of $G_j(q,a)I_j(\eta)$. So if we define, for $1 \le v \le 9$,

$$M_{3v} := \sum_{1\le q\le P} \frac{1}{\varphi(D)^9} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{ab}{q}\right) \times \int_{-\infty}^{\infty} e(-b\eta)\{\text{sum of the terms in the } v\text{th type}\}\, d\eta,$$

then by (3.10) we have

$$M_3 = \sum_{1\le v\le 9} M_{3v}. \tag{6.7}$$

For $1 \le v \le 9$, the contributions to M_{3v} from the terms of the vth type can be estimated in precisely the same way. So we only consider the contribution, denoted by M_{3v1}, from the typical term

$$\prod_{j=1}^{v}(-G_j(\widetilde{\chi}\chi_0, a)\widetilde{I}_j(\eta)) \prod_{j=v+1}^{9} (G_j(q,a)I_j(\eta)).$$

We have by definition,

$$\begin{aligned} M_{3v1} = (-1)^v &\sum_{\substack{1\le q\le P\\ \widetilde{r}\mid D}} \frac{1}{\varphi(D)^9} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{ab}{q}\right) \\ &\times \prod_{j=1}^{v} G_j(\widetilde{\chi}\chi_0, a) \prod_{j=v+1}^{9} G_j(q,a) \\ &\times \int_{-\infty}^{\infty} e(-b\eta) \prod_{j=1}^{v} \widetilde{I}_j(\eta) \prod_{j=v+1}^{9} I_j(\eta)\, d\eta. \end{aligned} \tag{6.8}$$

By (5.4) and the first equality for $\sum_1$ in Lemma 5.2, and then the definition of $\mathcal{L}(m_1, m_2, \ldots)$ in (6.4), the sum over q in (6.8) is

$$\begin{aligned} \varphi(k)^{-9}\widetilde{\sigma}\widetilde{r}'\left(\frac{\varphi((k,\widetilde{\sigma}\widetilde{r}'))}{\varphi(\widetilde{\sigma}\widetilde{r}')}\right)^9 &\mathcal{L}(1,2,\ldots,v) \prod_{p\nmid \widetilde{r}'} s(p) \\ &+ O(P^{-1}k^{11}\varphi(k)^{-9}L^{500}|a_1\cdots a_9|^{2.5}). \end{aligned} \tag{6.9}$$

Note that by (6.5) and (4.4) we have

$$\mathcal{P}(m_1, m_2, \ldots) \ll N^2|a_1\cdots a_9|^{-1/3}. \tag{6.10}$$

From (6.5) and Lemma 4.1 we see that the integral with respect to η in (6.8) is precisely $\mathcal{P}(1,2,\ldots,v)$. This together with (6.8)–(6.10) gives

$$M_{3v1} = (-1)^v \varphi(k)^{-9} \mathcal{L}(1,2,\dots,v)\mathcal{P}(1,2,\dots,v) \times \widetilde{\sigma}\widetilde{r}' \left(\frac{\varphi((k,\widetilde{\sigma}\widetilde{r}'))}{\varphi(\widetilde{\sigma}\widetilde{r}')} \right)^9 \prod_{p \nmid \widetilde{r}'} s(p) + O(N^2 P^{-1} k^{11} \varphi(k)^{-9} L^{500} |a_1 \cdots a_9|^{7/3}). \tag{6.11}$$

Now gathering together all the results similar to (6.11) for all $1 \le v \le 9$, and using (6.7), we arrive at

$$\begin{aligned} M_3 = {}& \varphi(k)^{-9} \widetilde{\sigma}\widetilde{r}' \left(\frac{\varphi((k,\widetilde{\sigma}\widetilde{r}'))}{\varphi(\widetilde{\sigma}\widetilde{r}')} \right)^9 \Big(\prod_{p \nmid \widetilde{r}'} s(p) \Big) \Big\{ - \sum_{1 \le m_1 \le 9} \mathcal{L}(m_1)\mathcal{P}(m_1) \\ & + \sum_{1 \le m_1 < m_2 \le 9} \mathcal{L}(m_1,m_2)\mathcal{P}(m_1,m_2) + \cdots \\ & + (-1)^t \sum_{1 \le m_1 < \cdots < m_t \le 9} \mathcal{L}(m_1,\dots,m_t)\mathcal{P}(m_1,\dots,m_t) + \cdots \\ & - \mathcal{L}(1,\dots,9)\mathcal{P}(1,\dots,9) \Big\} \\ & + O(N^2 P^{-1} k^{11} \varphi(k)^{-9} L^{500} |a_1 \cdots a_9|^{7/3}). \end{aligned} \tag{6.12}$$

On the other hand, by (5.4) and the second inequality for $\sum_1$ in Lemma 5.2, the sum over q in (6.8) is $\ll \varphi(k)^{-9} \widetilde{r}'^{-1} k^9 |a_1 \cdots a_9|^{1/2}$. This in combination with (6.8) and (6.10) yields

$$\begin{aligned} M_{3v1} &\ll \varphi(k)^{-9} k^9 \widetilde{r}'^{-1} |a_1 \cdots a_9|^{1/2} N^2 |a_1 \cdots a_9|^{-1/3} \\ &\ll N^2 \varphi(k)^{-9} k^9 \widetilde{r}'^{-1} |a_1 \cdots a_9|^{1/6}, \end{aligned}$$

and consequently by (6.7),

$$M_3 \ll N^2 \varphi(k)^{-9} k^9 \widetilde{r}'^{-1} |a_1 \cdots a_9|^{1/6}. \tag{6.13}$$

Moreover, similar to [10, (6.12)] we have

$$\prod_{p \mid \widetilde{r}'} s(p) = \widetilde{\sigma}\widetilde{r}' \left(\frac{\varphi((k,\widetilde{\sigma}\widetilde{r}'))}{\varphi(\widetilde{\sigma}\widetilde{r}')} \right)^9 \sum_{(\widetilde{\sigma}\widetilde{r}')} 1. \tag{6.14}$$

By (4.18), (4.21) and (6.14),

$$\begin{aligned} \mathfrak{S}(b) = \prod_p s(p) &= \prod_{p \mid \widetilde{r}'} s(p) \prod_{p \nmid \widetilde{r}'} s(p) \\ &= \widetilde{\sigma}\widetilde{r}' \left(\frac{\varphi((k,\widetilde{\sigma}\widetilde{r}'))}{\varphi(\widetilde{\sigma}\widetilde{r}')} \right)^9 \Big(\prod_{p \nmid \widetilde{r}'} s(p) \Big) \sum_{(\widetilde{\sigma}\widetilde{r}')} 1. \end{aligned} \tag{6.15}$$

Substituting this into (4.28), by (4.29), (6.12) and (6.5) we get

$$(6.16)\quad \begin{aligned} &M_1 + M_3 \\ &= N^2(3^9|a_9|)^{-1}\varphi(k)^{-9}\widetilde{\sigma}\widetilde{r}'\Big(\frac{\varphi((k,\widetilde{\sigma}\widetilde{r}'))}{\varphi(\widetilde{\sigma}\widetilde{r}')}\Big)^9\Big(\prod_{p\nmid\widetilde{r}'} s(p)\Big) \\ &\quad\times \int_{\mathcal{D}}\Big(\prod_{j=1}^{9} x_j^{-2/3}\Big)\Big\{\sum_{(\widetilde{\sigma}\widetilde{r}')} 1 - \sum_{1\le m_1\le 9}\mathcal{L}(m_1)(Nx_{m_1})^{(\widetilde{\beta}-1)/3} \\ &\quad+ \sum_{1\le m_1<m_2\le 9}\mathcal{L}(m_1,m_2)(Nx_{m_1}Nx_{m_2})^{(\widetilde{\beta}-1)/3} + \cdots \\ &\quad+ (-1)^t\sum_{1\le m_1<\cdots<m_t\le 9}\mathcal{L}(m_1,\ldots,m_t)(Nx_{m_1}\cdots Nx_{m_t})^{(\widetilde{\beta}-1)/3} \\ &\quad+ \cdots - \mathcal{L}(1,\ldots,9)(Nx_1\cdots Nx_9)^{(\widetilde{\beta}-1)/3}\Big\}\,dx_1\cdots dx_8 \\ &\quad+ O(N^2\varphi(k)^{-9}k^{11}P^{-1}L^{500}|a_1\cdots a_9|^{7/3}) \\ &\quad+ O(N^2\varphi(k)^{-9}P^{-1}k^2L^{500}|a_1\cdots a_9|^{5/3}). \end{aligned}$$

By (6.6) we see that the quantity in the above curly brackets equals

$$\sum_{(\widetilde{\sigma}\widetilde{r}')}\prod_{j=1}^{9}(1-\widetilde{\chi}'(\ell_j)\widetilde{\chi}''(c_j)(Nx_j)^{(\widetilde{\beta}-1)/3}) \ge ((1-\widetilde{\beta})\log P)^9\sum_{(\widetilde{\sigma}\widetilde{r}')} 1.$$

Thus (6.16) together with (6.15) and (4.4) leads to

$$(6.17)\quad \begin{aligned} M_1 + M_3 \ge{}& c_1N^2\varphi(k)^{-9}\mathfrak{S}(b)((1-\widetilde{\beta})\log P)^9|a_1\cdots a_9|^{-1/3} \\ &+ O(N^2\varphi(k)^{-9}k^{11}P^{-1}L^{500}|a_1\cdots a_9|^{7/3}), \end{aligned}$$

where c_1 is an absolute positive constant. On the other hand, the combination of (4.28), (4.29), (6.13) and (4.4) yields

$$(6.18)\quad \begin{aligned} M_1 + M_3 \ge{}& c_2N^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3} \\ &+ O(N^2\varphi(k)^{-9}P^{-1}k^2L^{500}|a_1\cdots a_9|^{5/3}) \\ &+ O(N^2\varphi(k)^{-9}k^9\widetilde{r}'^{-1}|a_1\cdots a_9|^{1/6}), \end{aligned}$$

where c_2 is an absolute positive constant.

Now we turn to the estimation of M_2, defined in (3.10). By definition there are 19171 terms in $\mathcal{J}_2$. The contribution to M_2 from each of them can be estimated in precisely the same way. So in view of (3.9) we only give the details for the contribution from the typical term

$$\widetilde{E}C_1(a,q,\eta)C_2(a,q,\eta)\cdots C_7(a,q,\eta)G_8(q,a)I_8(\eta)G_9(\widetilde{\chi}\chi_0,a)\widetilde{I}_9(\eta)$$

to illustrate the method. We denote this contribution to M_2 as M_{27}. Note that if $\chi \pmod q$ is induced by a primitive $\chi^* \pmod{q^*}$ with $q^* \mid q$ then the corresponding L-function $L(s,\chi^*)$ has the same set of nontrivial zeros. So in view of (3.7) we have $I_j(\chi,\eta) = I_j(\chi^*,\eta)$ for $1 \le j \le 9$. Then in view of (3.10) and (3.8) we have

$$\begin{aligned}(6.19)\quad M_{27} = &\sum_{r_1 \le kP} \sideset{}{^*}\sum_{\chi_1 \pmod{r_1}} \cdots \sum_{r_7 \le kP} \sideset{}{^*}\sum_{\chi_7 \pmod{r_7}} \sum_{\substack{1 \le q \le P \\ [r_1,\dots,r_7,\widetilde{r}] \mid D}} \frac{1}{\varphi(D)^9} \\ &\times \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(-\frac{ab}{q}\Big)\Big(\prod_{j=1}^{7} G_j(\overline{\chi}_j\chi_0, a)\Big) G_8(q,a) G_9(\widetilde{\chi}\chi_0, a) \\ &\times \int_{-\infty}^{\infty} e(-b\eta)\Big(\prod_{j=1}^{7} I_j(\chi_j,\eta)\Big) I_8(\eta)\widetilde{I}_9(\eta)\, d\eta,\end{aligned}$$

where the * indicates that the sums over $\chi_j \pmod{r_j}$ run through all the primitive characters. By the definition of $Y(q)$ in (5.7), it is trivial that $|Y(q)| \le q \sum_{(q)} 1$. Thus Lemma 5.2 implies

$$\begin{aligned}(6.20)\qquad \Big|\sum\nolimits_1\Big| &\le \sigma r' \Big(\frac{\varphi((k,\sigma r'))}{\varphi(\sigma r')}\Big)^9 \Big(\sum_{(\sigma r')} 1\Big) \prod_{p \nmid r'} s(p) \\ &\le \prod_{p \mid r'} s(p) \prod_{p \nmid r'} s(p) = \prod_p s(p) = \mathfrak{S}(b).\end{aligned}$$

From this and (5.4) we see that the absolute value of the sum over q in (6.19) is $\le \varphi(k)^{-9}\mathfrak{S}(b)$. So in view of the definition of $I_j(\chi,\eta)$ in (3.7), we obtain from (6.19), and Lemma 4.1,

$$\begin{aligned}(6.21)\quad &|M_{27}| \le N^2 \varphi(k)^{-9} \mathfrak{S}(b)(3^9|a_9|)^{-1} \\ &\times \int_{\mathcal{D}} \Big(\prod_{j=1}^{9} x_j^{-2/3}\Big) \prod_{j=1}^{7} \sum_{r_j \le kP} \sideset{}{^*}\sum_{\chi_j \pmod{r_j}} \sideset{}{'}\sum_{|\gamma_j| \le T} (Nx_j)^{(\beta_j - 1)/3}\, dx_1 \cdots dx_8,\end{aligned}$$

where $\beta_j + i\gamma_j$ are the nontrivial zeros of $L(s,\chi_j)$. The last triple sum can be estimated as $\ll \Omega^9 \exp(-c/\sqrt{\delta})$, where $\Omega = (1-\widetilde{\beta})\log P$ or 1 according as $\widetilde{\beta}$ exists or not, and $c > 0$ is an absolute constant. This in combination with (6.21) and (4.4) gives

$$M_{27} \ll \exp(-c/\sqrt{\delta})\Omega^{63} N^2 \varphi(k)^{-9} \mathfrak{S}(b)|a_1 \cdots a_9|^{-1/3},$$

and consequently,

$$(6.22)\qquad M_2 \ll \exp(-c/\sqrt{\delta})\Omega^9 N^2 \varphi(k)^{-9} \mathfrak{S}(b)|a_1 \cdots a_9|^{-1/3}.$$

Now we can complete the estimation of the major arc integrals. We separate the argument into three cases:

(i) If $\widetilde{\beta}$ does not exist, then $M_3 = 0$. So by (3.11), (4.26), (4.28), (4.29), (6.22), (2.2) and (4.4) we get, for δ small enough,

$$
\begin{aligned}
(6.23)\qquad \int_{\mathfrak{M}} &= N^2\varphi(k)^{-9}\mathfrak{S}(b)(3^9|a_9|)^{-1}\int_{\mathcal{D}}\prod_{j=1}^{9} x_j^{-2/3}\,dx_1\cdots dx_4\\
&\quad + O(N^2|a_1\cdots a_9|^{-1/3}P^{-27})\\
&\quad + O(\exp(-c/\sqrt{\delta})N^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3})\\
&\quad + O(N^2\varphi(k)^{-9}P^{-1}k^2L^{500}|a_1\cdots a_9|^{5/3})\\
&\ge cN^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3}.
\end{aligned}
$$

(ii) If $\widetilde{\beta}$ exists with $\widetilde{r}' \ge P^{1/100}$, then the combination of (3.11), (4.26), (6.18), (6.22) and (2.2) gives

$$
\begin{aligned}
(6.24)\qquad \int_{\mathfrak{M}} &\ge cN^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3}\\
&\quad + O(N^2\varphi(k)^{-9}P^{-1}k^2L^{500}|a_1\cdots a_9|^{5/3})\\
&\quad + O(N^2\varphi(k)^{-9}k^9P^{-1/100}|a_1\cdots a_9|^{1/6})\\
&\quad + O(N^2|a_1\cdots a_9|^{-1/3}P^{-27})\\
&\quad + O(\exp(-c/\sqrt{\delta})\Omega^5N^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3})\\
&\ge cN^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3}.
\end{aligned}
$$

(iii) If $\widetilde{\beta}$ exists with $\widetilde{r}' \le P^{1/100}$, then by (6.1) we have $\widetilde{r} = \widetilde{r}'\widetilde{r}'' \le kP^{1/100} \le P^{1/99}$. Thus by (3.5) we get

$$
(6.25)\qquad \Omega = (1-\widetilde{\beta})\log P \ge \frac{c}{\widetilde{r}^{1/2}\log^2\widetilde{r}} \ge P^{-1/197}.
$$

Thus by (3.11), (2.2), (4.26), (6.17) and (6.22) we get

$$
\begin{aligned}
(6.26)\qquad \int_{\mathfrak{M}} &\ge cN^2\varphi(k)^{-9}\mathfrak{S}(b)\Omega^9|a_1\cdots a_9|^{-1/3}\\
&\quad + O(N^2\varphi(k)^{-9}k^{11}P^{-1}L^{500}|a_1\cdots a_9|^{7/3})\\
&\quad + O(\exp(-c/\sqrt{\delta})\Omega^9N^2\varphi(k)^{-9}\mathfrak{S}(b)|a_1\cdots a_9|^{-1/3})\\
&\quad + O(N^2|a_1\cdots a_9|^{-1/3}P^{-27})\\
&\ge cN^2\varphi(k)^{-9}\mathfrak{S}(b)\Omega^9|a_1\cdots a_9|^{-1/3}.
\end{aligned}
$$

Finally, we conclude that (6.26) always holds with Ω having lower bound as in (6.25), and so we have

LEMMA 6.1. *Let $\mathfrak{M}$ be as defined in* (2.4). *Then*

$$\int_{\mathfrak{M}} e(-b\alpha)\prod_{j=1}^{9} S_j(\alpha)\,d\alpha \gg N^2\varphi(k)^{-9}\mathfrak{S}(b)P^{-9/197}|a_1\cdots a_9|^{-1/3}.$$

7. Proof of Theorems 1 and 2

LEMMA 7.1. *Let ℓ and k be integers satisfying $(\ell,k)=1$. For any positive integer λ and any real α with $|\alpha-a/q|\le q^{-2}$ and $(a,q)=1$, define*

$$S_\lambda(\alpha) := \sum_{\substack{n\le N\\ n\equiv \ell\,(\mathrm{mod}\,k)}} \Lambda(n)e(\alpha n^\lambda).$$

Then for any absolute $\varepsilon>0$, we have

$$S_\lambda(\alpha) \ll \frac{N^{1+\varepsilon}}{k^{1-\lambda 2^{1-\lambda}}}\left(\frac{1}{q}+\frac{1}{N^{1/3}}+\frac{q}{N^\lambda}\right)^{2^{2-2\lambda}}. \tag{7.1}$$

For a proof, one can see, e.g., [12, Theorem 4].

LEMMA 7.2. *Let $C(\mathfrak{M})$ be defined as in* (2.4). *Then for any positive ε we have*

$$\int_{C(\mathfrak{M})} \ll N^{2+\varepsilon}P^{-1/16}. \tag{7.2}$$

Proof. Using Lemma 7.1, this lemma can be proved in precisely the same way as [10, Lemma 2.1].

Finally, the combination of (6.26), (7.2) and the definition of $r(b)$ in (2.6) proves Theorems 1 and 2.

References

[1] A. Baker, *On some Diophantine inequalities involving primes*, J. Reine Angew. Math. 228 (1967), 166–181.

[2] S. K. K. Choi and J. Y. Liu, *Small prime solutions of quadratic equations II*, Proc. Amer. Math. Soc. 133 (2005), 945–951.

[3] H. Davenport, *Multiplicative Number Theory*, 2nd ed., Springer, 1980.

[4] L. K. Hua, *Additive Theory of Prime Numbers*, Science Press, Beijing, 1979.

[5] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer, 1982.

[6] M. C. Liu and K. M. Tsang, *Small prime solutions of linear equations*, in: Théorie des Nombres, J.-M. De Koninck and C. Levesque (eds.), de Gruyter, 1989, 595–624.

[7] —, —, *Small prime solutions of some additive equations*, Monatsh. Math. 111 (1991), 147–169.

[8] M. C. Liu and T. Z. Wang, *On the equation $a_1p_1+a_2p_2+a_3p_3=b$ with prime variables in arithmetic progressions*, in: CRM Proc. Lecture Notes 19, Amer. Math. Soc., 1999, 243–264.

[9] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford Univ. Press, 1986.

[10] T. Z. Wang, *Small solutions of quadratic equation with prime variables in arithmetic progressions*, to appear.

[11] Y. Wang, *Numbers representable by five prime squares with primes in an arithmetic progression*, Acta Arith. 90 (1999), 217–244.

[12] Z. F. Zhang, *Some additive problems with primes in arithmetic progressions*, Ph.D. thesis, Chinese Academy of Sciences, 2001.

Department of Applied Mathematics
Tongji University
Shanghai, P.R. China
E-mail: haigangz@sina.com

College of Mathematics and Information Sciences
Henan University
Kaifeng 475001, Henan, P.R. China
E-mail: wangtz@henu.edu.cn

Received on 25.4.2006 (5192)