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Four squares of primes and powers of 2

by

LILU ZHAO (Hefei)

1. Introduction. In 1950s, it was shown by Linnik [9, 10] that every sufficiently large integer can be represented as the sum of two primes and K powers of two, where K is an absolute number. In 1975, Gallagher [2] obtained a stronger result via a different approach. An explicit value of K was first obtained by Liu, Liu and Wang [12], who established that $K = 54000$ is acceptable. This value was subsequently improved by Li [6], Wang [18] and Li [7]. Recently, a rather different method was described by Heath-Brown and Puchta [3], and independently by Pintz and Ruzsa [15]. In particular, it was shown in [3] that $K = 13$ is acceptable, and it was claimed in [15] that $K = 8$ is acceptable.

In 1938, Hua [5] proved that all large integers congruent to 5 modulo 24 can be represented as the sum of five squares of primes. It seems reasonable to conjecture that every large integer congruent to 4 modulo 24 can be expressed as the sum of four squares of primes. This problem is still open, while Brüdern and Fouvry [1] established that every sufficiently large integer $n \equiv 4 \pmod{24}$ is the sum of four squares of almost primes.

In 1999, Liu, Liu and Zhan [13] investigated the expression

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{\nu_1} + \cdots + 2^{\nu_k}, \tag{1.1}$$

and proved that every sufficiently large even integer can be represented as the sum of four squares of primes and k powers of two. It was shown in [11] that $k = 8330$ is acceptable. This value was sharpened to $k = 165$ in [14] and $k = 151$ in [8]. The purpose of this paper is to establish the following result.

THEOREM 1.1. *Every sufficiently large even integer can be represented as a sum of four squares of primes and* 46 *powers of* 2.

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We establish Theorem 1.1 by means of the Hardy–Littlewood method in combination with the linear sieve. In order to bound the contributions of the minor arcs, in previous works [14, 8], an integral of the type $\int_0^1 |T(\alpha)G(\alpha)|^4\,d\alpha$ was used, where $T(\alpha)$ and $G(\alpha)$ are defined in (2.2) below. The above integral is no more than the number of solutions for $p_1^2+p_2^2-p_3^2-p_4^2=h$ with $h=2^{\nu_1}+2^{\nu_2}-2^{\nu_3}-2^{\nu_4}$, where $p_j^2\le N$ and $\nu_i\le L$. The contribution from $h=0$ can be obtained by Rieger's result [16]. Then, as was pointed out in [14], a crucial step is to bound from above the number of solutions of the equation $p_1^2+p_2^2-p_3^2-p_4^2=h$ with nonzero h. The machinery of Brüdern and Fouvry was employed directly to provide such an estimate, while the information on the powers of two was lost in the process.

Our approach is different. Instead of the integral $\int_0^1 |T(\alpha)G(\alpha)|^4\,d\alpha$, we investigate a new integral $\int_0^1 |T(\alpha)^4G(\alpha)^{14}|\,d\alpha$. Now the loss is that we need more variables for the powers of 2 in the mean value integral, while the gain is a situation where we can apply a linear sieve procedure to the equation involving four squares of primes and fourteen powers of two. This approach is motivated by the works of Wooley [19] and of Tolev [17]. In view of [1, 4, 19], it seems hard to solve the equation $p_1^2+p_2^2-p_3^2-x^2=h$ for nonzero h, while Wooley's argument works well to establish the asymptotic formula for the number of solutions of the equation $p_1^2+p_2^2-p_3^2-x^2+\sum_{j=1}^3(2^{u_j}-2^{v_j})=0$ in a suitable box, where x is a natural number. Motivated by Wooley's result, Tolev considered the exceptional set for the equation $p_1^2+p_2^2+p_3^2+x^2=n$ with x an almost prime, and his argument works for the equation $p_1^2+p_2^2-p_3^2-(dx)^2+\sum_{j=1}^t(2^{\nu_j}-2^{\mu_j})=0$ with a suitable t. The linear sieve was employed in place of the four-dimensional vector sieve, hence the quantity is comparable to one fourth of those in [14, 8].

2. Preliminary results. The letter ε denotes an arbitrary small positive constant. The letter N is a large integer and $L=(\log(N/\log N))/\log 2$. To apply the circle method, we set

$$P=N^{1/5-\varepsilon},\qquad Q=L^{-14}N/P.$$

Define

$$\mathcal{M}=\bigcup_{1\le q\le P}\ \bigcup_{\substack{1\le a\le q\\(a,q)=1}}\mathcal{M}(q,a)\quad\text{and}\quad C(\mathcal{M})=\left[\frac{1}{Q},1+\frac{1}{Q}\right]\setminus\mathcal{M},\tag{2.1}$$

where

$$\mathcal{M}(q,a)=\left\{\alpha:\left|\alpha-\frac{a}{q}\right|\le\frac{1}{qQ}\right\}.$$

Denote by $\mathcal{B}$ the interval $[\sqrt{(1/4-\eta)N},\sqrt{(1/4+\eta)N}\,]$, where $\eta\in\left(0,\frac{1}{10^{10}}\right)$ is a constant. Let

$$T(\alpha)=\sum_{p\in\mathcal{B}}(\log p)e(p^2\alpha),\qquad G(\alpha)=\sum_{4\le\nu\le L}e(2^\nu\alpha). \tag{2.2}$$

Then we have

$$\begin{aligned}R_k(N)&:=\sum_{\substack{p_1^2+p_2^2+p_3^2+p_4^2+2^{\nu_1}+\dots+2^{\nu_k}=N\\ p_j\in\mathcal{B}\,(1\le j\le 4),\,4\le\nu_1,\dots,\nu_k\le L}}\prod_{j=1}^{4}\log p_j\\ &=\int_0^1 T^4(\alpha)G^k(\alpha)e(-\alpha N)\,d\alpha\\ &=\int_{\mathcal{M}}T^4(\alpha)G^k(\alpha)e(-\alpha N)\,d\alpha+\int_{C(\mathcal{M})}T^4(\alpha)G^k(\alpha)e(-\alpha N)\,d\alpha.\end{aligned}$$

Let

$$C^*(q,a)=\sum_{\substack{m=1\\(m,q)=1}}^{q}e\left(\frac{am^2}{q}\right),\qquad B(n,q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}C^*(q,a)^4e\left(-\frac{an}{q}\right), \tag{2.3}$$

and

$$A(n,q)=\frac{B(n,q)}{\phi^4(q)},\qquad \mathfrak{S}(n)=\sum_{q=1}^{\infty}A(n,q). \tag{2.4}$$

For $n\equiv 4 \pmod{24}$, we have

$$1\ll\mathfrak{S}(n)\ll(\log\log n)^{11} \tag{2.5}$$

and

$$\mathfrak{S}(n)=24\prod_{p>3}(1+A(n,p)). \tag{2.6}$$

Define

$$\mathfrak{I}(h)=\int_{-\infty}^{\infty}\Big(\int_{\sqrt{1/4-\eta}}^{\sqrt{1/4+\eta}}e(x^2\beta)\,dx\Big)^4e(-h\beta)\,d\beta.$$

For the major arcs, we quote

Lemma 2.1 ([14, Lemma 2.1]). *For $2\le n\le N$, we have*

$$\int_{\mathcal{M}}T^4(\alpha)e(-\alpha n)\,d\alpha=\mathfrak{S}(n)\mathfrak{I}\left(\frac{n}{N}\right)N+O\left(\frac{N}{\log N}\right),$$

where $\mathfrak{S}(n)$ is given by (2.4).

The definition of $T(\alpha)$ in (2.2) is slightly different from that in [14], while the above result can be proved by the same argument.

3. An application of the linear sieve. Let

$$I=\int_0^1 |T(\alpha)^4 G(\alpha)^{14}|\,d\alpha. \tag{3.1}$$

The purpose of this section is to obtain an upper bound for I by using the linear sieve. We first give an auxiliary lemma.

LEMMA 3.1. *Let*

$$J=\sum_{\substack{x_1^2+x_2^2=x_3^2+x_4^2\\ 1\le x_1,x_2,x_3,x_4\le P}} \tau(x_1)\tau(x_2)\tau(x_3)\tau(x_4),$$

where $\tau(n)$ denotes the divisor function. Then

$$J\ll P^2(\log P)^{14}. \tag{3.2}$$

Proof. One has

$$\begin{aligned} J&=\sum_{\substack{x_1^2+x_2^2=x_3^2+x_4^2\\ x_1\ne x_3}} \tau(x_1)\tau(x_2)\tau(x_3)\tau(x_4)+\Big(\sum_{x_1}\tau^2(x_1)\Big)^2\\ &=: J_o+J_d. \end{aligned}$$

The diagonal contribution J_d is bounded by $P^2(\log P)^6$. It suffices to prove $J_o\ll P^2(\log P)^{14}$. We have

$$\begin{aligned} J_o&\le \sum_{\substack{x_1^2+x_2^2=x_3^2+x_4^2\\ x_1\ne x_3}} \tau^2(x_1)\tau^2(x_3)\\ &=2\sum_{x_1<x_3}\tau^2(x_1)\tau^2(x_3)\sum_{\substack{x_2,x_4\\ (x_2-x_4)(x_2+x_4)=x_3^2-x_1^2}} 1\\ &\le 2\sum_{x_1<x_3}\tau^2(x_1)\tau^2(x_3)\tau(x_3^2-x_1^2)\\ &\le 2\Big(\sum_{x_1<x_3}\tau^3(x_1)\tau^3(x_3)\Big)^{2/3}\Big(\sum_{x_1<x_3}\tau^3(x_3^2-x_1^2)\Big)^{1/3}. \end{aligned}$$

Note that $\sum_{1\le x_1<x_3\le P}\tau^3(x_3^2-x_1^2)\le \sum_{1\le a,b\le 2P}\tau^3(a)\tau^3(b)$. The desired result follows from the above easily. ∎

Let

$$g(\beta)=\int_{\sqrt{1/4-\eta}}^{\sqrt{1/4+\eta}} e(x^2\beta)\,dx \quad\text{and}\quad g^+(\beta)=\int_{\sqrt{1/4-\eta-\eta^2}}^{\sqrt{1/4+\eta+\eta^2}} e(x^2\beta)\,dx.$$

Note that

$$g(\beta),g^+(\beta)\ll \min\{1,|\beta|^{-1}\}.$$

We introduce two integrals:

$$\mathfrak{J}^+(h) = \int_{-\infty}^{\infty} g(\beta)^2 g(-\beta) g^+(-\beta) e(-h\beta)\, d\beta,$$

$$\mathfrak{J}(h) = \int_{-\infty}^{\infty} |g(\beta)|^4 e(-h\beta)\, d\beta.$$

Note that $\mathfrak{J}^+(h)$ and $\mathfrak{J}(h)$ are nonnegative constants depending on η. Moreover, $\mathfrak{J}(1) \le \mathfrak{J}(0) \le \mathfrak{J}^+(0) \le (1+O(\eta))\mathfrak{J}(1)$, where the O-constant is absolute. Let

$$\mathbf{S}(h) = \prod_{p>2} \left(1 + \frac{\mathbf{B}(p,h)}{(p-1)^4}\right), \tag{3.3}$$

where

$$\mathbf{B}(p,h) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |C^*(p,a)|^4 e(ah/p).$$

Lemma 3.2. *Let I be defined by* (3.1). *Then*

$$I \le 8(16+\varepsilon)\mathfrak{J}^+(0) N \sum_{h\ne 0} r_7(h)\mathbf{S}(h) + O(NL^{13}),$$

where

$$r_t(h) = \sum_{\substack{4\le \nu_j,\mu_j \le L \\ \sum_{j=1}^t (2^{\nu_j} - 2^{\mu_j}) = h}} 1, \qquad t \in \mathbb{N}.$$

Proof. Note that

$$I = \sum_{h\in\mathbb{Z}} r_7(h) \sum_{\substack{p_j \in \mathcal{B} \\ p_1^2 + p_2^2 - p_3^2 - p_4^2 = h}} \prod_{j=1}^4 \log p_j. \tag{3.4}$$

Let us introduce a smooth function $w : \mathbb{R}^+ \to [0,1]$ which is supported on the interval $[\sqrt{1/4-\eta-\eta^2}, \sqrt{1/4+\eta+\eta^2}\,]$ and satisfies $w(x) = 1$ for all $x \in [\sqrt{1/4-\eta}, \sqrt{1/4+\eta}\,]$. It is clear that

$$I \le I_w \log \sqrt{N}, \tag{3.5}$$

where

$$I_w = \sum_{h\in\mathbb{Z}} r_7(h) \sum_{\substack{p_1,p_2,p_3 \in \mathcal{B} \\ p_1^2 + p_2^2 - p_3^2 - p_4^2 = h}} w(p_4/\sqrt{N}) \prod_{j=1}^3 \log p_j. \tag{3.6}$$

Consider Rosser's weight $\lambda^+(d)$ of order $D = N^{1/16-\varepsilon}$. Let $z = D^{1/2}$ and $\Pi_z = \prod_{2<p<z} p$. Recalling the properties of Rosser's weights, we know

$|\lambda^+(d)| \le 1$, $\sum_{d|(n,\Pi_z)} \mu(d) \le \sum_{d|(n,\Pi_z)} \lambda^+(d)$, and

$$\lambda^+(d) = 0 \quad \text{if } \mu(d) = 0 \text{ or } d > D.$$

We have

$$\begin{aligned}
I_w &\le \sum_{h\in\mathbb{Z}} r_7(h) \sum_{\substack{p_1,p_2,p_3\in\mathcal{B},\,(y,\Pi_z)=1\\ p_1^2+p_2^2-p_3^2-y^2=h}} w(y/\sqrt{N}) \prod_{j=1}^{3} \log p_j \\
&\le \sum_{h\in\mathbb{Z}} r_7(h) \sum_{\substack{p_1,p_2,p_3\in\mathcal{B}\\ p_1^2+p_2^2-p_3^2-y^2=h}} w(y/\sqrt{N}) \Big(\sum_{d|(y,\Pi_z)} \lambda^+(d) \Big) \prod_{j=1}^{3} \log p_j \\
&= \sum_{d|\Pi_z} \lambda^+(d) \sum_{h\in\mathbb{Z}} r_7(h) \sum_{\substack{p_1,p_2,p_3\in\mathcal{B}\\ p_1^2+p_2^2-p_3^2-d^2x^2=h}} w(dx/\sqrt{N}) \prod_{j=1}^{3} \log p_j \\
&:= I_w^+.
\end{aligned}$$

Define

$$\begin{aligned}
f_d(\alpha) &= \sum_x w(dx/\sqrt{N}) e(d^2x^2\alpha), \\
F(\alpha) &= \sum_{d|\Pi_z} \lambda^+(d) f_d(\alpha).
\end{aligned}$$

Now I_w^+ can be represented as

$$I_w^+ = \int_0^1 T^2(\alpha)T(-\alpha)F(-\alpha)|G(\alpha)|^{14}\, d\alpha. \tag{3.7}$$

Let

$$\mathfrak{M} = \bigcup_{1\le q\le N^\eta} \bigcup_{\substack{1\le a\le q\\ (a,q)=1}} \mathfrak{M}(q,a), \tag{3.8}$$

where $\mathfrak{M}(q,a) = \{\alpha : |q\alpha - a| \le N^\eta N^{-1}\}$. Then we define

$$\mathfrak{m} = [N^\eta/N, 1 + N^\eta/N] \setminus \mathfrak{M}. \tag{3.9}$$

So we have

$$\begin{aligned}
I_w^+ &= \sum_{h\ne 0} r_7(h) \int_{\mathfrak{M}} T^2(\alpha)T(-\alpha)F(-\alpha)e(-h\alpha)\, d\alpha \\
&\quad + r_7(0) \int_{\mathfrak{M}} T^2(\alpha)T(-\alpha)F(-\alpha)\, d\alpha \\
&\quad + \int_{\mathfrak{m}} T^2(\alpha)T(-\alpha)F(-\alpha)|G(\alpha)|^{14}\, d\alpha.
\end{aligned} \tag{3.10}$$

We first consider the third integral on the right hand side of (3.10). By Hölder's inequality,

$$\int_{\mathfrak{m}} T^2(\alpha)T(-\alpha)F(-\alpha)|G(\alpha)|^{14}\,d\alpha \le \Big(\int_0^1 |T(\alpha)|^4\,d\alpha\Big)^{3/4}\Big(\int_{\mathfrak{m}} |F(\alpha)^4G(\alpha)^{56}|\,d\alpha\Big)^{1/4}.$$

In light of Rieger's result [16], $\int_0^1 |T(\alpha)|^4\,d\alpha \ll N\log^2 N$. Note that

$$\begin{aligned}\int_{\mathfrak{m}} |F(\alpha)^4G(\alpha)^{56}|\,d\alpha &= \sum_h r_{28}(h)\int_{\mathfrak{m}} |F(\alpha)^4|e(h\alpha)\,d\alpha\\ &= \sum_{h\ne 0} r_{28}(h)\int_{\mathfrak{m}} |F(\alpha)^4|e(h\alpha)\,d\alpha + r_{28}(0)\int_{\mathfrak{m}} |F(\alpha)^4|\,d\alpha.\end{aligned}$$

In view of the work of Heath-Brown and Tolev [4] (see also [17]), for $h\ne 0$ one has

$$\int_{\mathfrak{m}} |F(\alpha)^4|e(h\alpha)\,d\alpha \ll N^{1-\delta},$$

where $\delta>0$ is a small constant depending on η. Considering the underlying Diophantine equation, we have

$$\int_{\mathfrak{m}} |F(\alpha)^4|\,d\alpha \le \int_0^1 |F(\alpha)^4|\,d\alpha \le J,$$

where J is given by Lemma 3.1. Hence $\int_{\mathfrak{m}} |F(\alpha)^4|\,d\alpha \ll NL^{14}$ and

$$\int_{\mathfrak{m}} |F(\alpha)^4G(\alpha)^{56}|\,d\alpha \ll NL^{42}.$$

We conclude from the above that

$$\int_{\mathfrak{m}} T^2(\alpha)T(-\alpha)F(-\alpha)|G(\alpha)|^{14}\,d\alpha \ll NL^{12}.$$

The second integral in (3.10) can be handled similarly (and is actually easier). In particular, we have

$$r_7(0)\int_{\mathfrak{M}} T^2(\alpha)T(-\alpha)F(-\alpha)\,d\alpha \ll NL^{12}.$$

Now we turn to the first integral in (3.10), which is equal to

$$\sum_{d\mid \Pi_z} \lambda^+(d)\int_{\mathfrak{M}} T^2(\alpha)T(-\alpha)f_d(-\alpha)e(-h\alpha)\,d\alpha.$$

Let us introduce

$$\mathcal{S}_d(h) = \sum_{q=1}^{\infty} \frac{\mathcal{A}_d(q,h)}{q\phi^3(q)}, \qquad \mathcal{S}(h) = \mathcal{S}_1(h),$$

where

$$\mathcal{A}_d(q,h) = \sum_{\substack{a=1\\(a,q)=1}}^{q} C^*(q,a)^2 C^*(q,-a) C(q,-ad^2) e(-ah/q)$$

and

$$C(q,a) = \sum_{x=1}^{q} e(ax^2/q).$$

Define $\Omega(d) = \mathcal{S}_d(h)/\mathcal{S}(h)$ provided that $\mathcal{S}(h) \neq 0$, and $\Omega(d) = 1$ otherwise. Let

$$g_w^+(\beta) = \int_{\sqrt{1/4-\eta-\eta^2}}^{\sqrt{1/4+\eta+\eta^2}} w(x) e(x^2\beta)\,dx,$$

$$\mathfrak{J}^+(h) = \int_{-\infty}^{\infty} g(\beta)^2 g(-\beta) g_w^+(-\beta) e(-h\beta)\,d\beta.$$

The standard argument in the Waring–Goldbach problem implies the asymptotic formula

$$\int_{\mathfrak{M}} T^2(\alpha)T(-\alpha)f_d(-\alpha)e(-h\alpha)\,d\alpha = \frac{\Omega(d)}{d}\mathcal{S}(h)\mathfrak{J}_w^+(h/N)N + O(d^{-1}N\log^{-A} N).$$

In view of the properties of Rosser's weights, we have

$$\sum_{d|\Pi_z} \lambda^+(d) \int_{\mathfrak{M}} T^2(\alpha)T(-\alpha)f_d(-\alpha)e(-h\alpha)\,d\alpha \leq (\Phi(2)+\varepsilon) \prod_{2<p<z} (1-\Omega(p)/p)\mathcal{S}(h)\mathfrak{J}_w^+(h/N)N + O(N\log^{-A} N),$$

where $\Phi(s) = 2e^\gamma/s$ for $0 < s \leq 3$, and γ is Euler's constant. Note that $\Omega(2) = 0$ when h is even. Therefore we finally obtain

$$I_w^+ \leq \sum_{h\neq 0} r_7(h)(\Phi(2)+\varepsilon) \prod_{2<p<z} (1-\Omega(p)/p)\mathcal{S}(h)\mathfrak{J}_w^+(h/N)N + O(NL^{12}).$$

For $p > 2$, one has

$$\left(1-\frac{\Omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{\mathcal{A}_1(p,h)}{p(p-1)^3}\right) = 1 + \frac{\mathbf{B}(p,h)}{(p-1)^4}.$$

One also has

$$1 + \sum_{k=1}^{\infty} \frac{\sum_{a(2^k)^*} C^*(2^k,a)^2 C^*(2^k,-a) C(2^k,-a)}{2^k\phi^3(2^k)} = 4.$$

It is well-known that

$$\prod_{2\le p<z}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log z}\left(1+O\left(\frac{1}{\log z}\right)\right).$$

Now we conclude that

$$\begin{aligned}I_w^+ &\le 8(16+\varepsilon)(\log\sqrt{N})^{-1}\sum_{h\ne 0} r_7(h)\mathbf{S}(h)\mathfrak{J}_w^+(h/N)N+O(NL^{12})\\ &\le 8(16+\varepsilon)\mathfrak{J}^+(0)N(\log\sqrt{N})^{-1}\sum_{h\ne 0} r_7(h)\mathbf{S}(h)+O(NL^{12}).\end{aligned}$$

The desired conclusion now follows from (3.5) easily. ∎

LEMMA 3.3. *One has*

$$\int_{C(\mathcal{M})}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha\le 8(15+\varepsilon)(1+O(\eta))\mathfrak{J}(0)N\sum_{h\ne 0} r_7(h)\mathbf{S}(h)+O(NL^{13}).$$

Proof. Recalling (2.1) and (3.9), one has $C(\mathcal{M})\subseteq\mathfrak{m}$ and

$$\int_{C(\mathcal{M})}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha\le\int_{\mathfrak{m}}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha.$$

Note that

$$\int_{\mathfrak{M}}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha=\sum_{h\ne 0} r_7(h)\int_{\mathfrak{M}}|T(\alpha)^4|e(h\alpha)\,d\alpha+O(NL^9).$$

For $h\ne 0$, the standard argument provides

$$\int_{\mathfrak{M}}|T(\alpha)^4|e(h\alpha)\,d\alpha=8\mathbf{S}(h)\mathfrak{J}(h/N)N+O(NL^{-100}).$$

Therefore

$$\int_{\mathfrak{M}}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha=8\sum_{h\ne 0} r_7(h)\mathbf{S}(h)\mathfrak{J}(h/N)N+O(NL^9).$$

Recalling that $h\le NL^{-1}$, one has

$$\int_{\mathfrak{M}}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha=8\mathfrak{J}(0)(1+O(L^{-1}))N\sum_{h\ne 0} r_7(h)\mathbf{S}(h)+O(NL^9).$$

By Lemma 3.2, we obtain

$$\begin{aligned}\int_{\mathfrak{m}}|T(\alpha)^4G(\alpha)^{14}|\,d\alpha &=\int_0^1-\int_{\mathfrak{M}}\\ &\le 8(15+\varepsilon)\sum_{h\ne 0} r_7(h)\mathbf{S}(h)(1+O(\eta))\mathfrak{J}(0)N+O(NL^{13}).\end{aligned}$$

The desired conclusion is established. ∎

4. Numerical computations. Throughout this section, we use h to denote $\sum_{j=1}^{7}(2^{u_j}-2^{v_j})$. For odd q, denote by $\varrho(q)$ the smallest positive integer ϱ such that $2^{\varrho(q)}\equiv 1 \pmod q$.

Define

$$a(p)=\begin{cases}-(p+1)^2 & \text{if } p\equiv 3 \pmod 4,\\ 3p^2-2p-1 & \text{if } p\equiv 1 \pmod 4,\end{cases}$$
$$b(p)=\begin{cases}(p-1)(p+1)^2 & \text{if } p\equiv 3 \pmod 4,\\ (p-1)(p^2+6p+1) & \text{if } p\equiv 1 \pmod 4.\end{cases}$$

Then we define the multiplicative function $c(d)$ by

$$1+\frac{1}{c(p)}=\frac{1+\frac{b(p)}{(p-1)^4}}{1+\frac{a(p)}{(p-1)^4}},$$

where d is square-free and $(30,d)=1$.

Lemma 4.1. *Let* $c_0=\frac{25}{32}c_1+\left(\frac{3}{2}-\frac{25}{32}\right)c_2$, *where*

$$c_1:=\sum_{p\mid d\Rightarrow p>5}\frac{\mu^2(d)}{c(d)\varrho^{14}(3d)}\sum_{\substack{1\le u_j,v_j\le\varrho(3d),\,1\le j\le 7\\ 3d\mid h}}1,$$

$$c_2:=\sum_{p\mid d\Rightarrow p>5}\frac{\mu^2(d)}{c(d)\varrho^{14}(15d)}\sum_{\substack{1\le u_j,v_j\le\varrho(15d),\,1\le j\le 7\\ 15d\mid h}}1.$$

Then $c_0<0.69$.

Proof. The proof follows the lines of [12]. Set

$$\beta(d)=\left(\frac{1}{\varrho^{14}(3d)}\sum_{\substack{1\le u_j,v_j\le\varrho(3d),\,1\le j\le 7\\ 3d\mid h}}1\right)^{-1}.$$

Then

$$c_1=\sum_{p\mid d\Rightarrow p>5}\frac{\mu^2(d)}{c(d)}\int_{\beta(d)}^{\infty}\frac{dx}{x^2}=\int_{2}^{\infty}\sum_{\substack{p\mid d\Rightarrow p>5\\ \beta(d)\le x}}\frac{\mu^2(d)}{c(d)}\,\frac{dx}{x^2}.$$

Clearly $\beta(d)\ge\varrho(3d)$, so

$$\sum_{\substack{p\mid d\Rightarrow p>5\\ \beta(d)\le x}}\frac{\mu^2(d)}{c(d)}\le\sum_{\substack{p\mid d\Rightarrow p>5\\ \varrho(3d)\le x}}\frac{\mu^2(d)}{c(d)}.$$

Let $m(x)=\prod_{e\le x}(2^e-1)$. Then for $x\ge 3$ we have

$$\sum_{\substack{p\mid d\Rightarrow p>5\\ \beta(d)\le x}}\frac{\mu^2(d)}{c(d)}\le\sum_{\substack{p\mid d\Rightarrow p>5\\ 3d\mid m(x)}}\frac{\mu^2(d)}{c(d)}\le\prod_{\substack{p>5\\ p\mid m(x)}}\Big(1+\frac{1}{c(p)}\Big)$$
$$\le\prod_{p>5}\frac{1+\frac{1}{c(p)}}{1+\frac{1}{p-1}}\prod_{\substack{p>5\\ p\mid m(x)}}\Big(1+\frac{1}{p-1}\Big).$$

It was proved in [12] that $m(x)/\phi(m(x))\le e^\gamma\log x$ for $x\ge 9$. If $x\ge 9$, then

$$\sum_{\substack{p\mid d\Rightarrow p>3\\ \beta(d)\le x}}\frac{\mu^2(d)}{c(d)}\le\frac{8c_3}{15}e^\gamma\log x,$$

where $c_3=\prod_{p>5}\frac{1+\frac{1}{c(p)}}{1+\frac{1}{p-1}}\le 1.3904$. Let $M=40$. We have

$$c_1=\int_2^M\sum_{\substack{p\mid d\Rightarrow p>5\\ \beta(d)\le x}}\frac{\mu^2(d)}{c(d)}\frac{dx}{x^2}+\int_M^\infty\sum_{\substack{p\mid d\Rightarrow p>5\\ \beta(d)\le x}}\frac{\mu^2(d)}{c(d)}\frac{dx}{x^2}$$
$$\le\sum_{\substack{p\mid d\Rightarrow p>5\\ \beta(d)<M}}\frac{\mu^2(d)}{c(d)}\int_{\beta(d)}^M\frac{dx}{x^2}+\int_M^\infty\frac{8c_3}{15}e^\gamma\log x\,\frac{dx}{x^2}$$
$$=\sum_{\substack{p\mid d\Rightarrow p>3\\ \beta(d)<M}}\frac{\mu^2(d)}{c(d)}\Big(\frac{1}{\beta(d)}-\frac{1}{M}\Big)+\frac{8c_3}{15}e^\gamma\frac{1+\log M}{M}.$$

The constant c_2 can be handled in a similar way. Then numerical computations provide the desired result. ∎

In the following lemma, the condition (h) in $\sum_{(h)}$ means that the summation is taken over all $(u_1,\dots,u_7,v_1,\dots,v_7)$ satisfying $4\le u_j,v_j\le L$ and $h=\sum_{j=1}^7(2^{u_j}-2^{v_j})\ne 0$.

Lemma 4.2. *Let*

$$\kappa(h)=\begin{cases}\dfrac{25+15\big(\frac{h}{5}\big)}{32} & \text{if } 5\nmid h,\\[2mm] \dfrac{3}{2} & \text{if } 5\mid h.\end{cases}$$

Then

$$\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,3)}}\kappa(h)\prod_{\substack{p>5\\ p\mid h}}\Big(1+\frac{1}{c(p)}\Big)\le\Big(\frac{25}{32}c_1+\Big(\frac{3}{2}-\frac{25}{32}\Big)c_2+\varepsilon\Big)L^{14}. \tag{4.1}$$

Proof. The left hand side of (4.1) is equal to

$$\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,3)\\ 5\nmid h}} \frac{25+15\big(\frac{h}{5}\big)}{32}\prod_{\substack{p>5\\ p|h}}\Big(1+\frac{1}{c(p)}\Big)+\frac{3}{2}\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,15)}}\prod_{\substack{p>5\\ p|h}}\Big(1+\frac{1}{c(p)}\Big)$$

$$= \sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,3)\\ 5\nmid h}} \frac{25}{32}\prod_{\substack{p>5\\ p|h}}\Big(1+\frac{1}{c(p)}\Big)+\frac{3}{2}\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,15)}}\prod_{\substack{p>5\\ p|h}}\Big(1+\frac{1}{c(p)}\Big)+o(L^{14})$$

$$= \frac{25}{32}\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,3)}}\prod_{\substack{p>5\\ p|h}}\Big(1+\frac{1}{c(p)}\Big)+\Big(\frac{3}{2}-\frac{25}{32}\Big)\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,15)}}\prod_{\substack{p>5\\ p|h}}\Big(1+\frac{1}{c(p)}\Big)$$
$$+\,o(L^{14})$$

$$=: \frac{25}{32}\Sigma_1+\Big(\frac{3}{2}-\frac{25}{32}\Big)\Sigma_2+o(L^{14}).$$

Let us consider Σ_1. One has

$$\Sigma_1=\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,3)}}\sum_{\substack{d|h\\ p|d\Rightarrow p>5}}\frac{\mu^2(d)}{c(d)}=\sum_{\substack{(h)\\ h\equiv 0\,(\mathrm{mod}\,3)}}\sum_{\substack{d<N^{\varepsilon}\\ d|h\\ p|d\Rightarrow p>5}}\frac{\mu^2(d)}{c(d)}+O(N^{-\varepsilon})$$

$$\le \sum_{\substack{d<N^{\varepsilon}\\ p|d\Rightarrow p>5}}\frac{\mu^2(d)}{c(d)}\sum_{\substack{1\le u_j,v_j\le L\\ 3d|h}}1+O(N^{-\varepsilon})=:\Sigma_1'+O(N^{-\varepsilon}),$$

where

$$\Sigma_1'\le \sum_{\substack{d<N^{\varepsilon}\\ p|d\Rightarrow p>5\\ \varrho(3d)<L}}\frac{\mu^2(d)}{c(d)}\sum_{\substack{1\le u_j,v_j\le \varrho(3d)\\ 3d|h}}\Big(\frac{L}{\varrho(3d)}+O(1)\Big)^{14}+\sum_{\substack{d<N^{\varepsilon}\\ p|d\Rightarrow p>5\\ \varrho(3d)\ge L}}\frac{\mu^2(d)}{c(d)}L^{13}$$

$$\le L^{14}\sum_{\substack{d<N^{\varepsilon}\\ p|d\Rightarrow p>5}}\frac{\mu^2(d)}{c(d)\varrho(3d)^{14}}\sum_{\substack{1\le u_j,v_j\le \varrho(3d)\\ 3d|h}}1+O(\varepsilon)L^{14}.$$

Therefore $\Sigma_1\le (c_1+\varepsilon)L^{14}$. Similarly, $\Sigma_2\le (c_2+\varepsilon)L^{14}$. Now the desired conclusion is established. ■

Lemma 4.3. *Let* $\mathbf{S}(h)$ *be given by* (3.3). *Then*

$$\sum_{h\ne 0} r_7(h)\mathbf{S}(h)\le 3c_0L^{14},$$

where c_0 *is given by Lemma* 4.1.

Proof. Note that

$$\mathbf{B}(p,h)=\begin{cases} -(p+1)^2 & \text{if } p\equiv 3 \pmod 4 \text{ and } p\nmid h,\\ -(p^2+6p+1)-4p(p+1)\big(\frac{h}{p}\big) & \text{if } p\equiv 1 \pmod 4 \text{ and } p\nmid h,\\ (p-1)(p+1)^2 & \text{if } p\equiv 3 \pmod 4 \text{ and } p\,|\,h,\\ (p-1)(p^2+6p+1) & \text{if } p\equiv 1 \pmod 4 \text{ and } p\,|\,h. \end{cases}$$

Then we have

$$\mathbf{S}(h)\le 3\widetilde{\kappa}(h)\prod_{p>5}\left(1+\frac{a(p)}{(p-1)^4}\right)\prod_{\substack{5<p\\ p|h}}\frac{1+\frac{b(p)}{(p-1)^4}}{1+\frac{a(p)}{(p-1)^4}},$$

where $\widetilde{\kappa}(h)=\kappa(h)$ if $3\,|\,h$ and zero otherwise. One has

$$c_4=\prod_{p>5}\left(1+\frac{a(p)}{(p-1)^4}\right)\le 0.9743.$$

Therefore,

$$\mathbf{S}(h)\le 3c_4\,\widetilde{\kappa}(h)\prod_{\substack{5<p\\ p|h}}\left(1+\frac{1}{c(p)}\right).$$

The conclusion now follows from Lemmas 4.1–4.2. ∎

Let

$$\Xi(N,k)=\{n\ge 2: n=N-2^{\nu_1}-\cdots-2^{\nu_k},\ 4\le \nu_1,\ldots,\nu_k\le L\}$$

for positive integer k.

Lemma 4.4. *For $k\ge 35$ and $N\equiv 4 \pmod 8$, one has*

$$\frac{1}{8}\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4 \,(\mathrm{mod}\,24)}}\mathfrak{S}(n)\ge 0.9NL^k.$$

Proof. As shown in [14], for $p\equiv 1 \pmod 4$,

$$1+A(n,p)\ge 1-\frac{5p^2+10p+1}{(p-1)^4},$$

while for $p\equiv 3 \pmod 4$,

$$1+A(n,p)\ge 1-\frac{5p^2-2p+1}{(p-1)^4}.$$

We have the numerical inequalities

$$\prod_{\substack{17\le p<p_{5000}\\ p\equiv 1\,(\mathrm{mod}\,4)}}\left(1-\frac{5p^2+10p+1}{(p-1)^4}\right)\prod_{\substack{17\le p<p_{5000}\\ p\equiv 3\,(\mathrm{mod}\,4)}}\left(1-\frac{5p^2-2p+1}{(p-1)^4}\right)\ge 0.904923,$$

where p_r denotes the rth prime. Moreover

$$\begin{aligned}\prod_{p\ge p_{5000}}(1+A(n,p)) &\ge \prod_{p\ge p_{5000}}\Big(1-\frac{1}{(p-1)^2}\Big)^6\\ &\ge \prod_{m\ge p_{5000}}\Big(1-\frac{1}{(m-1)^2}\Big)^6=\Big(1-\frac{1}{p_{5000}-1}\Big)^6\\ &> 0.99994271.\end{aligned}$$

Thus

$$\prod_{p\ge 17}(1+A(n,p))\ge C_1:=0.904811.$$

Set $m_0=14$. Now we have

$$\begin{aligned}\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4\,(\mathrm{mod}\,24)}}\mathfrak{S}(n) &\ge 24C_1\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4\,(\mathrm{mod}\,24)}}\prod_{3<p<m_0}(1+A(n,p))\\ &=24C_1\sum_{1\le j\le q}\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4\,(\mathrm{mod}\,24)\\ n\equiv j\,(\mathrm{mod}\,q)}}\prod_{3<p<m_0}(1+A(n,p))\\ &=24C_1\sum_{1\le j\le q}\prod_{3<p<m_0}(1+A(j,p))\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4\,(\mathrm{mod}\,24)\\ n\equiv j\,(\mathrm{mod}\,q)}}1,\end{aligned}$$

where $q=\prod_{3<p<m_0}p$. For the inner sum, we have

$$\mathcal{S}:=\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4\,(\mathrm{mod}\,24)\\ n\equiv j\,(\mathrm{mod}\,q)}}1=\Big(\frac{L}{\varrho(3q)}+O(1)\Big)^k\sum_{\substack{1\le\nu_1,\ldots,\nu_k\le\varrho(3q)\\ 2^{\nu_1}+\cdots+2^{\nu_k}\equiv a_j\,(\mathrm{mod}\,3q)}}1,$$

where a_j is the natural number in $[1,3q]$ satisfying $a_j\equiv 0 \pmod 3$ and $a_j\equiv j \pmod q$. Note that

$$\mathcal{S}=\Big(\frac{L}{\varrho(3q)}+O(1)\Big)^k\frac{1}{3q}\sum_{t=0}^{3q-1}e\Big(\frac{ta_j}{3q}\Big)\Big(\sum_{1\le s\le\varrho(3q)}e\Big(\frac{t2^s}{3q}\Big)\Big)^k.$$

We arrive at

$$\begin{aligned}\mathcal{S}&\ge\Big(\frac{L}{\varrho(3q)}+O(1)\Big)^k\frac{1}{3q}\big(\varrho(3q)^k-(3q-1)(\max)^k\big)\\ &=\frac{L^k}{3q}\Big(1-(3q-1)\Big(\frac{\max}{\varrho(3q)}\Big)^k\Big)+O(L^{k-1}),\end{aligned}$$

where

$$\max = \max\left\{\left|\sum_{1\le s\le \varrho(3q)} e\left(\frac{j2^s}{3q}\right)\right| : 1 \le j \le 3q-1\right\}.$$

Note that $3q = 15015$ and $\varrho(3q) = 60$. With the help of a computer, it is not hard to check that

$$\max = 34.5\ldots < \ 34.6 \quad \text{and} \quad (3q-1)\left(\frac{\max}{\varrho(3q)}\right)^{35} < 10^{-7}.$$

Therefore

$$\mathcal{S} \ge \frac{(1-10^{-7})L^k}{3q} + O(L^{k-1}),$$

and

$$\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4 \,(\mathrm{mod}\, 24)}} \mathfrak{S}(n) \ge 24C_1 \sum_{j=1}^{p} \prod_{3<p<m_0} (1+A(j,p)) \frac{(1-10^{-7})L^k}{3q} + O(L^{k-1})$$

$$= \frac{8C_1(1-10^{-7})L^k}{q} \prod_{3<p<m_0} \left(\sum_{j=1}^{p}(1+A(j,p))\right) + O(L^{k-1}).$$

Observing that

$$\sum_{j=1}^{p}(1+A(j,p)) = p + \frac{1}{(p-1)^4} \sum_{1\le a\le p-1} C^4(p,a) \sum_{j=1}^{p} e(-aj/q) = p,$$

one has

$$\sum_{\substack{n\in\Xi(N,k)\\ n\equiv 4 \,(\mathrm{mod}\, 24)}} \mathfrak{S}(n) \ge 8C_1(1-10^{-7})L^k + O(L^{k-1}).$$

The proof is complete since L is sufficiently large. ∎

5. Proof of Theorem 1.1. As in [14], it suffices to prove that large even integers $N \equiv 4 \pmod 8$ can be represented as the sum of four squares of primes and 44 powers of 2, since for every even integer N, there exist $u_1, u_2 \in \{1,2,3\}$ such that $N - 2^{u_1} - 2^{u_2} \equiv 4 \pmod 8$. We set $k = 44$. Let

$$E(\lambda) = \{\alpha \in (0,1] : |G(\alpha)| \ge \lambda L\}.$$

By [14, Lemma 5.3], we know

$$|E(0.887167)| \ll N^{-3/4-10^{-10}}.$$

Let

$$\mathfrak{m}_1 = C(\mathcal{M}) \cap E(0.887167), \qquad \mathfrak{m}_2 = C(\mathcal{M}) \setminus \mathfrak{m}_1.$$

Following the lines of [14], we have

$$\begin{aligned}\Big|\int_{C(\mathcal{M})} T^4(\alpha)G^k(\alpha)e(-\alpha N)\,d\alpha\Big| \\ &\le \int_{\mathfrak{m}_1} |T^4(\alpha)G^k(\alpha)|\,d\alpha + \int_{\mathfrak{m}_2} |T^4(\alpha)G^k(\alpha)|\,d\alpha \\ &\le O(N^{1-\varepsilon}) + (0.887167L)^{k-14}\int_0^1 |T(\alpha)^4 G(\alpha)^{14}|\,d\alpha.\end{aligned}$$

By Lemmas 3.3 and 4.3,

$$\Big|\int_{C(\mathcal{M})}\Big| \le O(N) + (0.887167L)^{k-14}\ 3c_0 \times 8(15+O(\eta)+\varepsilon)\mathfrak{J}(0)NL^{14}.$$

On the major arcs, we have by Lemma 2.1,

$$\int_{\mathcal{M}} T^4(\alpha)G^k(\alpha)e(-\alpha N)\,d\alpha = \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 8\,(\mathrm{mod}\,24)}} \mathfrak{S}(n)\mathfrak{J}(n/N)N + O(NL^{k-1}).$$

For $n \in \Xi(N,k)$, one has $n/N = 1 + O(L^{-1})$. Applying Lemma 4.4, we get

$$\begin{aligned}\int_{\mathcal{M}} T^4(\alpha)G^k(\alpha)e(-\alpha N)\,d\alpha \\ &= \sum_{\substack{n\in\Xi(N,k)\\ n\equiv 8\,(\mathrm{mod}\,24)}} \mathfrak{S}(n)\mathfrak{J}(1)(1+O(L^{-1}))N + O(NL^{k-1}) \\ &\ge 0.9\times 8\mathfrak{J}(1)NL^k + O(NL^{k-1}).\end{aligned}$$

Therefore we have

$$\begin{aligned}R_{44}(N) &\ge 8NL^{44}\big(0.9\mathfrak{J}(1) - (0.887167)^{30}(45+\varepsilon)c_0(1+O(\eta))\mathfrak{J}(0)\big) \\ &> 0.00001\mathfrak{J}(0)NL^{44}\end{aligned}$$

provided that η is sufficiently small. The proof of Theorem 1.1 is complete.

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Lilu Zhao
School of Mathematics
Hefei University of Technology
Hefei 230009, People's Republic of China
E-mail: zhaolilu@gmail.com

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