

Substitutions and 1/2-discrepancy of $\{n\theta + x\}$

by

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1. Introduction. Given an irrational θ and some $x \in [0, 1) = S^1$ (all addition in S^1 is taken modulo one), let

$$(1.1) \quad f(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x).$$

With θ fixed, the 1/2-discrepancy sums of the sequence $\{x+i\theta\}$ are given by

$$S_n(x) = \sum_{i=0}^{n-1} f(x + i\theta).$$

Two results are classical in this setting, for any irrational θ and for all x :

$$(1.2) \quad S_n(x) \in o(n), \quad S_n(x) \notin O(1).$$

The first restriction is due to unique ergodicity of the underlying rotation, and the second is a theorem of Kesten [10].

We will use standard continued fraction notation; partial quotients are denoted $a_i(\theta)$, and convergents are denoted $p_i(\theta)/q_i(\theta)$. When θ is clear from context we will simply write a_i , p_i and q_i . The distance from x to the nearest integer is denoted $\|x\|$. As $\theta \in (0, 1)$ without loss of generality, we will assume that $a_0(\theta) = 0$ and omit this term, writing simply

$$\theta = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

All necessary background in continued fractions may be found in [11]. The *Gauss map* will be denoted by γ ; it acts as the noninvertible shift on the sequence of partial quotients:

$$(1.3) \quad \gamma(\theta) = \frac{1}{\theta} \bmod 1, \quad \gamma([a_1, a_2, \dots]) = [a_2, a_3, \dots].$$

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Our goal is to investigate what behavior is possible for the sequence $S_n(x)$ within the constraints of (1.2). Because S_n is not monotone, however, it will be more convenient to consider the following sequences, which track the *maximal* and *minimal* discrepancies, as well as the *range* of values taken:

$$(1.4) \quad M_n(x) = \max\{S_i(x) : i = 1, \dots, n-1\},$$

$$(1.5) \quad m_n(x) = \min\{S_i(x) : i = 1, \dots, n-1\},$$

$$(1.6) \quad \rho_n(x) = M_n(x) - m_n(x) + 1.$$

It is worth clarifying that m_n is taken as a minimum over *integers*, and as such can generally be expected to be negative. It is a matter of later convenience that $i = 0$ is not considered: for example, $M_1(0) = m_1(0) = S_1(0) = 1$.

We will develop a renormalization procedure through which the sequence of values $f(x + i\theta)$ can be determined from a sequence of substitutions. Let $\theta < 1/2$ and $A = [0, 1/2)$, $B = [1/2, 1 - \theta)$, $C = [1 - \theta, 1)$. If we wish to change which interval certain endpoints belong to (for example, if we wish for A to be closed and B to be open), we will say that we make a *change of endpoints* of the intervals A , B , and C . Our central result is the following:

THEOREM 1.1. *Given any irrational θ and any $x \in [0, 1)$, there is a sequence of words ω_i (some of which may be empty) and substitutions σ_i (infinitely many are not identity), both defined on the alphabet $\{A, B, C\}$, given by a dynamic process depending on x and θ , such that the infinite word given by*

$$(1.7) \quad \omega_0\sigma_0(\omega_1\sigma_1(\omega_2\sigma_2(\dots)))$$

encodes the orbit of x up to at most two errors. Alternately, the coding is exact up to a change of endpoints of the intervals A , B , and C . The dependence of σ_i on θ and of ω_i on (x, θ) is explicit.

There is one special point $x(\theta)$ for which all ω_i may be taken to be the empty word, in which case the infinite word

$$(1.8) \quad \lim_{n \rightarrow \infty} (\sigma_0 \circ \sigma_1 \circ \dots \circ \sigma_{n-1})(\omega)$$

will encode the orbit of $x(\theta)$ regardless of the choice of nonempty word ω . The orbit of zero can alternately be determined by

$$(1.9) \quad \lim_{n \rightarrow \infty} (\sigma'_0 \circ \sigma'_1 \circ \dots \circ \sigma'_{n-1})(\omega'_{n-1}),$$

where σ'_n are either substitutions or a different map. This distinction and the word ω'_n are explicitly presented.

We will include some remarks regarding the point $x(\theta)$ (including a complete characterization in Proposition 4.3 of those θ for which $x(\theta) = 0$), as

well as proving that the sequence of substitutions σ_i is eventually periodic if and only if θ is a quadratic surd (Proposition 4.4).

As $(0, 1/2) \subset A$ and $(1/2, 1) \subset (B \cup C)$, any change of endpoints is completely irrelevant to the asymptotic growth rates of $M_n(x)$, $m_n(x)$, and $\rho_n(x)$. While Theorem 1.1 provides a way to produce the orbit of an arbitrary point, computation of the words ω_i is a nontrivial task. However, for the special point $x(\theta)$ and for 0, the process is much simpler. We will show that given any growth condition that does not violate (1.2), such behavior is seen to be possible:

THEOREM 1.2. *Suppose that $\{c_n\}$ and $\{d_n\}$ are two increasing sequences of positive real numbers, both in $o(n)$, the differences*

$$\Delta c_n = c_{n+1} - c_n$$

are in $O(1)$ (similarly for $\{\Delta d_n\}$), and at least one of $\{c_n\}$, $\{d_n\}$ is divergent. Then there is a dense set of θ such that if $\{c_n\}$ is divergent, then

$$\limsup_{n \rightarrow \infty} \frac{M_n(0)}{c_n} = 1,$$

while if $\{c_n\}$ is bounded then so is $M_n(0)$. Similarly, if $\{d_n\}$ is divergent, then

$$\limsup_{n \rightarrow \infty} \frac{|m_n(0)|}{d_n} = 1,$$

while if $\{d_n\}$ is bounded then so is $m_n(0)$.

A closely related result concerns the sequence of values $M_n(x)/|m_n(x)|$:

THEOREM 1.3. *Let $0 \leq r_1 \leq r_2 \leq \infty$. Then there is a dense set of θ such that the set of accumulation points of the sequence*

$$\left\{ \frac{M_n(0)}{|m_n(0)|} : n = 0, 1, 2, \dots \right\}$$

is the interval $[r_1, r_2]$.

We will also include a partial rederivation of [4, Theorem 1] in Corollary 5.3: a characterization of those θ for which $S_n(\theta) \geq 0$ for all $n \geq 0$.

A classical application of the Denjoy–Koksma inequality is that if the $a_i(\theta)$ are drawn from a finite set (such θ are said to be *badly approximable* or *of finite type*), then $S_n(x) \in O(\log n)$.

THEOREM 1.4. *If θ is of finite type, then for all x we have $\rho_n(x) \sim \log n$, meaning that the ratio is bounded away from both zero and infinity.*

COROLLARY 1.5. *If θ is of finite type, then $|S_n(x)| \notin o(\log n)$ for every x , and*

$$m_n(x) \in o(\log n) \Rightarrow M_n(x) \sim \log n,$$

and vice versa.

If $A \cup B$ represents a single interval, then as S^1 has been partitioned into two intervals of length θ and $1 - \theta$, the analogous problem would be to encode the *Sturmian sequences*, and generating Sturmian sequences using a sequence of substitutions is intimately related to continued fraction expansions for numbers: see for example [8, Chapter 6]. The study of substitutions as they relate to discrepancy sequences of different intervals has been initiated before: in particular, our approach is similar to that developed in [1, 2] which generates a sequence of substitutions to encode the orbit of the origin for an arbitrary interval $[0, \beta)$. Herein the principal differences are:

- the interval $[0, 1/2]$ is fixed, but the technique will allow us to study arbitrary x (though $x = 0$ is one particularly nice case that we investigate);
- the renormalization procedure works directly through the continued fraction expansion of θ ;
- the techniques are constructive, providing a direct procedure for generating the θ for which M_n and m_n have desired growth rates.

The use of continued fraction coefficients to directly extract information also appears in [12, 13], including a study of one-sided boundedness of sums, the original motivation (see [4]) for the development of the techniques herein. Classical studies of discrepancy sums (using a supremum over all intervals of sequences of similar ergodic sums) have a long history: the reader should refer to the works of Dupain, Sós, and Schoissengeier (and their coauthors). A nonexhaustive list would begin with [6, 15, 16]. The present work approaches the problem from an ergodic-theoretic perspective and therefore fixes the interval in question; the term *local discrepancy* is used to describe this problem in [14], where the case of an interval of length one-half appears as a particular example.

2. Symbol spaces, encodings, and substitutions. All background material pertaining to common definitions in symbolic dynamics and substitution systems may be found in [8, Chapter 1]; we present here only a short summary of specific notation used herein. Let $\mathcal{A} = \{A, B, C\}$, and denote by \mathcal{A}^* the *free monoid* on \mathcal{A} . Given $\omega \in \mathcal{A}^*$, we denote

$$\omega = (\omega)_0(\omega)_1 \dots (\omega)_{n-1},$$

and say that ω is a *word of length n* with *letters* $(\omega)_i$ drawn from the *alphabet* \mathcal{A} . Note that ω_i will refer to a sequence of words indexed by i , while $(\omega)_i$ will denote the individual letters of a fixed word ω . This similarity is a potential source of confusion, but the latter notation is much more common in this work: we will rarely refer to specific letters in a given word.

Denote by $|\omega|$ the length of ω . Elements in \mathcal{A}^* multiply by concatenation, and we adopt power notation for this operation: $(AB)^3 = ABABAB$, for example. The empty word (the identity under concatenation) is denoted \emptyset . A *factor* of ω (of finite or infinite length) is some finite word ψ of length n such that there is some i for which

$$(\psi)_j = (\omega)_{i+j}, \quad j = 0, 1, \dots, n-1.$$

If $i = 0$ then we say ψ is a *left factor* of ω , and we say ψ is a *right factor* of ω if $(\psi)_{n-1} = (\omega)_{|\omega|-1}$. The factor ψ will be called *proper* if $\psi \notin \{\omega, \emptyset\}$.

Any map $\sigma : \mathcal{A} \rightarrow \mathcal{A}^*$ may be extended to a map on \mathcal{A}^* by requiring it to be a homomorphism. The following is nonstandard but natural. Endow $\mathcal{A}^{\mathbb{N}}$ with the cylinder topology, and let a finite word $\omega \in \mathcal{A}^*$ represent a clopen set: the set of all elements of $\mathcal{A}^{\mathbb{N}}$ with left factor ω . We may then further extend σ to a map on $\mathcal{A}^{\mathbb{N}}$ by defining

$$\sigma(\omega) = \bigcap_{i=0}^{\infty} \sigma((\omega)_0(\omega)_1 \dots (\omega)_{i-1}).$$

In all of these situations we refer to σ as a *substitution*.

Given a sequence of words $\omega_0, \omega_1, \dots$ such that ω_i is a left factor of ω_{i+1} , if

$$\bigcap_{i=0}^{\infty} \omega_i = \{x\},$$

then we say that $x \in \mathcal{A}^{\mathbb{N}}$ is the *limit* of the words ω_i .

Now consider the space $S^1 = [0, 1)$ with the map $R_\theta(x) = x + \theta \pmod{1}$ for some irrational θ . Suppose that X is partitioned into three intervals A , B , and C . Then given a word ω , we say that ω encodes the orbit of x if for all $i \leq |\omega| - 1$ we have

$$(\omega)_i = A \Leftrightarrow x + i\theta \in A,$$

and similarly for B and C . Given a partition, then, to each $x \in S^1$ we may assign an infinite word $\omega \in \Omega$, the one which encodes the (forward) orbit of x .

Let \mathcal{D} be the discontinuities of $(f \circ R_\theta^i)(x)$ for $i = 0, 1, 2, \dots$:

$$\mathcal{D} = \{-i\theta, -i\theta + 1/2\}, \quad i = 0, 1, 2, \dots$$

For each $x \in \mathcal{D}$, then, we replace $x \in S^1$ with two points, a right and left limit, denoted x^+ and x^- . We set

$$R_\theta(0^+) = R_\theta(1^-) = \theta,$$

and similarly for $(1/2)^\pm$; while this makes the rotation two-to-one at these points, note that with respect to the alphabet \mathcal{A} , the symbolic codings for the forward orbit of θ^+ and θ^- are identical, so we do not distinguish them.

We still denote our space by S^1 . We may now make each of A , B and C closed, although we have made S^1 totally disconnected.

Given an irrational θ , partition $S^1 = [0^+, 1^-]$ according to Table 1 and in a slight abuse of notation let S^1 be the set of all words which encode orbits with respect to these conventions.

Table 1. The partition $S^1 = A \cup B \cup C$ depending on θ

$\theta < 1/2$	$\theta > 1/2$
$A = [0^+, \frac{1}{2}^-]$	$C = [0^+, (1 - \theta)^-]$
$B = [\frac{1}{2}^+, (1 - \theta)^-]$	$B = [(1 - \theta)^+, \frac{1}{2}^-]$
$C = [(1 - \theta)^+, 1]$	$A = [\frac{1}{2}^+, 1^-]$

The following lemma is immediate, and immediately explains the apparent ambiguity in the statement of Theorem 1.1:

LEMMA 2.1. *If ω is an infinite word encoding the orbit of a point $x \in S^1$ under rotation by θ , then ω encodes the orbit of some $x \in S^1$ without the introduction of \mathcal{D} with at most two errors. Alternately the coding is exact up to a change of endpoints of the intervals A , B , and C .*

Proof. The orbit of any point can hit the endpoints of A , B and C at most twice. ■

3. The renormalization procedure. Recall γ , the Gauss map (1.3); we define a similar map. For irrational θ (whose continued fraction expansion does not terminate), define

$$(3.1) \quad g([a_1, a_2, a_3, \dots]) = \begin{cases} [a_3, a_4, \dots] = \gamma^2(\theta) & (a_1 = 0 \bmod 2), \\ [1, a_2, a_3, \dots] = \frac{1}{1 + \gamma(\theta)} & (a_1 = 1 \bmod 2, a_1 \neq 1), \\ [a_2 + 1, a_3, \dots] = 1 - \theta & (a_1 = 1). \end{cases}$$

Note that if $\theta > 1/2$, then necessarily $g(\theta) < 1/2$. For rational θ (whose continued fraction expansion terminates) the definition may be extended by considering the continued fraction expansion of θ whose final coefficient is of odd index (see e.g. [4]; rather than assume the final coefficient is not one as is standard, we assume it to be of odd index). If this index is at least three, the definition above may be applied without modification, and for length one expansions we fix $g(1/n) = g(0) = 0$. It will be convenient to define

$$(3.2) \quad E(x) = \max\{n \leq x : n \in \mathbb{Z}, n = 0 \bmod 2\}.$$

The triplet $\{X, \mu, T\}$ refers to a compact probability space $\{X, \mu\}$ and a continuous transformation T on X which preserves μ . We assume from this

point forward that $\theta \notin \mathbb{Q}$, and we denote

$$(3.3) \quad \theta_n = g^n(\theta), \quad \delta_n = 1 - E(a_1(\theta_n))\theta_n, \quad I_n = \{S^1, \mu, R_{\theta_n}\}.$$

Note that $\delta_n = 1$ if and only if $\theta > 1/2$; otherwise $\delta_n < 1/2$.

Partition each I_n into intervals A , B , and C according to Table 1, and recall that by convention we have disconnected each I_n such that all iterates of the characteristic functions of A , B , and C under $R_{\theta_n}^i$ are continuous. Given $\{X, \mu, T\}$ and a set $S \subset X$, the *return time to S* is given by

$$n(x) = \min\{n > 0 : T^n(x) \in S\}.$$

As irrational rotations are minimal, $n(x)$ will be defined for all $x \in S^1$ if S is an interval of positive length. The *induced system* on S is defined by

$$\{S, \mu|_S, T|_S\},$$

where $T|_S(x) = T^{n(x)}(x)$ for all $x \in S$. Define $I'_{n+1} \subset I_n$ by

$$I'_{n+1} = [0^+, \delta_n^-].$$

Finally, define the substitutions $\sigma_n = \sigma(\theta_n)$ according to Table 2, and define the functions $\varphi_n = \varphi(\theta_n)$ according to

$$(3.4) \quad \varphi(x) = \begin{cases} 1-x & (a_1(\theta) = 1), \\ \delta_n^{-1}x & (a_1(\theta) \neq 1). \end{cases}$$

Table 2. The substitution σ as a function of θ

Case	Substitution
$a_1 = 2k, a_3 \neq 1$	$A \rightarrow (A^{k+1}B^{k-1}C)(A^k B^{k-1}C)^{a_2-1}$
	$B \rightarrow (A^k B^k C)(A^k B^{k-1}C)^{a_2-1}$
	$C \rightarrow (A^k B^k C)(A^k B^{k-1}C)^{a_2}$
$a_1 = 2k, a_3 = 1$	$A \rightarrow (A^k B^k C)(A^k B^{k-1}C)^{a_2}$
	$B \rightarrow (A^{k+1}B^{k-1}C)(A^k B^{k-1}C)^{a_2}$
	$C \rightarrow (A^{k+1}B^{k-1}C)(A^k B^{k-1}C)^{a_2-1}$
$a_1 = 2k + 1$	$A \rightarrow A^k B^k C$
	$B \rightarrow A^{k+1} B^{k-1} C$
	$C \rightarrow A$
$a_1 = 1$	$A \rightarrow A$
	$B \rightarrow B$
	$C \rightarrow C$

LEMMA 3.1. *Suppose that $\theta < 1/2$, $E(a_1(\theta)) = 2k$, and*

$$(1 - 2k\theta)^+ \leq x \leq (1/2 - (k-1)\theta)^-.$$

Then the orbit of x begins with $A^k B^{k-1} C$.

Proof. The assumption $\theta < 1/2$ tells us how to partition S^1 according to Table 1 as well as guaranteeing that $k \geq 1$. Note that the lower inequality certainly ensures that

$$\frac{1}{2} - k\theta < x \leq (\frac{1}{2} - (k-1)\theta)^-,$$

which tells us that $x + i\theta \leq (1/2)^-$ for $i = 0, 1, \dots, k-1$, while $x + k\theta > 1/2$. So the coding of the orbit of x begins with exactly A^k before seeing either B or C . As

$$(1 - 2k\theta)^+ \leq x < 1 - (2k-1)\theta,$$

we know that $x + (2k-1)\theta < 1$, while $x + 2k\theta \geq 1^+$. Therefore, once we have accounted for the points $x + i\theta$ for $i = 0, 1, \dots, k-1$, the terms $i = k, k+1, \dots, 2k-1$ must all belong to either B or C . That C is an interval of length exactly θ guarantees that exactly the final term is C . The rest of the terms (if there are any) are therefore B . ■

PROPOSITION 3.2. *We have the measurable and continuous isomorphism*

$$\{I'_{n+1}, \mu|_{I'_{n+1}}, (R_{\theta_n})|_{I_{n+1}}\} \xrightarrow{\varphi_n} \{I_{n+1}, \mu, R_{\theta_{n+1}}\}.$$

Furthermore, for all $x \in A \subset I_{n+1}$, the word $\sigma_n(A)$ encodes the orbit of $\varphi^{-1}(x)$ through its return to I'_{n+1} (the encoding is with respect to the partition A, B, C in I_n), and similarly for B and C .

Proof. The use of the substitutions to encode the orbit of points through some length of the orbit corresponds to saying that σ_n correctly encodes *return words* in the sense of [7].

In the case that $\theta_n > 1/2$, we have $\theta_{n+1} = 1 - \theta_n$ and $I'_{n+1} = [0^+, 1^-]$. However, by referring to Table 1, we see that the intervals A , B , and C exactly reflect the reversal of orientation given by $\varphi_n(x) = 1 - x$, and the substitution σ_n is identity. So we proceed on the assumption that $\theta_n < 1/2$: in I_n we have

$$A = [0^+, 1/2^-], \quad B = [1/2^+, (1-\theta)^-], \quad C = [(1-\theta)^+, 1^-].$$

Then φ_n is scalar multiplication by δ_n^{-1} , so there are only two things to show:

- the first-return map $(R_{\theta_n})|_{I'_{n+1}}$ is rotation by θ_{n+1} , after rescaling by φ_n , and
- the substitution σ_n encodes the correct information.

There are three cases to consider: $a_1(\theta_n) = 1 \bmod 2$, or $a_1(\theta_n) = 0 \bmod 2$ with the subcases $a_3(\theta_n) = 1$ or $\neq 1$. Assume for now that $a_1(\theta_n) = 0 \bmod 2$

and $a_3(\theta_n) = 1$. Then we have $g(\theta_n) = \gamma^2(\theta_n) > 1/2$, so in I_{n+1} we get

$$C = [0^+, (1 - \theta_{n+1})^-], \quad B = [(1 - \theta_{n+1})^+, 1/2^-], \quad A = [1/2^+, 1^-],$$

with corresponding preimages in I'_{n+1} scaled by δ_n . We will first verify that the intervals have the desired return times (which may be read from the length of the words $\sigma_n(A)$, $\sigma_n(B)$, and $\sigma_n(C)$) and that the induced map is indeed rotation by θ_{n+1} (up to scale δ_n). As $E(a_1(\theta_n)) = a_1(\theta_n)$ we have

$$\delta_n = \|q_1(\theta_n) \cdot \theta_n\|,$$

from which it follows that the return time of 0 is

$$n(0) = q_2 = a_1 a_2 + 1,$$

and one may now verify that the entire interval $\varphi_n^{-1}(C)$ has this return time; the preimage of the right endpoint of C under φ_n is exactly $1 - (q_1 + q_2)\theta_n$. The remaining points in I'_{n+1} have return time $q_2 + q_1$ and the induced map is rotation by $q_2\theta_n$ on $[0^+, \delta_n^-]$; see Figure 1.

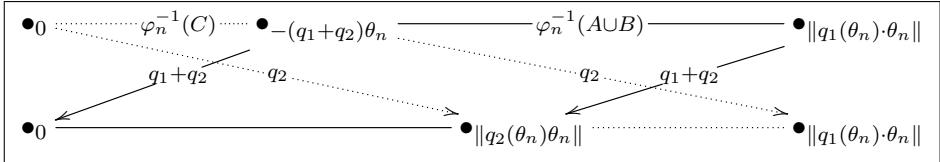


Fig. 1. Return times for $a_1(\theta_n) = 0 \bmod 2$, $a_3(\theta_n) = 1$

At this point we may verify that the rotation is by $g(\theta_n)$, up to scale:

$$\begin{aligned} \frac{\|q_2(\theta_n) \cdot \theta_n\|}{\delta_n} &= \frac{q_2(\theta_n) \cdot \theta_n - p_2(\theta_n)}{1 - q_1(\theta_n) \cdot \theta_n} = \frac{(a_1 a_2 + 1)\theta_n - a_2}{1 - a_1 \theta_n} \\ &= \frac{a_2(a_1 - 1/\theta_n) + 1}{1/\theta_n - a_1} = \frac{1 - a_2 \gamma(\theta_n)}{\gamma(\theta_n)} = \gamma^2(\theta_n). \end{aligned}$$

Now suppose that $x \in \varphi_n^{-1}(B)$, and for convenience let $E(a_1) = a_1 = 2k$. Clearly, the orbit of x begins with a point in A (in I_n , as $A = [0^+, 1/2^-]$ contains $[0^+, \delta_n^-]$). As $x < 1/2 - k\theta_n$, however, we have

$$(1 - 2k\theta_n)^+ \leq x + \theta_n \leq (1/2 - (k - 1)\theta_n)^-,$$

so by Lemma 3.1, we may concatenate the word $A^k B^{k-1} C$ to this initial A . Since $2k = a_1$, we now have

$$x + \theta_n + (2k\theta_n) < x + \theta_n \leq (1/2 - (k - 1)\theta_n)^-.$$

Either we have returned to I'_{n+1} , in which case we are done, or we have not, in which case we apply Lemma 3.1 again, repeating until we return to I'_{n+1} , which must take a total of $q_2 + q_1 = a_1(a_2 + 1) + 1$ steps.

For those points in the interval $\varphi_n^{-1}(a)$, note that the only discontinuity of $R_{\theta_n}^i$ for $i = 0, 1, \dots, q_2$ to distinguish the orbits compared to points in $\varphi_n^{-1}(A)$ is the point $1/2 - k\theta$, which will change the single term $x + k\theta$ from an ‘A’ to a ‘B’. Points in $\varphi_n^{-1}(C)$ are considered identically to those in $\varphi_n^{-1}(B)$, noting that the shorter return time requires one fewer concatenation of $A^{a_1}B^{a_1-1}C$.

The other cases are similarly considered; the case $a_1(\theta_n) = 0 \bmod 2$, $a_3(\theta_n) \neq 1$ is nearly identical, while for the case $a_1(\theta_n) = 1 \bmod 1, \neq 1$ we have $\delta_n > \theta_n$, so the return time of 0^+ is one, explaining the much shorter substitution $\sigma_n(C) = A$ in this case. ■

Denote the iterated pull-back of I_n into I_0 by

$$(3.5) \quad \tilde{I}_n = (\varphi_0^{-1} \circ \cdots \circ \varphi_{n-1}^{-1})(I_n).$$

COROLLARY 3.3. *We have the measurable and continuous isomorphism*

$$\{\tilde{I}_n, \mu|_{\tilde{I}_n}, (R_\theta)|_{\tilde{I}_n}\} \xrightarrow{(\varphi_{n-1} \circ \cdots \circ \varphi_0)} \{I_n, \mu, R_{\theta_n}\}.$$

Furthermore, for any $x \in A \subset I_n$, the word $(\sigma_0 \circ \cdots \circ \sigma_{n-1})(A)$ encodes the orbit of $(\varphi_0^{-1} \circ \cdots \circ \varphi_{n-1}^{-1})(x)$ in I_0 through its return to \tilde{I}_n , and similarly for B, C .

4. Proof of Theorem 1.1. The proof of (1.8) is immediate in light of Corollary 3.3; the point $x(\theta)$ is given by

$$x(\theta) = \bigcap_{i=0}^{\infty} \tilde{I}_i,$$

where the \tilde{I}_i were defined in (3.5). This intersection is nonempty as the sets are nested closed intervals in the compact space S^1 . The length of \tilde{I}_n is given by

$$\delta_0 \cdot \delta_1 \cdots \delta_{n-1},$$

and we have already remarked that for $\theta_n < 1/2$ we have $\delta_n < 1/2$. As no two successive terms in the sequence $\theta_0, \theta_1, \dots$ may be larger than one-half, the length tends to zero, and the intersection is either a singleton or a pair $\{x^-, x^+\}$. In the latter scenario, however, both x^- and x^+ would have identical coding of their forward orbits. As we did not ‘split’ the points $i\theta$ or $i\theta + 1/2$ for $i > 0$ when disconnecting S^1 , this is not possible.

As all nonidentity substitutions map each letter to a word beginning in A , and all nonidentity substitutions map A to a word of length at least three, and no two consecutive substitutions may be identity, it follows that the sequence of words

$$(\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1})(\omega)$$

has a limit regardless of the choice of nonempty ω , and Corollary 3.3 shows that this word must encode the orbit of $x(\theta)$ in the disconnected version of S^1 . Lemma 2.1 finishes the proof of this portion of Theorem 1.1.

Let us now turn our attention to constructing the orbit of an arbitrary $x_0 \in S^1$. Define

$$x_1 = x_0 + i\theta, \quad i \in \{j \geq 0 : x + j\theta \in I'_1\},$$

and let ω_0 be the word which encodes the orbit of x_0 through its arrival to x_1 ; if $x_0 \in I'_1$, we may set ω_0 to be the empty word (though we are not required to do so). We now pass to the system I_1 , letting $(x_1 \in I_1) = \varphi_0(x_1 \in I'_1)$. We set x_2 to be a point in I'_2 which is in the orbit of x_1 , and let ω_1 be the word encoding this finite portion of the orbit, then pass to I_2 , etc. Equation (1.7) now follows from Proposition 3.2 so long as infinitely many ω_n are nonempty. We only have the option of letting all but finitely many ω_n be empty if x is a preimage of $x(\theta)$; we have already remarked in this case that the limiting word may be found handily.

A potential source of confusion at this point is the desire to claim that $x(\theta) = 0$, as we always construct $I'_{n+1} = [0^+, \delta_n^-]$. However, $\varphi_n(x) = 1 - x$ for those n such that $\theta_n > 1/2$. So $\varphi_n^{-1} \circ \varphi_{n+1}^{-1}$ pulls back I_{n+2} to the interval $[(1 - \delta_{n+1})^+, 1^-] \subset I_n$. Those θ for which $x(\theta) = 0$ will be addressed in Proposition 4.3.

PROPOSITION 4.1. *Without loss of generality, ω_n may be required to be either empty, or a proper right factor of either $\sigma_n(A)$, $\sigma_n(B)$, or $\sigma_n(C)$.*

Proof. The images of $R_{\theta_n}^i(I'_{n+1})$ cover all of I_n through the return times, so any x may be viewed as returning to I'_{n+1} via a right factor of one of these words. If the return is through the entire word $\sigma_n(A)$, we would have begun with $x_n \in I'_{n+1}$ and could have set $\omega_n = \emptyset$. ■

REMARK. One could alternately require that ω_n be nonempty by allowing all nonempty right factors of $\sigma_n(A)$, $\sigma_n(B)$, and $\sigma_n(C)$; instead of $\omega_n = \emptyset$ for $x \in I'_{n+1}$, let ω_n be σ applied to the letter encoding whichever interval in I_{n+1} contains $\varphi_n(x)$.

In order to construct the orbit of zero we will side-step this computation altogether:

LEMMA 4.2. *Suppose that $\theta_n > 1/2$. Let Ω encode the orbit of 0^+ in the system I_n , and Υ encode the orbit of 0^+ in the system I_{n+1} . Then for all $i \geq 1$, $(\Omega)_i = (\Upsilon)_i$. For $i = 0$, $(\Omega)_0 = C$ while $(\Upsilon)_0 = A$.*

Proof. The isomorphism $\varphi_n(x) = 1 - x$ and the identity substitution σ_n ensure that Ω is identical to the coding of the orbit of 1^- in I_{n+1} . As the forward orbit of 0 under rotation by the irrational θ_n does not hit any other

endpoints of the intervals A , B , and C , we see that the orbits of 1^- and 0^+ in the system I_{n+1} are identical after this initial term. ■

With this lemma in mind, define the map $\Psi(\omega)$ on both \mathcal{A}^* and $\mathcal{A}^\mathbb{N}$:

$$(4.1) \quad (\Psi\omega)_i = \begin{cases} C & (i = 0), \\ \omega_i & (i \neq 0). \end{cases}$$

Define the maps $\sigma'_n = \sigma'(\theta_n)$:

$$(4.2) \quad \sigma'(\theta) = \begin{cases} \sigma(\theta) & (\theta < 1/2), \\ \Psi & (\theta > 1/2). \end{cases}$$

Then (1.9) follows if we appropriately choose the words ω'_n to accurately encode some string of the initial orbit of 0^+ in I_n . Then the resulting word

$$(\sigma'_0 \circ \sigma'_1 \circ \cdots \circ \sigma'_{n-1})(\omega'_n)$$

will accurately represent the initial orbit of 0^+ , but it is no longer guaranteed that the length of this word increases! For example, if $\theta = [3, 2, 2, 2, 2, \dots]$, then we will alternate between σ'_n being Ψ and a substitution which maps $C \rightarrow A$. Setting $\omega'_n = A$ for all those n for which $\theta_n < 1/2$ would therefore always map via this long string of compositions to

$$A \xrightarrow{\Psi} C \xrightarrow{\sigma} A \xrightarrow{\Psi} C \xrightarrow{\sigma} \cdots.$$

Define

$$(4.3) \quad \omega'_n = \begin{cases} A^{k+1}B^{k-1}C & (a_1(\theta_n) = 2k), \\ A^{k+1}B^k & (a_1(\theta) = 2k + 1), \\ \Psi(\omega'_{n+1}) & (a_1(\theta) = 1). \end{cases}$$

The reader may verify that the word ω'_n does accurately encode some initial portion of the orbit of 0^+ depending on the parity of $a_1(\theta_n)$. Note that whenever Ψ is applied, it affects only the first letter of its input. From this it follows that if $\omega = (\omega)_0\nu$, then

$$(4.4) \quad (\sigma'_0 \circ \cdots \circ \sigma'_{n-1})(\omega) = (\sigma'_0 \circ \cdots \circ \sigma'_{n-1})((\omega)_0)(\sigma_0 \circ \cdots \circ \sigma_{n-1})(\nu).$$

As ω'_n always has length larger than one, our previous reasoning now guarantees that the length of Ω'_n diverges, establishing (1.9) and completing the proof.

Before moving on to the study of the growth rates of discrepancy sums, we present a few observations about this process.

PROPOSITION 4.3. *Those θ for which $x(\theta) = 0$ ($= 0^+$) are exactly the set*

$$(4.5) \quad H = \{\theta : a_{2i-1}(\theta) = 0 \bmod 2, i = 1, 2, \dots\}.$$

Proof. We leave to the reader to verify that H is exactly the set of θ for which $g^n(\theta) < 1/2$ for every n . For those $\theta \in H$, then, we always have

$I'_{n+1} = [0^+, \delta_n^-]$, where $\delta_n < 1$, and we never need to apply the isomorphism $\varphi_n(x) = 1 - x$. That is,

$$0 \in (\varphi_0^{-1} \circ \cdots \circ \varphi_{n-1}^{-1})(I_n)$$

for all n , so $0 = x(\theta)$.

On the other hand, if n is the first index such that $\theta_n > 1/2$, we must have $\varphi_n(x) = 1 - x$. As $\theta_{n+1} < 1/2$, however, it follows that within I_n we have

$$\varphi_n^{-1} \circ \varphi_{n+1}^{-1}(I_{n+2}) = [(1 - \delta_{n+1})^+, 1^-],$$

from which it follows that

$$0 \notin (\varphi_0^{-1} \circ \cdots \circ \varphi_{n+1}^{-1})(I_{n+2}). \blacksquare$$

PROPOSITION 4.4. *The sequence of substitutions σ_n is eventually periodic if and only if θ is a quadratic surd.*

Proof. Clearly the sequence σ_n is eventually periodic if and only if the orbit of θ under g is eventually periodic. From the definition (3.1) of g we have, for all $i \geq 2$,

$$(4.6) \quad a_i(\theta_{n+1}) = a_{i+k}(\theta_n), \quad k = \begin{cases} 0 & (a_1(\theta_n) = 1 \bmod 2, \neq 1), \\ 1 & (a_1(\theta_n) = 1), \\ 2 & (a_1(\theta_n) = 0 \bmod 2). \end{cases}$$

So, if $a_i(\theta)$ are eventually periodic (Gauss' criteria for quadratic surds), we must have infinitely many n such that for all $i \geq 2$ we have for any j, k ,

$$a_i(\theta_{n_k}) = a_i(\theta_{n_j}).$$

Suppose that a period of $a_i(\theta)$ is given by the terms $\alpha_1, \dots, \alpha_N$, and assume without loss of generality that for $i \geq 2$,

$$a_i(\theta_{n_k}) = \alpha_{i \bmod N}.$$

Then $a_1(\theta_{n_k})$ is either 1, α_1 , or $\alpha_1 + 1$. Since the collection n_k was infinite, one value must be taken twice, giving a period in the orbit $g(\theta)$.

On the other hand, assume that $\theta_j = \theta_{j+nk}$ for $n = 0, 1, \dots$ and $k \neq 0$. From (4.6) it follows that $a_i(\theta)$ is eventually periodic. \blacksquare

REMARK. The periods under g and γ need not be the same, nor is one necessarily longer than the other. For example, the golden mean has period one under γ but period two under g , while $\theta = [2, 1, 2, 1, \dots]$ has period two under γ and period one under g . Furthermore, the sequence σ_n is purely periodic if and only if $\theta_n = \theta_0$ for some $n \neq 0$, which is not the same as the partial quotients of θ being purely periodic. Consider for example $\theta = [3, 2, 2, 2, \dots]$, whose partial quotients are clearly not purely periodic, but satisfy $\theta_2 = \theta_0$.

5. The arithmetic of our substitutions. Let $\theta_0 < 1/2$, so that

$$f(x) = \begin{cases} +1 & (x \in A), \\ -1 & (x \in B \cup C). \end{cases}$$

For $\theta_0 > 1/2$ we could repeat all future arguments with a sign change. Given $\omega \in \mathcal{A}^n$, define (consistent with existing notation)

$$\begin{aligned} S(\omega) &= \sum_{i=0}^{n-1} (\chi_A - \chi_{B \cup C}) \omega_i, \\ M(\omega) &= \max\{S(\omega_0 \dots \omega_{j-1}) : j = 1, \dots, n\}, \\ m(\omega) &= \min\{S(\omega_0 \dots \omega_{j-1}) : j = 1, \dots, n\}. \end{aligned}$$

Note that we *do not* include the empty word in determining $M(\omega)$, $m(\omega)$.

PROPOSITION 5.1. Suppose $|\omega| = n \neq 0$, $\omega \neq C$, $M(\omega) \geq 0$, ω does not have CC , CB , or BA as factors, and σ is a substitution given by Table 2, depending on θ . If $a_1(\theta) = 0 \bmod 2$ and $a_3(\theta) \neq 1$, or if $a_1(\theta) = 1$, then

$$S(\sigma(\omega)) = S(\omega), \quad M(\sigma(\omega)) = M(\omega) + E(a_1), \quad m(\sigma(\omega)) = m(\omega).$$

On the other hand, if $a_1(\theta) = 0 \bmod 2$ and $a_3(\theta) = 1$, then

$$S(\sigma(\omega)) = -S(\omega), \quad M(\sigma(\omega)) = -m(\omega) + E(a_1), \quad m(\sigma(\omega)) = -M(\omega).$$

Finally, if $a_1(\theta) = 1 \bmod 2$, $\neq 1$, and either

- $(\omega)_{n-1} \neq C$, or
- $(\omega)_{n-1} = C$, but there is some $j \neq n$ such that $S((\omega)_0(\omega)_1 \dots (\omega)_{j-1}) = m(\omega)$,

then also

$$S(\sigma(\omega)) = -S(\omega), \quad M(\sigma(\omega)) = -m(\omega) + E(a_1), \quad m(\sigma(\omega)) = -M(\omega).$$

If $a_1(\theta) = 1 \bmod 2$, $(\omega)_{n-1} = C$ and $S((\omega)_0 \dots (\omega)_{j-1}) > m(\omega)$ for all $j \neq n$, then

$$S(\sigma(\omega)) = -S(\omega), \quad M(\sigma(\omega)) = -m(\omega) - 1 + E(a_1), \quad m(\sigma(\omega)) = -M(\omega).$$

Proof. The prohibition on CB , CC and BA being factors of ω is necessary for ω to encode the orbit of any point under rotation by any θ , so this condition is not prohibitive in our setting.

In all cases, the statements regarding the value $S(\sigma(\omega))$ follow from examining $S(\sigma(x))$ for each $x \in \mathcal{A}$; the reader may consult Table 2 to verify that $S(\sigma(x)) = \pm S(x)$ as described, and the statement then follows from the fact that σ is a homomorphism. We will turn our attention, then, to the statements regarding $m(\sigma(\omega))$ and $M(\sigma(\omega))$. All cases but the last are considered similarly with the possible sign-change outlined above in mind.

For example, suppose that $a_1 = 0 \bmod 2$ and $a_3 \neq 1$. Let $\omega = v\psi$, where v is the largest left factor of ω such that $S(v) = M(\omega) - 1$; note that as

$M(\omega) \geq 0$ and the empty word was not considered in computation of $M(\omega)$, we have $(\psi)_0 = A$. As $S(\sigma(v)) = S(v) = M(\omega) - 1$ and $M(\sigma(A)) = E(a_1) + 1$, we know that

$$M(\sigma(\omega)) \geq M(\sigma(v)\psi) = M(\omega) + E(a_1).$$

Assume on the other hand that

$$\sigma(\omega) = \sigma(v)\nu\psi, \quad S(\sigma(v)\nu) > M(\omega) + E(a_1),$$

and v is of maximal length to allow such a decomposition. Note that $\nu \neq \emptyset$ as $S(\sigma(v)) = S(v) \leq M(\omega)$. As v is a proper factor, it is followed by a letter, and by maximality of the length of v , we see that ν is a proper left factor of either $\sigma(A)$, $\sigma(B)$, or $\sigma(C)$, and $E(a_1) \neq 0$. If v is followed by A in ω , then

$$S(\sigma(v)) = S(v) \leq M - 1.$$

On the other hand, $S(\nu) \leq E(a_1) + 1 = M(\sigma(A))$, contradicting the value $S(\sigma(v)\nu)$. The possibility of v followed by B or C is similarly considered; the larger possible $S(\sigma(v)) = M(\omega)$ is countered by $S(\nu) \leq E(a_1)$ in these cases.

The ambiguity in the situation when $a_1(\theta) = 1 \bmod 2, \neq 1$ is due to the substitution $\sigma(A) = C$, which does not achieve an intermediate sum of $E(a_1)$ (as does $\sigma(B)$). On the assumption that there is some proper left factor ψ of ω such that $S(\psi) = m(\omega)$, however, we know that the letter which follows ψ must be A ; similar computations to the above then apply. If the only left factor of ω which achieves a sum of $m(\omega)$ is in fact ω itself, then if the final letter of ω is B we again have no problem.

Assume, then, that $S(\omega) = m(\omega)$, there is no proper left factor with this sum, and ω ends with the letter C . As $M(\omega) \geq 0$ by assumption, there is a letter preceding this terminal C (that is, $\omega \neq C$). If this letter is A , then the left factor ψ such that $\omega = \psi AC$ has the minimal sum as its sum (even if it is empty), and the preceding reasoning applies. Therefore ω must be of the form ψBC (recall that CC is not a factor); considering $\sigma(B)$ following $S(\sigma(\psi)) = -m(\omega) - 2$ completes the proof. ■

For convenience, denote

$$(5.1) \quad \sigma^{(n)} = \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1},$$

$$(5.2) \quad \sigma'^{(n)} = \sigma'_0 \circ \sigma'_1 \circ \cdots \circ \sigma'_{n-1}.$$

Recall (4.3) and define for $n \geq 1$,

$$(5.3) \quad \Omega_n = \sigma^{(n)}(A), \quad \Omega'_n = \sigma'^{(n)}(\omega'(n)).$$

Define p_n to track the parity of how many θ_i are greater than $1/2$:

$$(5.4) \quad p_n = \left(\sum_{i=1}^{n-1} \chi_{(1/2, 1)}(\theta_i) \right) \bmod 2.$$

We now have all the tools necessary to precisely study the sequences $M_n(y)$ and $m_n(y)$ for $y \in \{x(\theta), 0\}$:

PROPOSITION 5.2. *Assume that $\theta_0 < 1/2$. Then*

$$S(\Omega_n) = (-1)^{p_n}, \quad S(\Omega'_n) = 1$$

and

$$\left| M(\Omega_n) - \left(1 + \sum_{\substack{i \leq n-1 \\ p_i=0}} E(a_1(\theta_i)) \right) \right| \leq 1, \quad M(\Omega'_n) = 1 + \sum_{\substack{i \leq n \\ p_i=0}} E(a_1(\theta_i)),$$

$$\left| m(\Omega_n) - \left(1 - \sum_{\substack{i \leq n-1 \\ p_i=1}} E(a_1(\theta_i)) \right) \right| \leq 1, \quad m(\Omega'_n) = 1 - \sum_{\substack{i \leq n \\ p_i=1}} E(a_1(\theta_i)).$$

Proof. The word Ω_n in (5.3) is formed by successive substitutions acting on the word A ; as such, it will always begin with A , so $M(\Omega_n) \geq 1$. We immediately see that all $S(\Omega_n)$ equal ± 1 according to the parity of p_n by applying Proposition 5.1 in succession. The ambiguous case in Proposition 5.1 arose when ω was a word which had a nonnegative maximal sum (as do all Ω_n) and whose minimum sum was only achieved as its total sum, with C as a terminal factor. Furthermore, we would need θ_n to have first partial quotient odd and larger than one. For this to happen with the restriction that all $S(\Omega_n)$ are ± 1 requires that $S(\Omega_n) = -1$ (otherwise the minimal sum is achieved by the proper left factor A), and therefore $S(\Omega_{n-1}) = 1$. This scenario *also* requires that $M(\Omega_{n-1}) = 1$ (otherwise $m(\Omega_n) < -1 \leq S(\Omega_n)$); so this situation can only occur in our scenario when $\Omega_{n-1} = A$: *this possible error of one may only appear once in the sequence of arithmetic computations from repeated application of Proposition 5.1.*

We leave to the reader the verification that the parity of p_n exactly dictates whether substitutions will add to the maximal values or subtract from the minimal values; refer to Proposition 5.1 again.

Let us now consider Ω'_n . Note that $\sigma'_j = \Psi$ exactly when $\theta_j > 1/2$, exactly when σ_{j-1} has the property that $S(\sigma_{j-1}(\omega)) = -S(\omega)$. Clearly we have $S(\Psi(\omega)) = S(\omega) - 2$ provided ω begins with A . Also note that if $S(\omega) = 1$, then if $m(\omega) = 1$ we must have $\omega_0 = A$: *it is never possible in our construction for ω to terminate with C , $S(\omega) = 1$, and $m(\Psi(\omega)) = S(\Psi(\omega))$ is the only time this value is reached.*

Our choice of $\omega'(n)$ always begins with A and has $S(\omega'(n)) = 1$, and for those σ_n such that $S(\sigma_n(A)) = -1$, the reader may verify that

$$S(\sigma_n(\Psi(\omega))) = 2 - S(\omega)$$

by applying Proposition 5.1. While this change will change the sum of $+1$ to -1 , it is immediately followed by a substitution which reverses the sign

of the sum: we maintain

$$S(\Omega'_n) = 1.$$

Furthermore, as $m(\omega'_n) = 1$ for all ω'_n , if we do apply Ψ (so $m(\Psi\omega) = -1$) followed by one of these sign-reversing substitutions σ , we see

$$M(\sigma(\Psi\omega)) \geq -m(\Psi\omega) + E(a_1) - 1 \geq 1 + E(a_1) - 1 \geq 0,$$

so we may always apply Proposition 5.1 without worrying about the possible error of one. ■

COROLLARY 5.3 ([4, Theorem 1, case $k = 2$]). *We have $S_n(\theta) \geq 0$ for all $n \geq 0$ if and only if $x(\theta) = 0$.*

Proof. By viewing the ergodic sums as an additive cocycle, for all $n > 0$ we have $S_n(\theta) = S_{n+1}(0) - 1$, so by Proposition 5.2,

$$S_{|\Omega'_n|-1}(\theta) = 0, \quad M_{|\Omega'_n|-1}(\theta) = \sum_{\substack{i \leq n \\ p_i=0}} E(a_1(\theta_i)), \quad m_{|\Omega'_n|-1}(\theta) = -\sum_{\substack{i \leq n \\ p_i=1}} E(a_1(\theta_i)).$$

So $S_n(\theta) \geq 0$ for all n if and only if $p_i = 0 \pmod{2}$ for all i such that $\theta_i < 1/2$, which is equivalent to $p_i = 0 \pmod{2}$ for all i . A direct inductive argument shows that $p_i = 0$ for all i if and only if $a_{2i-1}(\theta) = 0 \pmod{2}$ by considering the action of g (3.1), which corresponds by Proposition 4.3 to $x(\theta) = 0$. ■

REMARK. It is possible to show that this set of θ is also characterized by the following condition: among the points $i\theta \pmod{1}$ for $i = 0, 1, \dots, N$, the value closest to $1/2$ is always less than $1/2$. Assume that $a_{2n+1} = 0 \pmod{2}$ for all n , and recall that $q_{2n+1} = 0 \pmod{2}$ for all n , so close approximations below one-half occur for $N\theta$ with $N = q_{2n+1}/2$ (as $q_{2n+1}\theta$ is a close approximation to an integer from below) and a distance of $\|q_{2n+1}\theta\|/2$. As $q_{2n+2} = 1 \pmod{2}$, however, the nearest approximation to one-half from above, for any $N < q_{2n+1}$, occurs for

$$N = \frac{q_{2n+1}}{2} + q_{2n} < q_{2n+1}$$

with a distance of

$$\|q_{2n}\theta\| - \frac{\|q_{2n+1}\theta\|}{2} > \frac{\|q_{2n+1}\theta\|}{2}.$$

A proof of the converse direction may be constructed using similar reasoning and inducting on n in the inequality $N < q_{2n+1}$ to conclude that $a_{2n+1} = 0 \pmod{2}$; as this property is not directly relevant to the current topic, we leave this particular argument to the interested reader. A similar statement may be shown via the same argument for approximation to rationals with denominator k for those θ such that $a_{2n+1} = 0 \pmod{k}$ for all n .

REMARK. As σ are all homomorphisms, a more constructive version of (1.7) is

$$\omega_0\sigma^{(1)}(\omega_1)\sigma^{(2)}(\omega_2)\cdots\sigma^{(n)}(\omega_n)\cdots,$$

which allows a more direct way of computing the word through successive computation of the words ω_n (given the starting point x). The study of one-sided boundedness of ergodic sums has appeared in various contexts, with [13, 4] being most directly related to this work.

LEMMA 5.4. *We always have*

$$|\sigma^{(n)}(A)| = |\sigma^{(n)}(B)|,$$

and if we define the matrices $M_i = M(\theta_i)$ according to Table 3, then

$$M_{n-1}M_{n-2}\cdots M_1M_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} |\sigma^{(n)}(A)| \\ |\sigma^{(n)}(C)| \end{bmatrix}.$$

Table 3. The matrices $M(\theta)$ used to determine return times in the induced systems

Case	$M(\theta)$
$a_1(\theta) = 0 \bmod 2, a_3(\theta) \neq 1$	$\begin{bmatrix} (a_1 - 1)a_2 + 1 & a_2 \\ (a_1 - 1)a_2 + a_1 & a_2 + 1 \end{bmatrix}$
$a_1(\theta) = 0 \bmod 2, a_3(\theta) = 1$	$\begin{bmatrix} (a_1 - 1)a_2 + a_1 & a_2 + 1 \\ (a_1 - 1)a_2 + 1 & a_2 \end{bmatrix}$
$a_1(\theta) = 1 \bmod 2, \neq 1$	$\begin{bmatrix} a_1 - 1 & 1 \\ 1 & 0 \end{bmatrix}$
$a_1(\theta) = 1$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. The first claim follows directly from the observation that, for all substitutions σ , the words $\sigma(A)$ and $\sigma(B)$ are always of the same length and always contain the same number of letters drawn from $\{A, B\}$. That is, within

$$(\varphi_{n-1}^{-1} \circ \cdots \circ \varphi_0^{-1})(A \cup B) \subset \tilde{I}_n$$

the return time under R_{θ_0} to \tilde{I}_n is constant, and similarly on the pullback of C . One need only count the number of C and $\{A, B\}$ within $\sigma_n(C)$ and $\sigma_n(\{A, B\})$ to construct the relevant matrices. ■

LEMMA 5.5.

$$|\Omega_n| \leq |\Omega'_n| \leq |\Omega_{n+1}|.$$

Proof. The lower inequality is direct in light of (4.4), recalling that $(\omega'_n)_1 = A$. The upper bound follows from Lemma 5.4, noting that while ω'_n may or may not be a left factor of $\sigma_n(A)$, it does contain the same

number of $\{A, B\}$ versus C as a proper left factor of $\sigma_n(A)$. Furthermore, the only substitutions for which $|\sigma(C)| > |\sigma(A)|$ are those corresponding to $a_1 = 0 \bmod 2$, $a_3 \neq 0$; such substitutions are *not* followed by Ψ . That is,

$$|\sigma'^{(n)}(A)| \leq |\Omega_n|,$$

completing the proof of the upper bound. ■

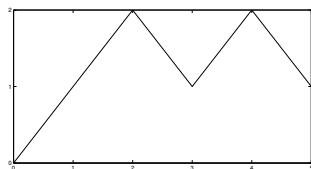
EXAMPLE 5.6. Let $\theta = \sqrt{2} \bmod 1 = [2, 2, 2, \dots]$. Then as θ is a quadratic irrational, the sequence of substitutions σ_i is eventually periodic by Proposition 4.4. As $g(\theta) = \theta$, the sequence of substitutions is periodic with period one, given by

$$\sigma : \begin{cases} A \rightarrow AACAC, \\ B \rightarrow ABCAC, \\ C \rightarrow ABCACAC. \end{cases}$$

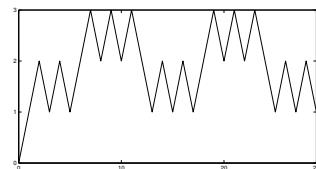
The point $x(\theta)$ is 0 by Proposition 4.3, so applying Theorem 1.1, the orbit of zero is given by the sequence

$$\lim_{n \rightarrow \infty} \sigma^n(A) = AACACAACACABCACACACAACABCACACAC \dots$$

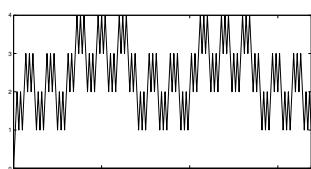
The self-similar structure of the sequence of ergodic sums $S_n(0)$ is not exact (as $\sigma(B) \neq \sigma(C)$), but nonetheless highly regular. This regularity was noticed by D. Hensley in [9, Figure 3.4]. We give several plots of $S_n(0)$ for different values of n in Figure 2. This same self-similarity for developing the orbit of $x(\theta)$ will be seen for any quadratic irrational θ in light of Proposition 4.4. Note that self-similar structure resulting from iterating a fixed transformation (like a substitution) is a well-studied topic, even in the particular case of discrepancy sums: see e.g. [1, 3, 5].



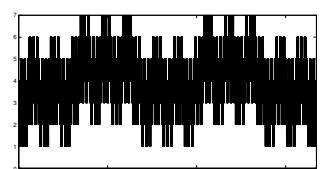
(a) $N = 5$, $\sigma(A) = AACAC$



(b) $N = 29$, $\sigma^2(A)$



(c) $N = 169$, $\sigma^3(A)$



(d) $N = 33461$, $\sigma^6(A)$

Fig. 2. Plots of $S_i(0)$ for different ranges of $0 \leq i \leq N$, where $\theta = \sqrt{2} - 1$

For quadratic irrational $\theta \notin H$, computation of the point $x(\theta)$ is not too difficult:

EXAMPLE 5.7. Let $\theta = [1, 1, \dots]$ be the golden mean. Recall that S^1 will be partitioned such that $A = [(1/2)^+, 1^-]$ as $\theta > 1/2$. As $g^2(\theta) = \theta$, and $a_1 = 1$ corresponds to the identity substitution, the only nonidentity substitution generated is

$$\sigma : \begin{cases} A \rightarrow ABCAC, \\ B \rightarrow AACAC, \\ C \rightarrow AAC. \end{cases}$$

So, the orbit of $x(\theta)$ is given by

$$\lim_{n \rightarrow \infty} \sigma^n(A) = ABCACAAACACAAACABCACAAAC \dots,$$

while the orbit of 0 is given by

$$\Psi(\sigma(\dots\Psi(AAC))) = CACABCACAAACABCACAAACAC \dots.$$

To compute the point $x(\theta)$, we need to determine the intervals \tilde{I}_n . For those $\theta_n = [2, 1, 1, \dots]$ we have

$$\delta_n = 1 - 2\theta_n = 1 - 2(1 - \theta) = 2\theta - 1.$$

Denote this quantity by δ for convenience. For this particular θ we do not ever have two consecutive $\theta_n < 1/2$, so the intervals $I'_{n+1} \subset I_n$ strictly alternate between $[0^+, \delta^-]$ and $[(1 - \delta)^+, 1^-]$ (for $n = 0 \bmod 2$; for odd n we have $\theta_n > 1/2$ and $I'_{n+1} = I_n$). So the sequence of preimages \tilde{I}_n (recall again (3.5)) is given by

$$[0^+, 1^-], [(1 - \delta)^+, 1^-], [(1 - \delta)^+, (1 - \delta + \delta^2)^-], \dots;$$

its intersection is given by the geometric series

$$x(\theta) = \sum_{i=0}^{\infty} (-1)^i \delta^i = \frac{1}{1 + (2\theta - 1)} = \frac{1}{2\theta}.$$

See Figure 3 for both of these orbits.

One particularly striking corollary of Proposition 5.2 is the following, which does not seem to be apparent from any other technique:

COROLLARY 5.8. *If θ is a quadratic irrational, then*

$$\lim_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} \in \mathbb{Q}^*,$$

where $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$, and $p/0 = \infty$ for any positive integer p . If $\theta_n = \theta_{n+k}$ is a minimal period under the orbit of g and $p_{n+k} = p_n + 1$, then the ratio tends to one. Furthermore, for any nonnegative $p/q \in \mathbb{Q}^*$, there is a quadratic irrational θ such that the above ratio has limit p/q .

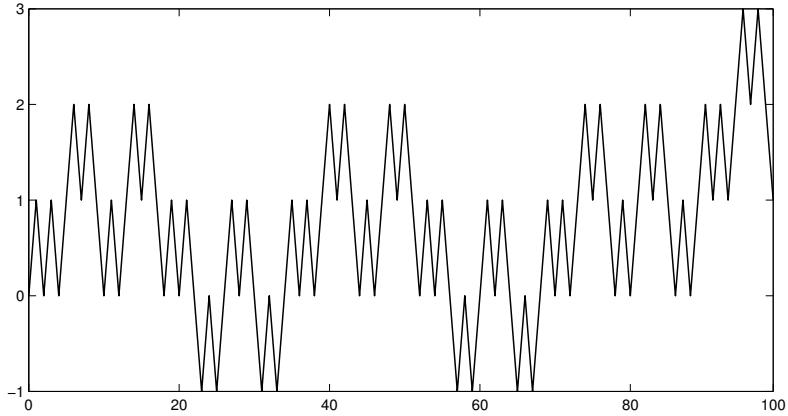
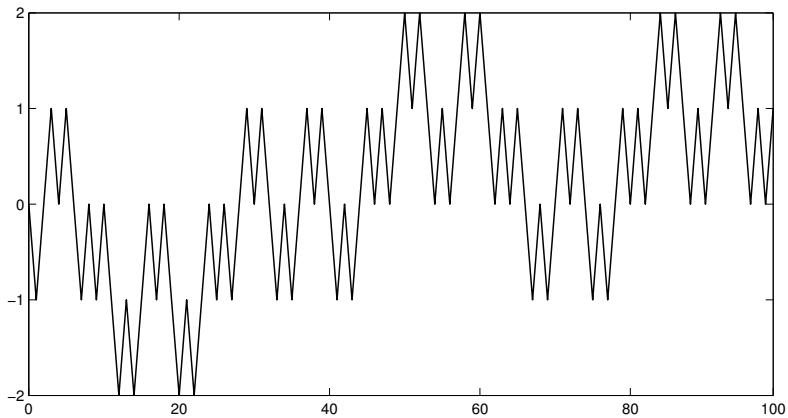
(a) $x = 0$, with orbit $CACABCACAAAC\dots$ (b) $x = x(\theta) = 1/(2\theta)$, with orbit $ABCACACAACAC\dots$

Fig. 3. Plots of $S_i(x)$ for $0 \leq i \leq 100$, where θ is the golden mean for the two given values of x . Note that as $\theta > 1/2$, we have $A \rightarrow -1$, $B, C \rightarrow +1$.

Proof. We have already shown that $g^n(\theta)$ is eventually periodic for such θ in Proposition 4.4. It follows from Proposition 5.2 that $M_n(0)$ and $m_n(0)$ see a periodic sequence of adjustments by bounded integer amounts, which must therefore have rational limit. If one period reflects a change in the parity of p , it will always be followed by the mirrored changes in M_n , m_n , producing a limit of one.

To produce quadratic irrationals with the desired limit, if $q = 0$ then $\theta \in H$ will suffice ($m_n(0) \equiv 1$, and $M_n(0)$ must therefore diverge), and for $p = 0$ any θ such that $a_1(\theta) = 1$ and $g(\theta) \in H$ will suffice (here $M_n(0) \equiv 1$).

For p/q with neither zero, just set

$$\theta = [2p, 1, 1, 2q - 1, 1, 1, 2p - 1, 1, 1, 2q - 1, 1, 1, \dots],$$

and verify that we will first add p to $M_n(0)$, then subtract q from $m_n(0)$, etc. ■

6. Proof of Theorem 1.2. Let c_n and d_n be divergent monotone sequences in $o(n)$ with bounded differences Δc_n , Δd_n ; we will construct a dense set of θ such that

$$\limsup_{n \rightarrow \infty} \frac{M_n(0)}{c_n} = \limsup_{n \rightarrow \infty} \frac{|m_n(0)|}{d_n} = 1.$$

Any irrational θ is completely determined by its sequence of partial quotients, which is equivalent to its orbit under g , and its orbit under g is completely determined by the sequence of values

$$a_1(\theta_i) \quad (a_1 = 1 \bmod 2), \quad a_1(\theta_i), a_2(\theta_i) \quad (a_1 = 0 \bmod 2).$$

Suppose, then, that the first finitely many partial quotients of θ are prescribed, such that the first n values of θ_i are fixed. Without loss of generality, insert an additional single term if necessary so that $p_n = 0$ (recall (5.4)). We are now completely free to choose k to construct ω'_n (refer to (4.3)). If we denote

$$M(\Omega'_n) = M, \quad m(\Omega'_n) = m, \quad |\Omega'_n| = L_n,$$

it follows from Proposition 5.2 that once we choose k , we will have

$$M(\Omega'_{n+1}) = M + k, \quad m(\Omega'_{n+1}) = m.$$

Consider $L_{n+1}(k) = |\Omega'_{n+1}|$ as a function of k .

Assume first that $M < c_{L_n}$, so we wish to increase the maximal sum compared to the sequence c_n . Then let $a_1(\theta_n)$ be odd, so

$$\omega'(n+1) = A^{k+1}B^k.$$

From (4.4) and the previous observation that $|\sigma^{(n)}(A)| = |\sigma^{(n)}(B)|$, it follows that

$$L_{n+1}(k) = |\tilde{\omega}| + 2k|\sigma^{(n+1)}(A)|,$$

where

$$\tilde{\omega} = \sigma'^{(n+1)}(A).$$

Consider, then, the proper left factors A^i of $\omega'(n+1)$ for $i = 1, \dots, k+1$. Applying Proposition 5.1, the new maximal sum $M + k$ is achieved at a time N , where

$$|\tilde{\omega}| + (k-1)|\sigma^{(n)}(A)| \leq N \leq |\tilde{\omega}| + k|\sigma^{(n)}(A)|.$$

As $c_n \in o(n)$, we may choose $k \geq 1$ to be minimal such that

$$\frac{M+k}{c(|\tilde{\omega}|+k|\sigma^{(n)}(A)|)} \geq 1.$$

If, however, we had $M \geq c_{L_n}$, then we would wish to not greatly increase M compared to c_n . In this case, let $\theta_n = [2, k, 1, \dots]$, and pass directly to considering the word

$$\sigma'^{(n+1)}(C) = \sigma'^{(n)}(A^{k+1}B^{k-1}C),$$

as C is always a left factor of $\omega'_{n+1} = \Psi(\omega'_{n+2})$ in this case. Then the maximal sum reached for this word is $M+1$, but its length is (similarly to before)

$$L_{n+1}(k) = |\tilde{\omega}| + 2k|\sigma^{(n)}(A)|.$$

We are now able to increase the length of the word *without* increasing the maximal sum of $M+1$, so as c_n is divergent, choose $k \geq 1$ minimal such that

$$\frac{M+1}{c(|\tilde{\omega}|+k|\sigma^{(n)}(A)|)} \leq 1.$$

After applying g twice (to skip past the next $\theta_k > 1/2$), then, we find it possible to manipulate the growth of the minimal sums $m(n)$. Continuing in this fashion, we construct a dense set of θ (as the initial string of partial quotients was arbitrary). That the limsups are actually 1 follows from the minimal choice of k and the fact that $\Delta c_n, \Delta d_n$ are bounded.

To prove the analogous statements where one of M_n, m_n is desired to remain bounded, one need only repeat the same arguments using $\theta_n \in H$ (recall (4.5)) so that the value p_n is eventually constant.

The statement of Theorem 1.2 applies as well to $M_n(x(\theta))$ and $m_n(x(\theta))$; the proof is simpler, in fact, as the map Ψ is not a concern, and the possible error of 1 from Proposition 5.2 is not an asymptotic concern. This process is highly amenable to diagonalization techniques. For example:

COROLLARY 6.1. *Given a countable collection of sequences $c_n^{(i)}$ and $d_n^{(i)}$, all of which are divergent and in $o(n)$, such that*

$$c_n^{(1)} \leq c_n^{(2)} \leq \dots, \quad d_n^{(1)} \geq d_n^{(2)} \geq \dots,$$

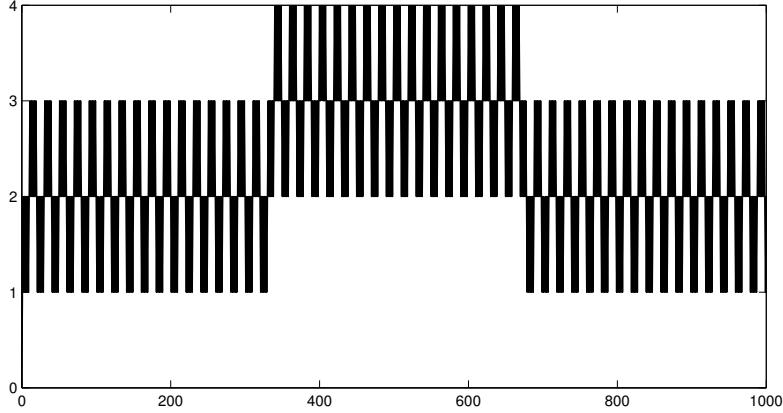
there is a dense set of θ for which

$$c_n^{(i)} \in o(M_n(0)), \quad |m_n(0)| \in o(d_n^{(i)})$$

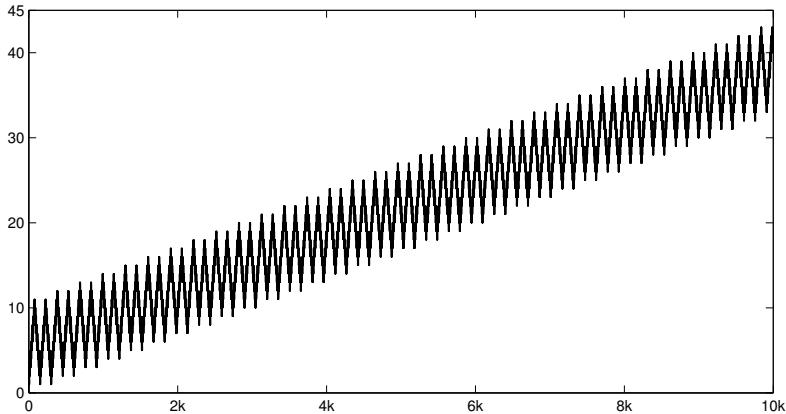
for all i .

Proof. Apply Theorem 1.2 after using a diagonalization process to construct c_n and d_n , both monotone, divergent, and in $o(n)$ such that

$$c_n^{(i)} \in o(c_n), \quad d_n \in o(d_n^{(i)}). \blacksquare$$



(a) θ exhibiting very slow growth of $M_n(0)$; this portion of the graph will repeat 2^{16} times with no additional growth.



(b) $\gamma(\theta)$ exhibiting very fast growth of $M_n(0)$; this sawtooth pattern will continue to climb by repeating itself $E(2^{16})/2$ times.

Fig. 4. Two different extreme growth rates for θ and $\gamma(\theta)$

Many variations of the above corollary are possible. For example, we may construct a dense set of θ such that the discrepancy sums grow in both directions faster than any $n^{1-\epsilon}$ (but necessarily in $o(n)$, of course!), or such that the discrepancy sums are bounded below, but $M_n(0)$ grows slower

than all iterated logarithms (but necessarily divergent, of course!), etc. See Figure 4 for an example where for both θ and $\gamma(\theta)$ we have $m_n \geq 1$, but $M_n(\theta) \notin o(n^{1-\epsilon})$ for any $\epsilon > 0$ while $M_n(\gamma(\theta)) \in o(\log^{(i)} n)$ for all i . In Figure 4 we set

$$\theta = [2, 2^2, 2, 2^{2^2}, 2, 2^{2^{2^2}}, 2, \dots].$$

Using diagonalization techniques one may similarly find a dense set of θ such that

$$\limsup_{i \rightarrow \infty} \frac{M_{n_i(j)}(0)}{c_{n_i(j)}^{(j)}} = 1$$

for an arbitrary collection of divergent sequences $c_n^{(j)}$ in $o(n)$ for different subsequences $n_i(j) \rightarrow \infty$ depending on j , and similarly for the $|m_n(0)|$ and a collection of sequences $d_n^{(j)}$.

Truly, beyond the constraints of (1.2), any asymptotic behavior desired is possible.

7. Proof of Theorem 1.3.

Suppose that

$$(7.1) \quad \liminf_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} = r_1, \quad \limsup_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} = r_2.$$

That the set of accumulation points of the sequence is the entire closed interval $[r_1, r_2]$ is straightforward and left to the reader. Let an arbitrary finite string of partial quotients a_1, \dots, a_N be given which determine θ_i for $i = 0, 1, \dots, n - 1$, and for convenience again assume without loss of generality that $p_n = 0$.

Now let c_n and d_n be arbitrary integer-valued strictly increasing sequences such that Δc_n and Δd_n are in $O(1)$ and

$$\liminf_{n \rightarrow \infty} \frac{c_n}{d_n} = \rho_1, \quad \limsup_{n \rightarrow \infty} \frac{c_n}{d_n} = \rho_2.$$

Furthermore, assume that $c_1 > M(\Omega'_n) = M$ and $d_1 > |m(\Omega'_n)| = m$.

Continue the continued fraction expansion of θ in the following way:

$$\theta_n = [2(c_1 - M) + 1, 2(d_1 - m), 2(c_2 - c_1), 2(d_2 - d_1), \dots].$$

Then Ω'_n will see the sequence of $M(\Omega'_{n+2k}) = c_k$ and $m(\Omega'_{n+2k}) = -d_k$; the bounded differences Δc_n and Δd_n ensure that the limiting behavior is the same as the limiting behavior along the subsequence of times $|\Omega'_n|$.

EXAMPLE 7.1. Suppose that $\theta = [1, 2, 3, 4, \dots]$. Then we begin computing the sequence of values $M_n(0)$ and $|m_n(0)|$ according to Proposition 5.2:

	$\theta_0 = [1, 2, 3, 4, \dots]$	$p = 0$	$E(a_1) = 0$	$(M, m) = (1, 1)$
	$\theta_1 = [3, 3, 4, 5, \dots]$	$p = 1$	$E(a_1) = 1$	$(M, m) = (1, 0)$
	$\theta_2 = [1, 3, 4, 5, \dots]$	$p = 1$	$E(a_1) = 0$	$(M, m) = (1, 0)$
	$\theta_3 = [4, 4, 5, 6, \dots]$	$p = 0$	$E(a_1) = 2$	$(M, m) = (3, 0)$
(7.2)	$\theta_4 = [5, 6, 7, 8, \dots]$	$p = 0$	$E(a_1) = 2$	$(M, m) = (5, 0)$
	$\theta_5 = [1, 6, 7, 8, \dots]$	$p = 0$	$E(a_1) = 0$	$(M, m) = (5, 0)$
	$\theta_6 = [7, 7, 8, 9, \dots]$	$p = 1$	$E(a_1) = 3$	$(M, m) = (5, 3)$
	\vdots	\vdots	\vdots	\vdots

The pattern is seen to continue in groups of five terms. Over the terms θ_{5k} through θ_{5k+4} , we will subtract $2k+1$ from m while adding $2(2k+2)$ to M . We therefore have $\rho_1 = \rho_2 = 2$, or

$$\lim_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} = 2.$$

See Figure 5 for this θ .

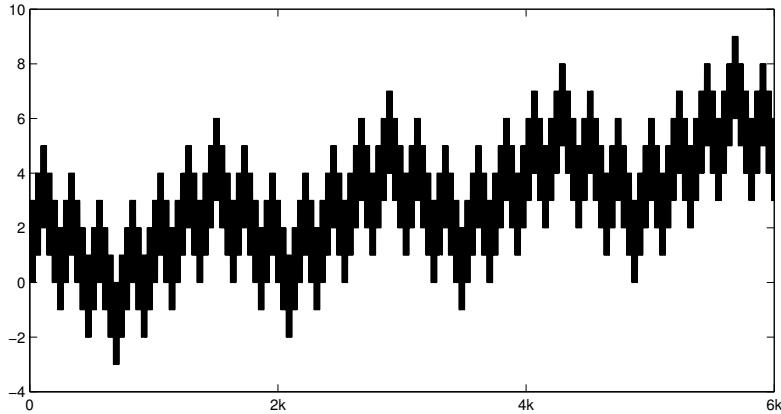


Fig. 5. A specific θ for which $M_n(0)/|m_n(0)|$ has limit two; refer to (7.2) and note the changes to M , m .

8. Proof of Theorem 1.4

LEMMA 8.1. *Suppose that $f(x)$ is a step function on S^1 with $k < \infty$ discontinuities, and denote by $V(f)$ the variation of f . Define $S_n(x)$, $M_n(x)$ and $m_n(x)$ as before. As we have not restricted f to be integer-valued, define*

$$\rho_N(x) = (M_N - m_N)(x).$$

Let n be such that $q_n \leq N < q_{n+1}$. Then for any $x, y \in S^1$,

$$\rho_N(y) \leq \rho_{q_{n+2}}(x) + a_{n+1}V(f).$$

Proof. Consider the set $\{x+i\theta\}$ for $i = 0, 1, \dots, q_n - 1$. Choose $0 \leq j < q_n$ such that $x + j\theta$ is closest to y . Then the distance between $x + j\theta$ and y is no larger than q_n^{-1} . For each discontinuity d_i there are therefore at most a_{n+1} preimages of d_i within this interval for time $L = 0, 1, \dots, q_{n+1} - 1$. It follows that $f(x + (j+i)\theta) = f(y + i\theta)$ for all but at most $k \cdot a_{n+1}$ of $i = 0, 1, \dots, N < q_{n+1}$. As $j+i$ is less than $q_n + q_{n+1} \leq q_{n+2}$, the lemma follows. ■

Assume that $a_i(\theta) \leq M$ for all i . Then (continuing with existing notation) we see that for some $C > 1$ independent of θ ,

$$(8.1) \quad C^{(n-1)/2} \leq |\Omega'_n| \leq (M+1)^{2n+2}.$$

The lower bound is due to the exponential decay in the length of the interval \tilde{I}_n (any $C < 2$ eventually suffices, as \tilde{I}_{n+1} is less than half as large as \tilde{I}_n at least half the time, with the $n-1$ accounting for the possibility that $I'_1 = I_0$, or $\theta_0 > 1/2$). The upper bound follows from Lemmas 5.4 and 5.5, and from the bound $a_i(\theta) \leq M$, while at the same time,

$$(8.2) \quad \frac{n-1}{2} \leq \rho_{|\Omega'_n|}(0) \leq \frac{nM}{2};$$

the lower inequality is due to the fact that at most half of the words satisfy $\Omega'_n = \Omega'_{n+1}$ (corresponding to those $\theta_n > 1/2$), and for the rest, $\rho(\Omega_{n+1}) \geq \rho(\Omega_n) + 1$, as $E(a_1) \geq 1$ for these $\theta_n < 1/2$. The upper bound follows as $E(a_i(\theta)) \leq M/2$ for all i .

Now, for any N let k be chosen such that

$$|\Omega_k| \leq N \leq |\Omega_{k+1}|.$$

From (8.1),

$$kC_1 \leq \log |\Omega'_k| \leq \log(N) \leq \log |\Omega'_{k+1}| \leq kC_2$$

for two constants C_1 and C_2 which do not depend on k . From (8.2),

$$\frac{(k+1)M}{2} \geq \rho_{|\Omega'_{k+1}|}(0) \geq \rho_N(0) \geq \rho_{|\Omega'_k|}(0) \geq \frac{k-1}{2},$$

so $\rho_n(0) \sim \log(n)$. The full theorem now follows from Lemma 8.1.

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References

- [1] B. Adamczewski, *Codages de rotations et phénomènes d'autosimilarité*, J. Théor. Nombres Bordeaux 14 (2002), 351–386.
- [2] B. Adamczewski, *Répartition des suites $(n\alpha)_{n \in \mathbb{N}}$ et substitutions*, Acta Arith. 112 (2004), 1–22.
- [3] B. Adamczewski, *Symbolic discrepancy and self-similar dynamics*, Ann. Inst. Fourier (Grenoble) 54 (2004), 2201–2234.
- [4] M. Boshernitzan and D. Ralston, *Continued fractions and heavy sequences*, Proc. Amer. Math. Soc. 137 (2009), 3177–3185.
- [5] J.-M. Dumont et A. Thomas, *Systèmes de numération et fonctions fractales relatifs aux substitutions*, Theoret. Comput. Sci. 65 (1989), 153–169.
- [6] Y. Dupain and V. T. Sós, *On the one-sided boundedness of discrepancy-function of the sequence $\{n\alpha\}$* , Acta Arith. 37 (1980), 363–374.
- [7] F. Durand, *A characterization of substitutive sequences using return words*, Discrete Math. 179 (1998), 89–101.
- [8] N. P. Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, edited by V. Berthé et al., Lecture Notes in Math. 1794, Springer, Berlin, 2002.
- [9] D. Hensley, *Continued Fractions*, World Sci., Hackensack, NJ, 2006.
- [10] H. Kesten, *On a conjecture of Erdős and Szusz related to uniform distribution mod 1*, Acta Arith. 12 (1966/1967), 193–212.
- [11] A. Ya. Khinchin, *Continued Fractions*, Dover, Mineola, NY, 1997.
- [12] C. Pinner, *On sums of fractional parts $\{n\alpha + \gamma\}$* , J. Number Theory 65 (1997), 48–73.
- [13] C. G. Pinner, *On the one-sided boundedness of sums of fractional parts $(\{n\alpha + \gamma\} - 1/2)$* , J. Number Theory 81 (2000), 170–204.
- [14] L. Roçadas and J. Schoißengeier, *On the local discrepancy of $(n\alpha)$ -sequences*, J. Number Theory 131 (2011), 1492–1497.
- [15] J. Schoißengeier, *On the discrepancy of $(n\alpha)$* , Acta Arith. 44 (1984), 241–279.
- [16] V. T. Sós, *On the discrepancy of the sequence $\{n\alpha\}$* , in: Topics in Number Theory (Debrecen, 1974), Colloq. Math. Soc. János Bolyai 13, North-Holland, Amsterdam, 1976, 359–367.

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