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# Transcendence results on the generating functions of the characteristic functions of certain self-generating sets

by

Peter Bundschuh (Köln) and Keijo Väänänen (Oulu)

**1. Introduction and results.** Quite recently, Dilcher and Stolarsky [11, Definition 5.1], introduced the two power series

$$(1.1)\qquad \begin{aligned} F(z) &:= 1+z+z^2+z^5+z^6+z^8+z^9+z^{10}+z^{21}+z^{22}+z^{24}+\cdots, \\ G(z) &:= 1+z+z^3+z^4+z^5+z^{11}+z^{12}+z^{13}+z^{16}+z^{17}+z^{19}+\cdots, \end{aligned}$$

holomorphic on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and proved several functional equations relating them. Here the sequences of integers appearing in the exponents are examples of so-called self-generating sequences. In fact, if $\Phi$ and $\Gamma$ are minimal sets of non-negative integers such that $0 \in \Phi \cap \Gamma$ and

$$\Phi \supset (4\Phi + 1) \cup 2\Gamma, \qquad \Gamma \supset (2\Phi + 1) \cup 4\Gamma,$$

then $\Phi, \Gamma$ are the sets of exponents of $F, G$, respectively. For properties of similarly defined sequences, see [4], for instance.

The main aim of the present paper is to study transcendence and algebraic independence over $\mathbb{C}(z)$ of some functions connected with the above functions $F(z)$ and $G(z)$ but also the analogous questions over $\mathbb{Q}$ if the variable $z$ is specialized to non-zero algebraic points $\alpha$ in $\mathbb{D}$. Already here we want to point out that our key result is Theorem 1.11, to the effect that $F(z)$ and $F(z^4)$ are algebraically independent over $\mathbb{C}(z)$. For this claim, we will present (in Section 5) an elementary algebraic proof which can be adapted to give also the algebraic independence of $G(z)$ and $G(z^4)$ over $\mathbb{C}(z)$. This last result was proved by a completely different method in Adamczewski's paper [1].

Besides the functions $F(z)$ and $G(z)$ in (1.1), we will consider the quotients

$$(1.2)\qquad U(z) := \frac{F(z)}{G(z^2)} \quad \text{and} \quad V(z) := \frac{G(z)}{F(z^2)},$$

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which, again, are holomorphic on $\mathbb{D}$ since $0 \notin F(\mathbb{D}) \cup G(\mathbb{D})$ (see [11, Proposition 6.4]). The two quotients in (1.2) are intimately linked to the continued fractions

$$C_{\mathrm{ev}}(z) := \lim_{\substack{n\to\infty \\ n\,\mathrm{even}}}[z, z^2, z^4, \ldots, z^{2^n}] \quad \text{and} \quad C_{\mathrm{od}}(z) := \lim_{\substack{n\to\infty \\ n\,\mathrm{odd}}}[z, z^2, z^4, \ldots, z^{2^n}]$$

converging in $\mathbb{D}\setminus\{0\}$: namely, in Proposition 6.3 of [11], one finds the relation

$$(1.3) \qquad C_{\mathrm{ev}}(z) = zU(z^3), \quad C_{\mathrm{od}}(z) = z^{-2}V(z^3) \quad \text{in } \mathbb{D}\setminus\{0\}.$$

We will start our investigations by determining, for both of the power series in (1.1), an infinite sequence of increasingly long gaps, which immediately leads to the following function-theoretical result.

Theorem 1.1. *The functions $F(z)$ and $G(z)$ cannot be analytically continued beyond the unit circle.*

Remark 1.2. The pure transcendence of $F(z)$ and $G(z)$ over $\mathbb{C}(z)$ was first established in [9, Theorem 4.1].

Remark 1.3. In the proof of Theorem 1.1, we will see that the sequence of power series coefficients of $F(z)$ (and $G(z)$) contains infinitely many 1's and arbitrarily long strings of 0's. This implies immediately that $F(1/b)$ (and $G(1/b)$) is irrational for any integer $b \geq 2$. In the next theorem these numbers will turn out to be transcendental. Moreover, we claim the following.

Theorem 1.4. *If $a, b \in \mathbb{Z}$ satisfy $b > 0$ and $0 < |a| < b^{1/2}$, then $F(a/b)$ and $G(a/b)$ are transcendental numbers but not U-numbers.*

Remark 1.5. In 1932, Mahler subdivided the set of all real transcendental numbers into three pairwise disjoint classes (called S-, T-, and U-classes) according to their properties of approximation by algebraic numbers. For more details (and an analogous subdivision of the complex transcendental numbers) see, e.g., Chapter III of [7].

Before stating our next result, let us recall the definition of the *irrationality exponent* $\mu(\sigma)$ of a number $\sigma \in \mathbb{R}\setminus\mathbb{Q}$. This is the infimum of the set of all $\mu \in \mathbb{R}$ for which the inequality $|\sigma - p/q| \leq q^{-\mu}$ has at most finitely many solutions $(p, q) \in \mathbb{Z}^2$ with $q > 0$ (with the convention $\mu(\sigma) := +\infty$ if this set is empty).

Theorem 1.6. *Under the hypotheses of Theorem 1.4, $F(a/b)$ and $G(a/b)$ have irrationality exponent bounded above by $(5-2\lambda)/(1-2\lambda)$, where $\lambda := (\log|a|)/\log b$ (which lies in $[0, 1/2[$).*

We next turn to the functions $U(z), V(z)$ defined in (1.2) and discuss first an important function-theoretical property of these.

Theorem 1.7. *The functions $U(z)$ and $V(z)$ cannot be analytically continued beyond the unit circle.*

Already the transcendence of $U(z)$ and $V(z)$ over $\mathbb{C}(z)$, combined with a classical transcendence criterion of Mahler from 1929 (see Lemma M in Section 3), will be enough to demonstrate the following.

THEOREM 1.8. *For any* (1) $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, *the numbers* $U(\alpha)$ *and* $V(\alpha)$ *are transcendental.*

COROLLARY 1.9. *For any* $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, *the continued fractions* $C_{\mathrm{ev}}(\alpha)$ *and* $C_{\mathrm{od}}(\alpha)$ *are transcendental.*

REMARK 1.10. This is Theorem 1.2 in [1], where the author deduced it from his $G$-analogue of our subsequent Theorem 1.12. He quoted Mahler's Lemma M but applied it only to prove the transcendence of the continued fraction

$$C(z) := [z, z^2, z^4, \ldots, z^{2^n}, \ldots]$$

(converging outside the closed unit disk) at all points $\alpha \in \overline{\mathbb{Q}}$ with $|\alpha| > 1$.

As we said earlier, our most basic result reads as follows.

THEOREM 1.11. *The functions* $F(z)$ *and* $F(z^4)$ *are algebraically independent over* $\mathbb{C}(z)$.

The whole Section 5 will be devoted to the proof of this theorem, and, at the end of that section, we will add some hints how to prove Adamczewski's analogue for $G$ along our lines. We point out that it would be possible to give a proof of Theorem 1.11 along the lines of [1], but we choose a different approach which we believe will be useful in the future. A deduction of Theorem 1.11 from its $G$-analogue (or conversely) does not seem to be easy if possible at all. Theorem 1.11 will serve us as the crucial argument in the proof of the following arithmetical statement to be established in Section 4.

THEOREM 1.12. *For any* $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, *the numbers* $F(\alpha)$ *and* $F(\alpha^4)$ *are algebraically independent.*

COROLLARY 1.13. *For any* $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, *the numbers* $F(\alpha)$ *and* $G(\alpha)$ *are transcendental.*

REMARK 1.14. Note that Theorem 1.12 is the $F$-analogue of Adamczewski's Proposition 3.1 from which he deduced the claims of Corollary 1.13 and of Theorem 1.8.

PROBLEM 1.15. (1) It would be interesting to see a direct proof of the transcendence results in Corollary 1.13, i.e., without detour via algebraic independence results as stated, e.g., in Theorem 1.12. Clearly, Theorem 1.4 is a moderate contribution in this direction.

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(1) As usual, $\overline{\mathbb{Q}}$ denotes the field of all complex algebraic numbers.

(2) Is it true that $F(z)$ and $G(z)$ are algebraically independent over $\mathbb{C}(z)$? Same question for $U(z)$ and $V(z)$. One could imagine that the method of proof of Theorem 1.11 presented in Section 5 can also be applied to progress on these questions.

(3) What about algebraic independence over $\mathbb{Q}$ of $F(\alpha), G(\alpha)$ (or $U(\alpha)$, $V(\alpha)$) at points $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$?

**2.** $F(z)$ **and** $G(z)$ **as gap series.** Concerning these two functions, we first need to explain their link to certain other mathematical objects. Namely, a few years ago, Dilcher and Stolarsky [10] introduced and studied the polynomial sequence $(a_n(z))_{n=0,1,\dots}$ defined by $a_0(z) := 0$, $a_1(z) := 1$ and, for $n \in \mathbb{N} := \{1, 2, \dots\}$, by

$$a_{2n}(z) := a_n(z^2) \quad \text{and} \quad a_{2n+1}(z) := z\, a_n(z^2) + a_{n+1}(z^2) \tag{2.1}$$

implying $a_n(0) = 1$ for any $n \in \mathbb{N}$. This polynomial sequence generalizes Stern's sequence $(a_n)_{n=0,1,\dots}$, where $a_n := a_n(1)$ for any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The latter one is related to the classical Fibonacci sequence $(F_n)_{n=0,1,\dots}$ defined by $F_0 := 0$, $F_1 := 1$, $F_n := F_{n-1} + F_{n-2}$ $(n \geq 2)$ through the formula

$$\max\{a_m : 2^{n-2} \leq m \leq 2^{n-1}\} = F_n$$

for any $n \in \mathbb{N}$. It is known that, given $n \geq 2$, this maximum is attained twice, and in fact at the subscripts

$$e_n := \tfrac{1}{3}(2^n - (-1)^n) \quad \text{and} \quad \overline{e}_n := \tfrac{1}{3}(5 \cdot 2^{n-2} + (-1)^n). \tag{2.2}$$

Using these positive integers $e_n, \overline{e}_n$, Dilcher and Stolarsky introduced in Definition 3.1 of [11] two more polynomial sequences by

$$f_n(z) := a_{e_n}(z) \quad (n \geq 0) \quad \text{and} \quad \overline{f}_n(z) := a_{\overline{e}_n}(z) \quad (n \geq 2). \tag{2.3}$$

In Proposition 3.2 of [11], they proved that all these polynomials have coefficients 0 or 1, exactly determined their degrees, and showed that the subsequences $(f_{2k}(z))$ and $(\overline{f}_{2k+1}(z))$ compactly converge on $\mathbb{D}$ to $F(z)$, whereas both of the subsequences $(f_{2k+1}(z))$ and $(\overline{f}_{2k}(z))$ compactly converge on $\mathbb{D}$ to $G(z)$.

Our subsequent auxiliary result directly prepares the proofs of Theorems 1.1, 1.4, and 1.6; more precisely, it is the crucial fact behind formula (2.6) to be used for Theorems 1.4 and 1.6.

Lemma 2.1 (Gap Lemma). *If $(\varphi_n)_{n=0,1,\dots}$ and $(\gamma_n)_{n=0,1,\dots}$ denote the sequences of power series coefficients (at $z = 0$) of $F(z)$ and $G(z)$, respectively, then for any $k \in \mathbb{N}$ one has*

$$\begin{aligned} &\varphi_n = 0 \quad \text{for } e_{2k-1} \leq n < e_{2k} \quad \text{but} \quad \varphi_{e_{2k-1}-1} = 1, \quad \varphi_{e_{2k}} = 1, \\ &\gamma_n = 0 \quad \text{for } e_{2k} < n < e_{2k+1} \quad \text{but} \quad \gamma_{e_{2k}} = 1, \qquad \gamma_{e_{2k+1}} = 1. \end{aligned}$$

*Proof.* According to formula (3.11) of [11], for any $n \in \mathbb{N}$ we have

$$f_{n+1}(z) = f_{n-1}(z) + z^{e_{n-1}} f_n(z).$$

This implies that $f_{2k}(z) + z^{e_{2k}}, f_{2k+2}(z), f_{2k+4}(z), \dots$ agree up to and including the $e_{2k}$th power of $z$, whence the same holds for $f_{2k}(z) + z^{e_{2k}}$ and $F(z)$. By (3.18) of [11], we have $\deg f_{2k}(z) = e_{2k-1} - 1$ and $f_n(0) = 1$ (by $a_{e_n}(0) = 1$, see after (2.1)) for any $n \in \mathbb{N}$, proving the first part of our lemma. Note that we already mentioned after (2.3) that all $f_n(z)$ have coefficients 0 or 1 only. For the second part of the lemma, we argue similarly but with $f_{2k+1}(z) + z^{e_{2k+1}}$ and $G(z)$ using $\deg f_{2k+1}(z) = e_{2k}$. ∎

*Proof of Theorem 1.1.* The length of the gaps in the power series of $F(z)$ exhibited in the first part of the Gap Lemma equals

$$e_{2k} - e_{2k-1} = \tfrac{2}{3}(2^{2(k-1)} - 1).$$

Similarly we obtain

$$e_{2k+1} - e_{2k} - 1 = \tfrac{1}{3}(2^{2k} - 1)$$

for the length of the gaps in $G(z)$. In both cases, the power series have infinitely many coefficients 1 but arbitrarily long strings of coefficients 0. Hence none of the sequences $(\varphi_n), (\gamma_n)$ is (ultimately) periodical. Then Szegő's theorem [17] tells us the following: *If a power series has only finitely many distinct coefficients, then either it represents a rational function* (in which case the coefficient sequence is periodical), *or it cannot be analytically continued beyond the unit circle.* ∎

The proof of Theorem 1.4 will essentially depend on the following common generalization of results of Baker [5] and Ridout [16] and is due to Adamczewski and Bugeaud [2, Théorème 3.1]. We quote only its part that is subsequently needed.

LEMMA 2.2 (Lemma AB). *Let $\sigma \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_+$, and assume $\mathcal{S}$ to be a finite set of distinct prime numbers. Suppose that there exists an infinite sequence $(p_k/q_k)_{k=1,2,\dots}$ of rational numbers in lowest terms with $2 \le q_1 < q_2 < \cdots$ and such that, for any $k \in \mathbb{N}$,*

$$0 < \Big(\prod_{s \in \mathcal{S}} |p_k|_s \cdot |q_k|_s\Big) \cdot \Big|\sigma - \frac{p_k}{q_k}\Big| < q_k^{-2-\varepsilon}, \tag{2.4}$$

*where $|\cdot|_s$ denotes the $s$-adic absolute value normalized by $|s|_s = s^{-1}$. If, moreover,*

$$\limsup_{k\to\infty} \frac{\log q_{k+1}}{\log q_k} < \infty, \tag{2.5}$$

*then $\sigma$ is either an $S$- or a $T$-number.*

*Proof of Theorem 1.4.* Note first that, without loss of generality, we may assume that $a$ and $b$ are coprime. Namely, if $a, b$ satisfy $|a| < b^{1/2}$ and have a common factor $d \in \mathbb{N} \setminus \{1\}$, then the integers $\overline{a} := a/d$, $\overline{b} := b/d$ satisfy $\overline{b} > 0$ and $0 < |\overline{a}| < d^{-1/2}\overline{b}^{1/2} < \overline{b}^{1/2}$.

We now apply Lemma AB to $\sigma := F(a/b)$ leaving the case of $G(a/b)$ to the reader. The equation

$$\sigma - \frac{1}{b^{e_{2k-1}-1}} \sum_{n=0}^{e_{2k-1}-1} \varphi_n a^n b^{e_{2k-1}-1-n} = \sum_{n \geq e_{2k}} \varphi_n \Big(\frac{a}{b}\Big)^n \tag{2.6}$$

suggests to take

$$q_k := b^{e_{2k-1}-1}, \quad p_k := \sum_{n=0}^{e_{2k-1}-1} \varphi_n a^n b^{e_{2k-1}-1-n}, \tag{2.7}$$

which, in particular, implies condition (2.5), by the first equation in (2.2). Thus, we have $p_k \equiv a^{e_{2k-1}-1} \pmod b$ implying the coprimality of $p_k, q_k$. Taking for $\mathcal{S}$ the set of all distinct prime divisors of $b$, condition (2.4) is equivalent to

$$0 < \Big|\sigma - \frac{p_k}{q_k}\Big| < q_k^{-1-\varepsilon}. \tag{2.8}$$

If $a > 0$ we bound below the difference in (2.6) simply by $(a/b)^{e_{2k}}$. If $a < 0$ we estimate as follows:

$$\begin{aligned} \Big|\sum_{n \geq e_{2k}} \varphi_n \Big(\frac{a}{b}\Big)^n\Big| &> \Big(\frac{|a|}{b}\Big)^{e_{2k}} \Big(1 - \frac{|a|}{b} - \Big(\frac{|a|}{b}\Big)^2 - \cdots\Big) \\ &= \frac{b - 2|a|}{b - |a|} \Big(\frac{|a|}{b}\Big)^{e_{2k}} \end{aligned} \tag{2.9}$$

since $|a| \leq b/2$ is implied by $|a| < b^{1/2}$ (this is also true if $b$ is 2 or 3). Note that the above *strong* inequality holds since we have many $n \geq e_{2k}$ with $\varphi_n = 0$. Therefore we have only to guarantee that, given $\varepsilon \in \mathbb{R}_+$, the right-hand side inequality in (2.8) is valid for any large $k \in \mathbb{N}$.

For this purpose, we need to bound above the sum in (2.9):

$$\Big|\sigma - \frac{p_k}{q_k}\Big| < \Big(\frac{|a|}{b}\Big)^{e_{2k}} \Big(1 + \frac{|a|}{b} + \Big(\frac{|a|}{b}\Big)^2 + \cdots\Big) \leq 2\Big(\frac{|a|}{b}\Big)^{e_{2k}}. \tag{2.10}$$

Hence the right-hand inequality in (2.8) is satisfied if $2q_k^{1+\varepsilon} \leq b^{(1-\lambda)e_{2k}}$, where $\lambda \in [0, 1/2[$ is as in Theorem 1.6. By our definition of $q_k$ in (2.7), the last inequality is equivalent to

$$(1 + \varepsilon)(e_{2k-1} - 1) \log b + \log 2 \leq (1 - \lambda) e_{2k} \log b. \tag{2.11}$$

From $e_n = \frac{1}{3} 2^n + \mathrm{O}(1)$ we deduce the following: Assuming $\varepsilon \in ]0, 1 - 2\lambda[$, we can determine $k_0 \in \mathbb{N}$ such that (2.11) holds for any integer $k \geq k_0$. This gives Theorem 1.4. ∎

To prepare the proof of Theorem 1.6, we quote the following approximation lemma [3, Lemma 4.1].

LEMMA 2.3 (Lemma AR). *Let $\sigma, \delta, \rho, \theta \in \mathbb{R}$ satisfy $0 < \delta \leq \rho$ and $\theta \geq 1$. Suppose that there exists a sequence $(p_k/q_k)_{k=1,2,\dots}$ of rational numbers and some constants $c_0, c_1, c_2 \in \mathbb{R}_+$ such that the inequalities*

$$q_k < q_{k+1} \leq c_0 q_k^\theta \quad \text{and} \quad c_1 q_k^{-1-\rho} \leq |\sigma - p_k/q_k| \leq c_2 q_k^{-1-\delta} \tag{2.12}$$

*are satisfied. Then*

$$\mu(\sigma) \leq (1+\rho)\theta/\delta,$$

*and if, moreover, $p_k$ and $q_k$ are coprime for any $k$ large enough, then $\theta \geq \delta$ and*

$$\mu(\sigma) \leq \max(1+\rho, 1+\theta/\delta).$$

*Proof of Theorem 1.6.* As in the proof of Theorem 1.4, it is no loss of generality to assume that $a, b$ are coprime. Namely, otherwise there would be some $d \in \mathbb{N} \setminus \{1\}$ such that $\overline{\lambda} := (\log|\overline{a}|)/\log \overline{b}$ satisfies $0 \leq \overline{\lambda} < \lambda < 1/2$, where $\overline{a}, \overline{b}$ are as at the beginning of the proof of Theorem 1.4. But then the quotient $(5-2\overline{\lambda})/(1-2\overline{\lambda})$ is less than the corresponding one without bars.

Now we closely follow the proof of Theorem 1.4, again restricting ourselves to the case $F(a/b)$ $(=:\sigma)$. First, taking $c_0 := b^2$ and $\theta := 4$ we conclude from (2.7) that $q_{k+1} = c_0 q_k^\theta$ for any $k \in \mathbb{N}$. Thus, the first condition in (2.12) is satisfied. For the one concerning $|\sigma - p_k/q_k|$, we use

$$c_4 \left(\frac{|a|}{b}\right)^{e_{2k}} < \left|\sigma - \frac{p_k}{q_k}\right| < 2\left(\frac{|a|}{b}\right)^{e_{2k}}; \tag{2.13}$$

compare (2.9) and (2.10). According to the remark before (2.9), we may take $c_4 := 1$ if $a > 0$ but $c_4 := (b-2|a|)/(b-|a|)$ if $a < 0$ *and* $|a| < b/2$ such that, so far, $c_4$ is a positive constant depending only on $a, b$. The remaining case $a < 0$ *and* $|a| = b/2$ is, by the coprimality assumption on $a, b$, equivalent to $a = -1$, $b = 2$ and will be deferred to the end.

Next, we transform the expression $(b/|a|)^{e_{2k}}$ into a power of $q_k$ by considering

$$\left(\frac{b}{|a|}\right)^{e_{2k}} = b^{(1-\lambda)(2(e_{2k-1}-1)+1)} = \frac{b}{|a|} q_k^{2(1-\lambda)},$$

taking $e_{2k} = 2(e_{2k-1}-1)+1$ into account. With this we see from (2.13) that the second condition in (2.12) is fulfilled with $\delta := \rho := 1 - 2\lambda \in \,]0,1]$ and $c_1 := c_4|a|/b$, $c_2 := 2|a|/b$. Thus, Lemma AR tells us

$$\mu(\sigma) \leq \max\left(2(1-\lambda), \frac{5-2\lambda}{1-2\lambda}\right),$$

giving the assertion of Theorem 1.6 in the general case.

We finally treat the excluded case $a=-1$, $b=2$ in which bounding (2.6) from below according to (2.9) would lead to the useless condition $c_4=0$. So we consider

$$\Big|\sum_{n\geq e_{2k}}\varphi_n\Big(\frac{-1}{2}\Big)^n\Big| > \Big|\sum_{n=e_{2k}}^{e_{2k+1}-1}\varphi_n\Big(\frac{-1}{2}\Big)^n\Big| - \sum_{n\geq e_{2k+2}} 2^{-n}$$
$$\geq 2^{-(e_{2k+1}-1)} - 2^{1-e_{2k+2}} > 2^{-e_{2k+1}}$$

using the first large gap in the series remainder $\sum_{n\geq e_{2k}}\varphi_n z^n$. We thus have to replace the left-hand side of (2.13) by the slightly weaker bound $2^{-e_{2k+1}}$ which equals $1/(8q_k^4)$ (notice here that $e_{2k+1}-3=4(e_{2k-1}-1)$). To satisfy here the condition (2.12), we may take $c_1:=1/8$ and $\rho:=3$ on the left-hand side, whereas the right-hand side remains unchanged, and our proof is complete. ■

## 3. On the functions $U(z)$ and $V(z)$

*Proof of Theorem 1.7.* The main tool here is a classical theorem of Carlson [8] to the effect that *a power series from $\mathbb{Z}[[z]]$ converging in $\mathbb{D}$ defines a function which is either rational or cannot be analytically continued beyond the unit circle.*

By (1.2), the remarks there, and the fact $F(0)=G(0)=1$, we know that both functions $U(z)$ and $V(z)$ satisfy all hypotheses of Carlson's theorem. Thus, to prove Theorem 1.7, we have only to exclude that $U(z)$ or $V(z)$ is rational. To this end, we quote the formulas (5.3) and (5.4) from [11] as

$$F(z)=G(z^2)+zF(z^4),\qquad G(z)=zF(z^2)+G(z^4), \tag{3.1}$$

leading to

$$U(z)=1+z/V(z^2),\qquad V(z)=z+1/U(z^2),$$

whence, by combining these formulas, we find

$$U(z)=1+\frac{z}{z^2+1/U(z^4)}=\frac{1+(z+z^2)U(z^4)}{1+z^2U(z^4)} \tag{3.2}$$

and a similar expression for $V(z)$ in terms of $z$ and $V(z^4)$ to be given in (3.5) in an equivalent form.

To complete our proof of Theorem 1.7, we assume $U\in\mathbb{C}(z)$ and try to come to a contradiction (the case of $V$ being left to the reader). By our assumption, we have coprime $u,w\in\mathbb{C}[z]\setminus\{0\}$ such that, by (3.2),

$$u(z^4)(z^2u(z)-(z+z^2)w(z))=w(z^4)(w(z)-u(z)), \tag{3.3}$$

which easily leads to $\deg u=\deg w$, but of course, $u\neq w$. Since $u(z^4)$ and $w(z^4)$ are coprime, too, we deduce from (3.3) the divisibility relation $u(z^4)\,|\,(w(z)-u(z))$ (in $\mathbb{C}[z]$), thus $4\deg u\leq\deg u$ and consequently $\deg u=0$

($=\deg w$), whence $U(z)$ is a constant $c \neq 0$, say. Therefore, (3.2) leads to the fact that the polynomial $c(c-1)z^2 - cz + (c-1)$ vanishes identically hence $c = 0$ *and* $c = 1$, a contradiction. ∎

To prepare the proof of Theorem 1.8, we briefly recall the notion of the resultant of two polynomials at a level of generality which will also be sufficient for the purposes of Section 5. If $R$ denotes a unique factorization domain, then, for

$$g = g_0 + \cdots + g_m y^m, \qquad h = h_0 + \cdots + h_n y^n \in R[y]$$

with positive integers $m$ and $n$, the *resultant* of $g, h$ is defined by

$$\operatorname{Res}(g,h) := \det \begin{pmatrix} g_m & g_{m-1} & \cdots & g_0 & & \\ & \ddots & \ddots & & \ddots & \\ & & g_m & g_{m-1} & \cdots & g_0 \\ h_n & h_{n-1} & \cdots & h_0 & & \\ & \ddots & \ddots & & \ddots & \\ & & h_n & h_{n-1} & \cdots & h_0 \end{pmatrix}.$$

Here the square matrix consists of $n$ rows with the successive coefficients of $g$, and then $m$ rows with the coefficients of $h$; all entries outside the two indicated ‘parallelograms’ are 0’s.

With this notation, we next quote the one-dimensional version of a transcendence criterion going back to Mahler [12] (see also Theorem 1.2 in [15]). Let $K$ be an algebraic number field and $O_K$ its ring of integers. Assume that $f(z) \in K[[z]]$ has convergence radius $r > 0$ and satisfies a functional equation

$$f(z^t) = \frac{g_0(z) + \cdots + g_m(z) f(z)^m}{h_0(z) + \cdots + h_m(z) f(z)^m} \tag{3.4}$$

with $t \in \mathbb{N} \setminus \{1\}$, $m \in \{1, \ldots, t-1\}$, $g_\mu, h_\mu \in O_K[z]$ $(\mu = 0, \ldots, m)$, $(g_m, h_m) \neq (0,0)$. If $\Delta(z)$ denotes the resultant of the two polynomials

$$g_0(z) + \cdots + g_m(z) y^m \quad \text{and} \quad h_0(z) + \cdots + h_m(z) y^m$$

with respect to the indeterminate $y$, then the following holds.

LEMMA 3.1 (Lemma M). *Assume that $K, f, \Delta$ are as before, and that $f(z)$ is transcendental over $K(z)$. If $\alpha \in \overline{\mathbb{Q}}^\times$ satisfies $|\alpha| < \min(1, r)$ and $\Delta(\alpha^{t^j}) \neq 0$ for any $j \in \mathbb{N}_0$, then $f(\alpha)$ is transcendental.*

*Proof of Theorem 1.8.* Since (3.2) and the corresponding equation for $V$ can be rewritten as

$$U(z^4) = \frac{1 - U(z)}{z^2 U(z) - z - z^2}, \qquad V(z^4) = \frac{z^3 - z^2 V(z)}{V(z) - 1 - z}, \tag{3.5}$$

we apply Lemma M to $f(z) = U(z)$ (letting aside the case of $V$), which has been recognized as transcendental over $\mathbb{C}(z)$. We may take $K = \mathbb{Q}$, $r = 1$, and, as (3.4) reads here as the first equation in (3.5), $t = 4$, $m = 1$, $g_0(z) = 1$, $g_1(z) = -1$, $h_0(z) = -z - z^2$, $h_1(z) = z^2$, whence the resultant becomes

$$\Delta(z) = \det \begin{pmatrix} -1 & 1 \\ z^2 & -z - z^2 \end{pmatrix} = z,$$

which ends our proof. ∎

*Proof of Corollary 1.9.* Using (1.2) we deduce from Theorem 1.8 that, for any $\beta \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, the numbers

$$\beta U(\beta^3) = \beta \frac{F(\beta^3)}{G(\beta^6)} \quad \text{and} \quad \beta^{-2} V(\beta^3) = \frac{G(\beta^3)}{\beta^2 F(\beta^6)}$$

are transcendental. By (1.3), these numbers equal $C_{\mathrm{ev}}(\beta)$ and $C_{\mathrm{od}}(\beta)$, respectively. ∎

**4. Algebraic independence of $F(\alpha)$ and $F(\alpha^4)$.** The proof of this statement depends on two main ingredients, namely, firstly on the algebraic independence of $F(z)$ and $F(z^4)$ over $\mathbb{C}(z)$ (its proof will be deferred to Section 5), and secondly, on the following algebraic independence criterion of Nishioka [15, Theorem 4.2.1] (see also [14, Corollary 2]).

LEMMA 4.1 (Lemma N). *Let $K$ denote an algebraic number field, and let $t \in \mathbb{N} \setminus \{1\}$. Suppose that $f_1, \dots, f_m \in K[[z]]$ converge in some disc $U \subset \mathbb{D}$ about the origin, where they satisfy the matrix functional equation*

$${}^{\tau}(f_1(z^t), \dots, f_m(z^t)) = \mathcal{A}(z) \cdot {}^{\tau}(f_1(z), \dots, f_m(z))$$

*with $\mathcal{A}(z) \in \mathrm{Mat}_{m,m}(K(z))$, $\tau$ denoting the matrix transpose. If $\alpha \in \overline{\mathbb{Q}}^{\times} \cap U$ is such that none of the $\alpha^{t^j}$ $(j \in \mathbb{N}_0)$ is a pole of the entries of $\mathcal{A}(z)$, then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(f_1(\alpha), \dots, f_m(\alpha)) \geq \operatorname{trdeg}_{K(z)} K(z)(f_1(z), \dots, f_m(z)). \tag{4.1}$$

*Proof of Theorem 1.12.* According to formula (5.5) in [11], $F$ satisfies the functional equation

$$z^4 F(z^{16}) - p(z) F(z^4) + F(z) = 0 \quad \text{with } p(z) := 1 + z + z^2 \tag{4.2}$$

in $\mathbb{D}$. If we set $F(z^4) =: H(z)$, the functions $F, H$ satisfy the system of functional equations

$$F(z^4) = H(z), \quad z^4 H(z^4) = -F(z) + p(z) H(z), \tag{4.3}$$

equivalent to (4.2). Therefore, we may apply Lemma N with $K = \mathbb{Q}$, $m = 2$, $f_1 = F$, $f_2 = H$, $t = 4$, $U = \mathbb{D}$ and

$$\mathcal{A}(z) = \begin{pmatrix} 0 & 1 \\ -z^{-4} & z^{-4} p(z) \end{pmatrix}.$$

Thus, for every $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, we have $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(F(\alpha), F(\alpha^4)) \geq 2$, by Lemma N and Theorem 1.11, ending our proof of Theorem 1.12. Of course, we took here into account the standard fact that, in the transcendence degree on the right-hand side of (4.1), we can replace $K(z)$ by $\mathbb{C}(z)$ since the power series involved are from $K[[z]]$. ∎

*Proof of Corollary 1.13.* We have to show the transcendence of $G(\alpha)$. But this follows by combining the first formula in (3.1) and Theorem 1.12. ∎

**5. Algebraic independence of $F(z)$ and $F(z^4)$.** In the middle of Section 3, we already encountered the notion of resultant of two polynomials. In the proof of Theorem 1.11, we will need, inter alia, some properties of such resultants, collected in the following lemma.

LEMMA 5.1 (Resultant Lemma). *Let $R$ be a unique factorization domain, and assume $g = g_0 + \cdots + g_m y^m$, $h = h_0 + \cdots + h_n y^n \in R[y]$ with $m, n \in \mathbb{N}$. Then* $\operatorname{Res}(g,h)$ *has the following properties:*

(i) *There are $\varphi, \psi \in R[y]$ satisfying* $\operatorname{Res}(g,h) = \varphi g + \psi h$.

(ii) *If $(g_m, h_n) \neq (0,0)$, then*

$$\operatorname{Res}(g,h) = 0 \ \Leftrightarrow\ g, h \textit{ have a common non-constant factor from } R[y].$$

*Proof.* See, e.g., [6, Section 4.4] or [13, Chapter III.6], where the underlying ring $R$ can even be more general. ∎

*Proof of Theorem 1.11.* Assume that the functions $F(z), H(z) := F(z^4)$ satisfying the system (4.3) are algebraically dependent over $\mathbb{C}(z)$. This means that there exists some $P \in \mathbb{C}[z,x,y] \setminus \{0\}$ satisfying

$$P(z, F(z), H(z)) = 0 \tag{5.1}$$

which we may assume to be irreducible in $\mathbb{C}[z,x,y]$ and depending on $x$ *and* on $y$, by the transcendence of $F$ and $H$ (as functions). If $L$ $(\geq 1)$ denotes the total degree in $x, y$, then we may write

$$\begin{aligned} P(z,x,y) &= \sum_{\ell=0}^{L} P_\ell(z,x,y), \\ P_\ell(z,x,y) &= \sum_{k=0}^{\ell} P_{\ell,k}(z) x^{\ell-k} y^k \qquad (0 \leq \ell \leq L), \end{aligned} \tag{5.2}$$

where $P_L(z,x,y) \neq 0$. Let

$$P^*(z,x,y) = z^L P(z,x,y) = \sum_{\ell=0}^{L} z^{L-\ell} P_\ell(z, zx, zy).$$

By (5.1) and (4.3), we have

$$P^*(z^4, F(z^4), H(z^4)) = \sum_{\ell=0}^{L} z^{4(L-\ell)} P_\ell\big(z^4, z^4 H(z), -F(z) + p(z)H(z)\big) = 0$$

in $\mathbb{D}$, giving rise to

$$\begin{aligned} Q(z,x,y) &:= P^*(z^4, y, -z^{-4}x + p(z)z^{-4}y) \\ &= \sum_{\ell=0}^{L} z^{4(L-\ell)} P_\ell(z^4, z^4 y, -x + p(z)y), \end{aligned} \tag{5.3}$$

which in turn implies

$$Q(z, F(z), H(z)) = 0 \tag{5.4}$$

in $\mathbb{D}$. As a consequence of the irreducibility hypothesis on $P$, we conclude from the Resultant Lemma that *$Q(z,x,y)$ is divisible by $P(z,x,y)$* (in $\mathbb{C}[z,x,y]$).

To see this, we apply the lemma with $R = \mathbb{C}[z,x]$. Assume first that $\mathrm{Res}(P,Q) \in R \setminus \{0\} = \mathbb{C}[z,x] \setminus \{0\}$. Then, by Lemma 5.1(i) combined with (5.1) and (5.4), we deduce that $\mathrm{Res}(P,Q)|_{x=F(z)}$ vanishes on $\mathbb{D}$, contrary to the fact that $F(z)$ is not an algebraic function of $z$. Therefore, $\mathrm{Res}(P,Q)$ must be $0 \in R$. Since $\deg_y P \geq 1$ we may apply (ii) of the lemma to infer that $P, Q$ have a common factor from $R[y] = \mathbb{C}[z,x,y]$ of positive degree in $y$. Then the irreducibility of $P$ yields $P \mid Q$ in $R[y]$.

Next we deduce from (5.3) that, if $Q \neq 0$, then its total degree in $x, y$ is not greater than $L$. Hence, by writing this $Q$ analogously to $P$ in (5.2), as

$$\begin{aligned} Q(z,x,y) &= \sum_{\ell=0}^{L} Q_\ell(z,x,y), \\ Q_\ell(z,x,y) &= \sum_{k=0}^{\ell} Q_{\ell,k}(z) x^{\ell-k} y^k \qquad (0 \leq \ell \leq L), \end{aligned} \tag{5.5}$$

we obtain, for $0 \leq k \leq \ell \leq L$, after some minor computation,

$$Q_{\ell,k}(z) = (-1)^{\ell-k} z^{4(L-\ell)} \sum_{j=0}^{k} \binom{\ell-k+j}{\ell-k} P_{\ell,\ell-k+j}(z^4) z^{4(k-j)} p(z)^j. \tag{5.6}$$

Since $P_L(z,x,y) \neq 0$, not all $P_{L,0}(z), \ldots, P_{L,L}(z)$ vanish identically, and we let $J, K \in \{0, \ldots, L\}$ denote the greatest, resp. smallest, subscript satisfying $P_{L,J}(z) \cdot P_{L,K}(z) \neq 0$. Soon we will see $J = L - K$.

Namely, on taking (5.6) for $\ell = L$ and $k = 0, \ldots, L-J$, we find that

$$Q_{L,0}(z) = \cdots = Q_{L,L-J-1}(z) = 0, \qquad Q_{L,L-J}(z) \neq 0, \tag{5.7}$$

whence, by (5.5), $Q_L(z,x,y) \neq 0$, hence $Q(z,x,y) \neq 0$ and the total degree of $Q$ equals $L$.

As already stated above, we know that $P \mid Q$ (in $\mathbb{C}[z,x,y]$), so there exists some $S \in \mathbb{C}[z,x,y] \setminus \{0\}$ with $Q = S \cdot P$, implying that the total degree of $S$ in $x, y$ is zero. We thus have $S \in \mathbb{C}[z] \setminus \{0\}$ and

$$Q_{\ell,k}(z) = S(z) P_{\ell,k}(z) \qquad (0 \le k \le \ell \le L), \tag{5.8}$$

implying, by (5.7),

$$P_{L,0}(z) = \cdots = P_{L,L-J-1}(z) = 0, \qquad P_{L,L-J}(z) \neq 0,$$

thus $L - J = K$.

To obtain more information on $S(z)$, we combine (5.6) and (5.8) to get

$$S(z)P_{\ell,k}(z) = (-1)^{\ell-k} z^{4(L-\ell)} \sum_{j=0}^{k} \binom{\ell-k+j}{\ell-k} P_{\ell,\ell-k+j}(z^4) z^{4(k-j)} p(z)^j \tag{5.9}$$

for $0 \le k \le \ell \le L$. On applying this, for fixed $\ell \in \{0, \dots, L\}$, successively for $k = 0, \dots, \ell$, we obtain

$$\begin{aligned} S(z)P_{\ell,0}(z) &= (-1)^{\ell} z^{4(L-\ell)} P_{\ell,\ell}(z^4), \\ S(z)P_{\ell,1}(z) &= (-1)^{\ell-1} z^{4(L-\ell)} \left( P_{\ell,\ell-1}(z^4) z^4 + \binom{\ell}{\ell-1} P_{\ell,\ell}(z^4) p(z) \right), \\ &\ \ \vdots \end{aligned}$$

Here the first equation implies $S(z) \mid z^{4(L-\ell)} P_{\ell,\ell}(z^4)$, whereas the second one gives $S(z) \mid z^{4(L-\ell+1)} P_{\ell,\ell-1}(z^4)$. Inductively one shows $S(z) \mid z^{4(L-k)} P_{\ell,k}(z^4)$ for any $k \in \{0, \dots, \ell\}$ and therefore

$$S(z) \mid z^{4L} P_{\ell,k}(z^4) \qquad (0 \le k \le \ell \le L). \tag{5.10}$$

Since the $P_{\ell,k}(z)$ are coprime, by the irreducibility of $P$, the same holds for the $P_{\ell,k}(z^4)$, and consequently

$$S(z) \mid z^{4L}. \tag{5.11}$$

Thus the degree of $S(z)$ is bounded above by $4L$.

The crucial point in our proof of Theorem 1.11 is the following detailed consideration of the degrees of the polynomials entering (5.9) in case $\ell = L$, which we rewrite as

(5.12)
$$(-1)^{L-k} S(z) P_k(z) = \sum_{j=0}^{k} \binom{L-k+j}{L-k} P_{L-k+j}(z^4) z^{4(k-j)} p(z)^j \quad (0 \le k \le L),$$

where $P_k(z)$ denotes $P_{L,k}(z)$ from (5.2). We write the degrees of $S(z)$ and $P_k(z)$ as $d$ and $d_k$, respectively, where we follow the convention to assign degree $-\infty$ to the zero polynomial. In particular, $d_0 = \cdots = d_{K-1} = d_{L-K+1} = \cdots = d_L = -\infty$, but $d_K, d_{L-K} \ge 0$.

The equation (5.12) gives, for $k = K$,

$$d + d_K = 4d_{L-K} + 4K. \tag{5.13}$$

Further, for fixed $k \ge K$, the degrees, possibly $\ne -\infty$, of the summands on the right-hand side of (5.12) are

$$4d_{L-k} + 4k, \dots, 4d_{L-k+j} + 4k - 2j, \dots, 4d_{L-K} + 2k + 2K. \tag{5.14}$$

If $T \in \{L-k, \dots, L-K\}$ satisfies $P_T(z) \ne 0$, then the degree of the polynomial $P_T(z^4)z^{4(L-T)}p(z)^{T+k-L}$ equals

$$4d_T + 2(L + k - T). \tag{5.15}$$

For $k = L$ this is $4d_T + 4L - 2T$; assume that $L - K \ge T_1 > \dots > T_v \ge K$ are exactly those values of $T$ where $4d_T + 4L - 2T$ attains its maximum. Note that we must have $v \le [L/2] + 1 - K$, since $2T \equiv 0$ or $\equiv 2 \pmod 4$ according as $T$ is even or odd.

Next we prove that *the degree*

$$d + d_k \tag{5.16}$$

*of the left-hand side of* (5.12) *attains its maximal value*

$$4d_{T_1} + 2(L + k - T_1) \tag{5.17}$$

(compare (5.15)) *for some* $k \ge L - v + 1$ (implying $P_k(z) \ne 0$). Namely, we consider the right-hand side of (5.12) for $k = L - u$, $u = 0, \dots, v-1$ ($\le [L/2] - K$), which is of the form

$$\sum_{j=0}^{L-u} \binom{u+j}{u} P_{u+j}(z^4) z^{4(L-u-j)} p(z)^j, \tag{5.18}$$

and, for fixed $u$ as above, we must precisely determine those $j \in \{0, \dots, L\text{–}u\}$ where $u + j \in \{u, \dots, L\}$ is one of the distinct $T_v, \dots, T_1$. By $u \le v - 1 \le K + v - 1 \le T_1$, we know that at least the largest $T$ appears. Then we may rewrite (5.18) in the form

$$\sum_{i=1}^{v} \sum_{\substack{j=0 \\ u+j=T_i}}^{L-u} \binom{T_i}{u} P_{T_i}(z^4) z^{4(L-T_i)} p(z)^j + \sum_{\substack{j=0 \\ u+j \text{ is not a } T_i}}^{L-u} \dots \qquad (u = 0, \dots, v-1). \tag{5.19}$$

Clearly, the degree of each term $P_{T_i}(z^4)z^{4(L-T_i)}p(z)^j$ in the double sum equals

$$\begin{aligned} 4d_{T_i} + 4(L - T_i) + 2(T_i - u) &= 4d_{T_i} + 4L - 2T_i + 2(k - L) \\ &= 4d_{T_1} + 4L - 2T_1 + 2(k - L) \end{aligned}$$

for all $i \in \{1, \ldots, v\}$. If $c_i$ $(\neq 0)$ denotes the leading coefficient of $P_{T_i}(z)$, then the 'formal' leading coefficient of (5.19) equals

$$\sum_{i=1}^{v} \binom{T_i}{u} c_i \quad (u = 0, \ldots, v-1). \tag{5.20}$$

Since

$$\det\left(\binom{T_i}{u}\right)_{i=1,\ldots,v;\, u=0,\ldots,v-1}$$

is non-zero (in fact, it equals $(\prod_{u=0}^{v-1} u!)^{-1} \cdot \prod_{i<j}(T_j - T_i)$), not all sums (5.20) can vanish, leading to our last intermediate result.

Finally, we show that this result provides a contradiction. As $P_{L-K+1}(z) = \cdots = P_L(z) = 0$, the sum (5.16) attains its maximal value (5.17) in some row (5.14) with $L - v + 1 \le k \le L - K$ and $P_k(z) \neq 0$. We take this special $k$ and consider the term with $j = k - K$ in the sum in (5.12) to obtain, by (5.13),

$$d + d_k \ge 4d_{L-K} + 4K + 2(k-K) = d + d_K + 2(k-K).$$

Thus

$$d_k \ge d_K + 2(k-K) \ge 2(L - v + 1 - K) \ge 2(L - ([L/2] + 1 - K) + 1 - K) \ge L,$$

where we used the inequalities $d_K \ge 0$, $v \le [L/2] + 1 - K$. Furthermore, the choice $j = 2k - L \in \{0, \ldots, k\}$ in (5.12) gives

$$d + d_k \ge 4d_k + 4(L-k) + 2(2k-L) = 4d_k + 2L \ \Leftrightarrow\ d \ge 3d_k + 2L,$$

whence $d \ge 3L + 2L = 5L$, contradicting $d \le 4L$; see after (5.11). ∎

*Sketch of proof of the G-analogue of Theorem 1.11* along lines similar to the preceding case of $F$. We assume that $G(z)$ and $H(z) := G(z^4)$ are algebraically dependent over $\mathbb{C}(z)$, and note that they satisfy the system

$$G(z^4) = H(z), \qquad H(z^4) = -zG(z) + p(z)H(z), \tag{5.21}$$

which is equivalent to the single equation $G(z^{16}) - p(z)G(z^4) + zG(z) = 0$ (compare [11, (5.6)]). Then one has an irreducible $P \neq 0$ such that (5.1) has to be replaced by $P(z, G(z), H(z)) = 0$. Making here the substitution $z \mapsto z^4$, we find $P(z^4, H(z), -zG(z) + p(z)H(z)) = 0$, giving rise to

$$Q(z, x, y) := P(z^4, y, -zx + p(z)y),$$

which in turn implies $Q(z, G(z), H(z)) = 0$ as the analogue of (5.4). Again we find $P \mid Q$.

On writing the new $Q(z, x, y)$ as in (5.5), the analogue of the important formula (5.6) for the $Q_{\ell,k}(z)$ now becomes considerably simpler, and this extends also to the analogue of (5.9), leading to $S(z) \mid z^L$ instead of (5.11). The case of $G$ is technically simpler than that of $F$ because, in the terminology of linear homogeneous differential equation systems of order two, the system

(4.3) for $F$ has an irregular singular point at the origin, whereas (5.21) has an ordinary point there. ■

## References


[1] B. Adamczewski, *Non-converging continued fractions related to the Stern diatomic sequence*, Acta Arith. 142 (2010), 67–78.

[2] B. Adamczewski et Y. Bugeaud, *Mesures de transcendance et aspects quantitatifs de la méthode de Thue–Siegel–Roth–Schmidt*, Proc. London Math. Soc. (3) 101 (2010), 1–26.

[3] B. Adamczewski and T. Rivoal, *Irrationality measures for some automatic real numbers*, Math. Proc. Cambridge Philos. Soc. 147 (2009), 659–678.

[4] J.-P. Allouche, J. Shallit, and G. Skordev, *Self-generating sets, integers with missing blocks, and substitutions*, Discrete Math. 292 (2005), 1–15.

[5] A. Baker, *On Mahler's classification of transcendental numbers*, Acta Math. 111 (1964), 97–120.

[6] S. Bosch, *Algebra*, Springer, Berlin, 1993.

[7] Y. Bugeaud, *Approximation by Algebraic Numbers*, Cambridge Tracts in Math. 160, Cambridge Univ. Press, Cambridge, 2004.

[8] F. Carlson, *Über Potenzreihen mit ganzzahligen Koeffizienten*, Math. Z. 9 (1921), 1–13.

[9] M. Coons, *The transcendence of series related to Stern's diatomic sequence*, Int. J. Number Theory 6 (2010), 211–217.

[10] K. Dilcher and K. B. Stolarsky, *A polynomial analogue to the Stern sequence*, Int. J. Number Theory 3 (2007), 85–103.

[11] K. Dilcher and K. B. Stolarsky, *Stern polynomials and double-limit continued fractions*, Acta Arith. 140 (2009), 119–134.

[12] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366; Berichtigung, ibid. 103 (1930), 532.

[13] M. Mignotte, *Mathématiques pour le calcul formel*, Presses Universitaires de France, Paris, 1989.

[14] K. Nishioka, *New approach in Mahler's method*, J. Reine Angew. Math. 407 (1990), 202–219.

[15] K. Nishioka, *Mahler Functions and Transcendence*, Lecture Notes in Math. 1631, Springer, Berlin, 1996.

[16] D. Ridout, *Rational approximations to algebraic numbers*, Mathematika 4 (1957), 125–131.

[17] G. Szegő, *Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten*, Sitzungsber. Preuß. Akad. Wiss. Math.-Phys. Kl. 1922, 88–91.



Peter Bundschuh
Mathematisches Institut
Universität zu Köln
Weyertal 86-90
50931 Köln, Germany
E-mail: pb@math.uni-koeln.de

Keijo Väänänen
Department of Mathematical Sciences
University of Oulu
P.O. Box 3000
90014 Oulu, Finland
E-mail: keijo.vaananen@oulu.fi



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