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The prime number theorem in short intervals for automorphic L-functions

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1. Introduction. The well known Legendre conjecture states that there is at least one prime number between n^2 and $(n+1)^2$ for each positive integer n. A related problem is the existence of primes in short intervals. Denote, as usual, by $\zeta(s)$ the Riemann zeta-function, and define the von Mangoldt function $\Lambda(n)$ by

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (\sigma > 1),$$

where $s = \sigma + i\tau$. Then

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\nu \text{ with } \nu \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Write

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

It is known that, under the Riemann Hypothesis (RH in brief) for $\zeta(s)$,

$$\psi(x) = x + O\big(x^{1/2}(\log x)^2\big) \qquad (x \geq 2).$$

From this we immediately deduce that, under RH,

$$\psi(x + h(x)) - \psi(x) \sim h(x) \qquad (x \to \infty) \tag{1.1}$$

for any increasing functions $h(x) \leq x$ satisfying

$$\frac{h(x)}{x^{1/2}(\log x)^2} \to \infty \qquad \text{as } x \to \infty.$$

It seems an interesting problem to determine how short $h(x)$ can be. According to Cramér's model, we could take $h(x)/(\log x)^2 \to \infty$ as $x \to \infty$. In

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1943, Selberg [12] partially confirmed this by showing that under RH the asymptotic relationship

$$\int_1^X |\psi(x+h(x)) - \psi(x) - h(x)|^2\,dx = o(h(X)^2X) \quad (X\to\infty) \tag{1.2}$$

holds for any increasing function $h(x) \le x$ satisfying

$$h(x)/(\log x)^2 \to \infty \quad \text{as } x\to\infty. \tag{1.3}$$

This shows that, under RH, (1.1) holds for almost all $x \ge 2$ provided (1.3) is satisfied.

In order to better understand the connection between the distribution of zeros of $\zeta(s)$ and that of primes, Montgomery [9] introduced the pair correlation function

$$F_T(X) := \sum_{0<\gamma_1,\gamma_2\le T} W(\gamma_1-\gamma_2)e^{2\pi iX(\gamma_1-\gamma_2)}, \tag{1.4}$$

where

$$W(u) := \frac{4}{4+u^2}$$

and γ runs over the imaginary parts $\Im m\,\rho$ of the nontrivial zeros ρ of $\zeta(s)$ (counted according to multiplicity). Assuming RH and that

$$F_T\biggl(\frac{\log x}{2\pi}\biggr) \ll T\log T \tag{1.5}$$

uniformly for $x(\log x)^{-3} \le T \le x$, Heath-Brown [2] showed that (1.2) holds for any increasing function $h(x) \le x$ satisfying

$$h(x)/\log x \to \infty \quad \text{as } x\to\infty. \tag{1.6}$$

In this paper, we shall investigate analogues of (1.2) for automorphic L-functions. Let us fix our notation first. To each irreducible unitary cuspidal representation $\pi = \otimes\pi_p$ of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \ge 2$, one can attach a global L-function

$$L(s,\pi) = \prod_{p<\infty} L_p(s,\pi_p) \tag{1.7}$$

converging for $\sigma > 1$ (see [5]), where the local factors are given by

$$L_p(s,\pi_p) = \prod_{j=1}^m (1-\alpha_\pi(p,j)p^{-s})^{-1}. \tag{1.8}$$

The complete L-function $\Phi(s,\pi)$ is defined by

$$\Phi(s,\pi) = L(s,\pi)L_\infty(s,\pi_\infty), \tag{1.9}$$

where $N_\pi \geq 1$ is an integer called the *arithmetic conductor* of π, and

$$L_\infty(s, \pi_\infty) := \left(\frac{N_\pi}{\pi^m}\right)^{s/2} \prod_{j=1}^{m} \Gamma\left(\frac{s + \mu_\pi(j)}{2}\right) \tag{1.10}$$

is the Archimedean local factor. Here $\{\alpha_\pi(p, j)\}_{j=1}^m$ and $\{\mu_\pi(j)\}_{j=1}^m$ are complex numbers associated with π_p and π_∞, respectively, according to the Langlands correspondence. Good bounds for these local parameters are of fundamental importance for the study of automorphic L-functions. Thanks to the work of Luo–Rudnick–Sarnak [8], it is known that

$$\begin{cases} |\alpha_\pi(p, j)| \leq p^\theta & \text{if } \pi \text{ is unramified at } p, \\ |\Re e\, \mu_\pi(j)| \leq \theta & \text{if } \pi \text{ is unramified at } \infty, \end{cases} \tag{1.11}$$

with $\theta = 1/2 - 1/(m^2 + 1)$. The *Generalized Ramanujan Conjecture* (GRC in brief) asserts that (1.11) should hold with $\theta = 0$. It also follows from work of Shahidi [13–16] that the complete L-function $\Phi(s, \pi)$ has an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Phi(s, \pi) = \varepsilon_\pi \Phi(1 - s, \widetilde{\pi}), \tag{1.12}$$

where ε_π is the root number satisfying $|\varepsilon_\pi| = 1$, and $\widetilde{\pi}$ is the representation contragredient to π. The important quantity

$$Q_\pi := N_\pi \prod_{j=1}^{m} (3 + |\mu_\pi(j)|)$$

is named the *conductor* of π.

Similarly to the classical case, we define $\Lambda_\pi(n)$ by taking logarithmic differentiation in (1.7).

$$-\frac{L'}{L}(s, \pi) = \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s} \qquad (\sigma > 1). \tag{1.13}$$

With the help of (1.8), it is easy to see that

$$\Lambda_\pi(n) = \begin{cases} \sum_{j=1}^m \alpha_\pi(p, j)^\nu \log p & \text{if } n = p^\nu \text{ with } \nu \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.14}$$

The prime number theorem for $L(s, \pi)$ concerns the asymptotic behavior of the counting function

$$\psi(x, \pi) := \sum_{n \leq x} \Lambda_\pi(n). \tag{1.15}$$

This problem was first studied by Liu & Ye [7] and Qu [10, 11]. In particular Qu [11] proved that, under the Generalized Riemann Hypothesis (GRH in brief) for $L(s,\pi)$, we have

$$\int_1^X |\psi(x+h(x),\pi)-\psi(x,\pi)|^2\,dx = o(h(X)^2X) \quad (X\to\infty) \tag{1.16}$$

for any increasing function $h(x)\le x$ satisfying

$$\frac{h(x)}{x^\theta(\log x)^2}\to\infty \quad \text{as } x\to\infty,$$

where θ is given by (1.11).

The first aim of this paper is to improve the above result by removing x^θ, which offers an exact generalization of Selberg's (1.2) and (1.3) to automorphic L-functions.

THEOREM 1.1. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with $m\ge 2$. Assume GRH for $L(s,\pi)$. Then for $X\ge 2$ we have*

$$\int_1^X |\psi(x+h(x),\pi)-\psi(x,\pi)|^2\,dx \ll h(X)X\log^2(Q_\pi X)+\left(\frac{\log Q_\pi}{\log X}\right)^4 \tag{1.17}$$

for any increasing function $h(x)\le x$, where the implied constant depends only on m. In particular, (1.16) *holds for any increasing function $h(x)\le x$ satisfying*

$$h(x)/(\log x)^2\to\infty \quad \textit{as } x\to\infty. \tag{1.18}$$

Our second aim in this paper is to consider the analogue of Heath-Brown's (1.2) and (1.6). Similar to (1.4), we can also define

$$F_T^\pi(X) := \sum_{0<\gamma_1,\gamma_2\le T} W(\gamma_1-\gamma_2)e^{2\pi iX(\gamma_1-\gamma_2)},$$

where γ runs over imaginary parts $\Im m\,\rho$ of the nontrivial zeros ρ of $L(s,\pi)$ (counted according to multiplicity).

THEOREM 1.2. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with $m\ge 2$. Assume GRH for $L(s,\pi)$, and*

$$F_T^\pi\left(\frac{\log X}{2\pi}\right)\ll T\log(Q_\pi T) \tag{1.19}$$

uniformly for $T\le (X\log X)^2$. Then for $X\ge 2$ we have

$$\int_1^X |\psi(x+h(x),\pi)-\psi(x,\pi)|^2\,dx \ll h(X)X\log(Q_\pi X) + X\log^2(Q_\pi X)+\left(\frac{\log Q_\pi}{\log X}\right)^4 \tag{1.20}$$

for any increasing functions $h(x) \le x$, where the implied constant depends only on m. In particular, (1.16) *holds for any increasing function $h(x) \le x$ satisfying*

$$h(x)/\log x \to \infty \quad \textit{as } x \to \infty. \tag{1.21}$$

Theorem 1.1 (resp. Theorem 1.2) shows that, under GRH (resp. under GRH and (1.19)) for almost all $x \ge 2$, we have

$$\psi(x + h(x), \pi) - \psi(x, \pi) = o(h(x)) \quad (x \to \infty),$$

provided (1.18) (resp. (1.21)) is satisfied. Thus the sequence $\{\Lambda_\pi(n)\}_{n\ge1}$ changes sign (unlike in the classical case $\{\Lambda(n)\}_{n\ge1}$). Very recently, Liu, Qu & Wu [6] showed that if $\Lambda_\pi(n)$ is real for all $n \ge 1$, then there is some n satisfying

$$n \ll_{m,\varepsilon} Q_\pi^{1+\varepsilon}$$

such that $\Lambda_\pi(n) < 0$. The implied constant depends only on m and ε. In particular, this result is true for any self-contragredient irreducible unitary cuspidal representation π for $GL_m(\mathbb{A}_\mathbb{Q})$ with trivial central character.

The main new ideas for proving Theorems 1.1 and 1.2 are a delicate application of Iwaniec–Kowalski's mean value estimate (cf. (3.3) below) and an explicit formula in a more precise form adapted to our purpose (cf. Lemma 3.1 below).

2. Preliminary lemmas. In view of (1.10) and the fact that $L(s,\pi)$ and $\Phi(s,\pi)$ are entire, it is not difficult to see that the trivial zeros of $L(s,\pi)$ and the poles of $L_\infty(1-s,\widetilde{\pi}_\infty)$ are

$$\mu := -2n - \mu_\pi(j) \qquad \text{for } n = 0,1,\ldots;\ j = 1,\ldots,m, \tag{2.1}$$

$$P_{n,j} := 2n + 1 + \mu_{\widetilde{\pi}}(j) \qquad \text{for } n = 0,1,\ldots;\ j = 1,\ldots,m, \tag{2.2}$$

respectively. As in [7], we let $\mathbb{C}(m)$ denote the complex plane with the discs

$$|s - P_{n,j}| < (8m)^{-1} \qquad \text{for } n = 0,1,\ldots;\ j = 1,\ldots,m$$

removed. Thus, for any $s \in \mathbb{C}(m)$, the quantity $(1 - s + \mu_{\widetilde{\pi}}(j))/2$ is away from all poles of $\Gamma(s)$ by at least $(16m)^{-1}$. For $j = 1,\ldots,m$, denote by $\beta(j)$ the fractional part of $\Re e\, \mu_{\widetilde{\pi}}(j)$. In addition, let $\beta(0) = 0$ and $\beta(m+1) = 1$. Then all $\beta(j)$ are in $[0,1]$, and hence there exist $\beta(j_1), \beta(j_2)$ such that $\beta(j_2) - \beta(j_1) \ge 1/(3m)$ and there is no $\beta(j)$ lying between $\beta(j_1)$ and $\beta(j_2)$. Consequently, for all $n = 0,1,\ldots,$ the strips

$$S_{-n} := \{s \in \mathbb{C} : -n + \beta(j_1) + (8m)^{-1} \le \Re e\, s \le -n + \beta(j_2) - (8m)^{-1}\} \tag{2.3}$$

are subsets of $\mathbb{C}(m)$.

The following assertions (i) and (ii) are Lemmas 4.3(d) and 4.4 of [7], respectively.

Lemma 2.1. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \ge 2$.*

(i) *For $T \ge 2$, there exists τ_T with $T \le \tau_T \le T+1$ such that*

$$-\frac{L'}{L}(\sigma \pm i\tau_T, \pi) \ll \log^2(Q_\pi T) \qquad (|\sigma| \le 2).$$

(ii) *If s is in some strip S_{-n} as in (2.3) with $n \ge 2$, then*

$$-\frac{L'}{L}(s,\pi) \ll 1.$$

The implied constants depend only on m.

The next lemma is about the distribution of zeros of $L(s,\pi)$. For its proof, one is referred to Lemma 4.3 of Liu & Ye [7], or Theorem 5.8 of Iwaniec & Kowalski [3].

Lemma 2.2. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \ge 2$. All the nontrivial zeros of $\Phi(s,\pi)$ are in the critical strip $0 \le \sigma \le 1$. Let $N(T,\pi)$ be the number of its nontrivial zeros within the rectangle $0 \le \sigma \le 1$ and $|\tau| \le T$. Then*

$$N(T,\pi) \ll T\log(Q_\pi T), \tag{2.4}$$

$$N(T+1,\pi) - N(T,\pi) \ll \log(Q_\pi T), \tag{2.5}$$

where the implied constants are absolute.

3. An explicit formula. Explicit formulae of different forms were established by many authors. In particular, under GRC, explicit formulae for general L-functions were proved in [3, (5.53)]. The explicit formula below is unconditional, and plays a key role in the proofs of Theorems 1.1 and 1.2.

Lemma 3.1. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \ge 2$, and let $A > 0$. Then, for $x \ge 2$ and $2 \le T \le x^A$, we have*

$$\psi(x,\pi) = -\sum_{|\gamma|\le T}\frac{x^\rho}{\rho} - \sum_{\substack{\kappa'<\lambda<\kappa \\ |\nu|\le T}}\frac{x^\mu}{\mu} - \frac{L'}{L}(0,\pi) + O(R_\pi(x,T)), \tag{3.1}$$

where

$$R_\pi(x,T) := \sum_{|n-x|\le x/\sqrt{T}} |\Lambda_\pi(n)| + \frac{x(\log Q_\pi)\log(Q_\pi x)}{\sqrt{T}} + \frac{x\log^2(Q_\pi x)}{T} + \frac{\log T}{x},$$

$-2 < \kappa' < -1$, $\kappa = 1 + 1/\log x$, *and μ (resp. ρ) runs over the trivial zeros $\mu = \lambda + i\nu$ (resp. the nontrivial zeros $\rho = \beta + i\gamma$) of $L(s,\pi)$. The implied constant depends only on A and m.*

Proof. Since the series (1.13) converges absolutely for $\sigma > 1$, we can apply the Perron formula [17, Theorem II.2.2] with $\kappa = 1 + 1/\log x$, so that

$$\psi(x,\pi) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} -\frac{L'}{L}(s,\pi)\frac{x^s}{s}\,ds + O\Big(x^\kappa \sum_{n=1}^{\infty} \frac{|\Lambda_\pi(n)|}{n^\kappa(1+T|\log(x/n)|)}\Big). \tag{3.2}$$

In order to treat the O-term, we split the sum into two parts according to whether

$$|x-n| \le x/\sqrt{T} \quad \text{or} \quad |x-n| > x/\sqrt{T}.$$

By the Cauchy–Schwarz inequality, it follows that

$$\sum_{|x-n|>x/\sqrt{T}} \frac{|\Lambda_\pi(n)|}{n^\kappa(1+T|\log(x/n)|)} \ll \frac{1}{\sqrt{T}}\sum_{n=1}^{\infty}\frac{|\Lambda_\pi(n)|}{n^\kappa} \ll \frac{1}{\sqrt{T}}\Big(\sum_{n=1}^{\infty}\frac{|\Lambda_\pi(n)|^2}{n^\kappa}\Big)^{1/2}\Big(\sum_{n=1}^{\infty}\frac{1}{n^\kappa}\Big)^{1/2}.$$

According to [3, (5.48)], we have

$$\sum_{n\le u}|\Lambda_\pi(n)|^2 \ll m^2 u \log^2(Q_\pi u) \qquad (u \ge 1), \tag{3.3}$$

where the implied constant is absolute. Thus a simple integration by parts gives us

$$\sum_{n=1}^{\infty}\frac{|\Lambda_\pi(n)|^2}{n^\kappa} = \int_{1-}^{\infty}\frac{1}{u^\kappa}\,d\Big(\sum_{n\le u}|\Lambda_\pi(n)|^2\Big) \ll_m \int_{1}^{\infty}\frac{\log^2(Q_\pi u)}{u^\kappa}\,du \ll_m \frac{\log^2 Q_\pi}{\kappa-1} + \frac{1}{(\kappa-1)^3}.$$

Similarly but more easily, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^\kappa} \ll \frac{1}{\kappa-1}.$$

Combining these estimates, we find that

$$x^\kappa \sum_{|x-n|>x/\sqrt{T}} \frac{|\Lambda_\pi(n)|}{n^\kappa(1+T|\log(x/n)|)} \ll \frac{x(\log x)\log(Q_\pi x)}{\sqrt{T}}. \tag{3.4}$$

Next, we shall evaluate the integral on the right-hand side of (3.2). For this purpose, we shift the contour of integration to the left. Choose κ' with $-2 < \kappa' < -1$ such that the vertical line $\sigma = \kappa'$ is contained in the strip $S_{-2} \subset \mathbb{C}(m)$; this is guaranteed by the structure of $\mathbb{C}(m)$. Without loss of generality, let $T \geq 2$ be a large number such that T and $-T$ can be taken as the τ_T in Lemma 2.1(i). Now we consider the contour $\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3$ with

$$\mathscr{L}_1 := [\kappa' - iT, \kappa - iT], \quad \mathscr{L}_2 := [\kappa' - iT, \kappa' + iT], \quad \mathscr{L}_3 := [\kappa' + iT, \kappa + iT].$$

By Lemma 2.2 and (2.1), certain nontrivial zeros $\rho = \beta + i\gamma$ and trivial zeros $\mu = \lambda + i\nu$ of $L(s, \pi)$, as well as $s = 0$ are passed by the shifting of the contour.

Computing the residues, we have

$$\frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} -\frac{L'}{L}(s, \pi) \frac{x^s}{s}\, ds = -\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq T}} \frac{x^\mu}{\mu} - \frac{L'}{L}(0, \pi) - \frac{1}{2\pi i} \int_{\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3} -\frac{L'}{L}(s, \pi) \frac{x^s}{s}\, ds. \tag{3.5}$$

The integral on $\mathscr{L}_1$ can be estimated by Lemma 2.1(i) as

$$\frac{1}{2\pi i} \int_{\mathscr{L}_1} -\frac{L'}{L}(s, \pi) \frac{x^s}{s}\, ds \ll \int_{\kappa'}^{\kappa} \log^2(Q_\pi T) \frac{x^\sigma}{T}\, d\sigma \ll \frac{x \log^2(Q_\pi T)}{T},$$

and the same upper bound also holds for the integral on $\mathscr{L}_3$. By Lemma 2.1(ii),

$$\frac{1}{2\pi i} \int_{\mathscr{L}_2} -\frac{L'}{L}(s, \pi) \frac{x^s}{s}\, ds \ll \int_{-T}^{T} \frac{x^{\kappa'}}{|\tau| + 1}\, dt \ll \frac{\log T}{x}.$$

Therefore, (3.5) becomes

$$\frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} -\frac{L'}{L}(s, \pi) \frac{x^s}{s}\, ds = -\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq T}} \frac{x^\mu}{\mu} - \frac{L'}{L}(0, \pi) + O\left(\frac{x(\log x)\log(Q_\pi x)}{\sqrt{T}} + \frac{x \log^2(Q_\pi T)}{T} + \frac{\log T}{x}\right).$$

Inserting the above formula and (3.4) into (3.2), we obtain the required result. ∎

4. Gallagher lemma and proof of Theorem 1.1. Our main tool is the following lemma of Gallagher [1, Lemma 1].

Lemma 4.1. *Let $U > 0$ and $\delta = \vartheta/U$ with $0 < \vartheta < 1$, and let*

$$S(u) := \sum_\nu c(\nu) e^{2\pi i \nu u}$$

be absolutely convergent, where $c(\nu) \in \mathbb{C}$, and the frequencies ν run over an arbitrary sequence of real numbers. Then

$$\int_{-U}^{U} |S(u)|^2\,du \ll_\vartheta U^2 \int_{-\infty}^{+\infty} \Big|\sum_{t<\nu\le t+\delta} c(\nu)\Big|^2\,dt.$$

Now we prove Theorem 1.1. Let $10^4 \le X \le x \le 2X$, and take $T = (X\log X)^2$ in the explicit formula (3.1) of Lemma 3.1. Since the length of the interval $(x - x/(X\log X), x + x/(X\log X)]$ is $2x/(X\log X) \le 1/2$, this interval contains at most one integer; we denote this possible integer by n_x. Thus our explicit formula becomes

$$\psi(x,\pi) = -\sum_{|\gamma|\le (X\log X)^2} \frac{x^\rho}{\rho} - \sum_{\substack{\kappa'<\lambda<\kappa\\ |\nu|\le (X\log X)^2}} \frac{x^\mu}{\mu} - \frac{L'}{L}(0,\pi)$$
$$+ O\bigg(|\Lambda_\pi(n_x)| + \log(Q_\pi X) + \frac{(\log Q_\pi)^2}{X(\log X)^2}\bigg),$$

where the implied constant depends only on m. From this we can write

$$\psi(x+h,\pi) - \psi(x,\pi) = A + B + O\bigg(C + \log(Q_\pi X) + \frac{(\log Q_\pi)^2}{X(\log X)^2}\bigg),$$

where $h \le 2X \le 2x$ and

$$A := -\sum_{|\gamma|\le (X\log X)^2} \frac{(x+h)^\rho - x^\rho}{\rho},$$
$$B := -\sum_{\substack{\kappa'<\lambda<\kappa\\ |\nu|\le (X\log X)^2}} \frac{(x+h)^\mu - x^\mu}{\mu},$$
$$C := |\Lambda_\pi(n_{x+h})| + |\Lambda_\pi(n_x)|.$$

Clearly,

$$\int_X^{2X} |\psi(x+h,\pi) - \psi(x,\pi)|^2\,dx \ll \int_X^{2X} (|A|^2 + |B|^2 + |C|^2)\,dx + X\log^2(Q_\pi X) + \frac{(\log Q_\pi)^4}{X(\log X)^4}. \tag{4.1}$$

We start from A. In A, we split the sum over $|\gamma|$ at T, with $4 \le T \le (X\log X)^2$ a parameter that will be specified later, and define

$$S_1(y) := \sum_{|\gamma|\le T} y^{i\gamma} \quad \text{and} \quad S_2(y) := \sum_{T<|\gamma|\le (X\log X)^2} \frac{y^{i\gamma}}{\rho}.$$

Under GRH, the sum in A runs over the nontrivial zeros $\rho = 1/2 + i\gamma$ of $L(s,\pi)$ with $|\gamma|$ up to $(X\log X)^2$. Thus we can write

$$\begin{aligned}
A &= -\sum_{|\gamma|\le T} \frac{(x+h)^\rho - x^\rho}{\rho} - \sum_{T<|\gamma|\le (X\log X)^2} \frac{(x+h)^\rho - x^\rho}{\rho} \\
&= -\sum_{|\gamma|\le T} \int_x^{x+h} y^{\rho-1}\,dy - \sum_{T<|\gamma|\le (X\log X)^2} \frac{(x+h)^{1/2+i\gamma} - x^{1/2+i\gamma}}{\rho} \\
&= -\int_x^{x+h} \frac{S_1(y)}{y^{1/2}}\,dy - (x+h)^{1/2} S_2(x+h) + x^{1/2} S_2(x) =: A_1 + A_2 + A_3,
\end{aligned}$$

say. By the Cauchy–Schwarz inequality,

$$|A_1|^2 \le h \int_x^{x+h} \frac{|S_1(y)|^2}{y}\,dy.$$

In view of $h \le 2X$, the contribution from $|A_1|^2$ is estimated as

$$\int_X^{2X} |A_1|^2\,dx \ll h \int_X^{2X} \bigg(\int_x^{x+h} \frac{|S_1(y)|^2}{y}\,dy \bigg)\,dx \ll h^2 \int_X^{4X} \frac{|S_1(y)|^2}{y}\,dy.$$

Changing variable $y = Xe^{2\pi u}$ and applying Lemma 4.1 and (2.5), it follows

$$\begin{aligned}
\int_X^{2X} |A_1|^2\,dx &\ll h^2 \int_0^{(\log 2)/\pi} \Big| \sum_{|\gamma|\le T} X^{i\gamma} e^{2\pi i\gamma u} \Big|^2\,du \\
&\ll h^2 \int_{-\infty}^{+\infty} \Big(\sum_{|\gamma|\le T,\, t<\gamma\le t+1} 1 \Big)^2\,dt \\
&\ll h^2 \int_0^T \Big(\sum_{t<\gamma\le t+1} 1 \Big)^2\,dt \ll h^2 T \log^2(Q_\pi T).
\end{aligned} \tag{4.2}$$

The contribution from $|A_2|^2$ can be estimated as

$$\begin{aligned}
\int_X^{2X} |A_2|^2\,dx &\ll X^2 \int_X^{4X} \frac{|S_2(x)|^2}{x}\,dx \\
&= X^2 \int_0^{(\log 2)/\pi} \Big| \sum_{T<|\gamma|\le (X\log X)^2} \frac{X^{i\gamma}}{\rho} e^{2\pi i\gamma u} \Big|^2\,du \\
&\ll X^2 \int_{-\infty}^{+\infty} \Big(\sum_{T<|\gamma|\le (X\log X)^2,\, t<\gamma<t+1} \frac{1}{|\gamma|} \Big)^2\,dt \\
&\ll X^2 \int_{T-1}^{(X\log X)^2} \Big(\sum_{t<\gamma\le t+1} \frac{1}{|\gamma|} \Big)^2\,dt.
\end{aligned} \tag{4.3}$$

By using (2.5) and (2.4) of Lemma 2.2, a simple integration by parts gives

$$\sum_{t<\gamma\le t+1} \frac{1}{|\gamma|} = \int_t^{t+1} \frac{1}{u}\, dN(u,\pi) \ll \frac{\log(Q_\pi t)}{t}.$$

Thus

$$\int_X^{2X} |A_2|^2\, dx \ll X^2 \int_{T-1}^{(X\log X)^2} \frac{\log^2(Q_\pi t)}{t^2}\, dt \ll \frac{X^2\log^2(Q_\pi T)}{T}. \tag{4.4}$$

Similarly, after taking $x+h=y$, we have

$$\int_X^{2X} |A_3|^2\, dx \ll \frac{X^2\log^2(Q_\pi T)}{T}. \tag{4.5}$$

We conclude from (4.2), (4.4) and (4.5) that

$$\int_X^{2X} |A|^2\, dx \ll h^2 T\log^2(Q_\pi T) + \frac{X^2\log^2(Q_\pi T)}{T}. \tag{4.6}$$

For the mean-value of $|B|^2$, we apply (2.1) and (1.11) to get

$$\begin{aligned} \int_X^{2X} |B|^2\, dx &= \int_X^{2X} \Bigg| \sum_{\substack{\kappa'<\lambda<\kappa \\ |\nu|\le(X\log X)^2}} \frac{(x+h)^\mu - x^\mu}{\mu}\Bigg|^2 dx \\ &\ll \int_X^{2X} \Big(\sum_{\substack{\kappa'<\lambda<\kappa \\ |\nu|\le(X\log X)^2}} x^{\lambda-1}h\Big)^2 dx \ll \int_X^{2X} (x^{\theta-1}h)^2\, dx \ll h^2. \end{aligned} \tag{4.7}$$

It remains to estimate the contribution of $|C|^2$. We have

$$\begin{aligned} \int_X^{2X} |C|^2\, dx &= \int_X^{2X} (|\Lambda_\pi(n_{x+h})| + |\Lambda_\pi(n_x)|)^2\, dx \\ &\ll \sum_{j=[X]}^{[2X]} \int_j^{j+1} (|\Lambda_\pi(n_{x+h})|^2 + |\Lambda_\pi(n_x)|^2)\, dx. \end{aligned}$$

Since $h(x)$ is increasing and $h(x)\le x$, we have trivially, for $j\le x\le j+1$,

$$j-1\le n_{x+h(x)}\le 2(j+2), \qquad j-1\le n_x\le j+2.$$

Thus,

$$\int_X^{2X} |C|^2\, dx \ll \sum_{j=[X]-1}^{3[2X]} |\Lambda_\pi(j)|^2 \ll X\log^2(Q_\pi X), \tag{4.8}$$

by applying (3.3).

Finally inserting (4.6)–(4.8) to (4.1), and taking $T = X/h(2X)$, we find

$$\int_X^{2X} |\psi(x+h,\pi) - \psi(x,\pi)|^2\,dx \ll h(2X)X\log^2(Q_\pi X) + \frac{(\log Q_\pi)^4}{X(\log X)^4}$$

for any increasing function $h(x)$ satisfying $1 \le h(x) \le x$. A splitting-up argument then gives the required inequality (1.17).

5. Pair correlation of zeros and proof of Theorem 1.2. The proof of Theorem 1.2 is very similar to that of Theorem 1.1. The only difference is in estimating the contribution of $|A_i|^2$ with the help of hypothesis (1.19) instead of Gallagher's lemma and Lemma 2.2. We retain the notation of Section 4. According to the first line of (4.2), we have

$$\int_X^{2X} |A_1|^2\,dx \ll h^2 \int_0^{(\log 2)/\pi} \Big|\sum_{|\gamma|\le T} X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\,du.$$

In view of the trivial inequality $e^{-4\pi|u|} \gg 1$ $(0 \le u \le 1)$ and the classical formula

$$2\pi\int_{-\infty}^{+\infty} e^{-4\pi|u|+2\pi itu}\,du = W(t),$$

we can deduce

$$\begin{aligned}
\int_X^{2X} |A_1|^2\,dx &\ll h^2 \int_0^{(\log 2)/\pi} e^{-4\pi|u|}\Big|\sum_{|\gamma|\le T} X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\,du \\
&\ll h^2 \int_{-\infty}^{+\infty} e^{-4\pi|u|}\Big|\sum_{0<\gamma\le T} X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\,du \\
&\ll h^2 \sum_{0<\gamma_1,\gamma_2\le T} X^{i(\gamma_1-\gamma_2)}\int_{-\infty}^{+\infty} e^{-4\pi|u|}e^{2\pi i(\gamma_1-\gamma_2)u}\,du \\
&\ll h^2 \sum_{0<\gamma_1,\gamma_2\le T} X^{i(\gamma_1-\gamma_2)}W(\gamma_1-\gamma_2) \\
&= h^2 F_T^\pi\Big(\frac{\log X}{2\pi}\Big).
\end{aligned} \tag{5.1}$$

Assuming (1.19), we have

$$\int_X^{2X} |A_1|^2\,dx \ll h^2 T\log(Q_\pi T). \tag{5.2}$$

Next we estimate the contribution of $|A_2|^2$. By partial summation,

$$\sum_{T<\gamma\le(X\log X)^2}\frac{X^{i\gamma}}{\rho}e^{2\pi i\gamma u}=\int_T^{(X\log X)^2}\frac{1}{1/2+it}\,d\Big(\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\Big)$$
$$=\sum_{\gamma\le(X\log X)^2}\frac{X^{i\gamma}e^{2\pi i\gamma u}}{1/2+i(X\log X)^2}-\sum_{\gamma\le T}\frac{X^{i\gamma}e^{2\pi i\gamma u}}{1/2+iT}$$
$$+\,i\int_T^{(X\log X)^2}\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\frac{dt}{(1/2+it)^2}.$$

Thus

$$\Big|\sum_{T<\gamma\le(X\log X)^2}\frac{X^{i\gamma}}{\rho}e^{2\pi i\gamma u}\Big|^2\ll\frac{1}{X^4}\Big|\sum_{\gamma\le(X\log X)^2}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2$$
$$+\frac{1}{T^2}\Big|\sum_{\gamma\le T}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2$$
$$+\int_T^{(X\log X)^2}\Big|\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\frac{\log^2(2t/T)}{t^3}\,dt,$$

where we have used the estimate

$$\Big(\int_T^{(X\log X)^2}\Big|\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\Big|\frac{dt}{t^2}\Big)^2$$
$$\ll\int_T^{(X\log X)^2}\frac{dt}{t\log^2(2t/T)}\int_T^{(X\log X)^2}\Big|\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\frac{\log^2(2t/T)}{t^3}\,dt$$
$$\ll\int_T^{(X\log X)^2}\Big|\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\frac{\log^2(2t/T)}{t^3}\,dt.$$

In view of the first two lines of (4.3) and the estimate above, we can write

$$\int_X^{2X}|A_2|^2\,dx\ll X^2\Big\{\frac{1}{X^4}\int_0^{(\log 2)/\pi}\Big|\sum_{\gamma\le(X\log X)^2}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\,du$$
$$+\frac{1}{T^2}\int_0^{(\log 2)/\pi}\Big|\sum_{\gamma\le T}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\,du$$
$$+\int_T^{(X\log X)^2}\Big(\int_0^{(\log 2)/\pi}\Big|\sum_{\gamma\le t}X^{i\gamma}e^{2\pi i\gamma u}\Big|^2\,du\Big)\frac{\log^2(2t/T)}{t^3}\,dt\Big\}.$$

From this, a similar argument to (5.1) allows us to deduce

$$\int_X^{2X} |A_2|^2\,dx \ll X^2\biggl(\frac{1}{T} + \int_T^{(X\log X)^2} \log^2\Bigl(\frac{2t}{T}\Bigr)\frac{dt}{t^2}\biggr) \sup_{T\le t\le (X\log X)^2} \frac{1}{t}F_t^\pi\Bigl(\frac{\log X}{2\pi}\Bigr)$$
$$\ll \frac{X^2}{T} \sup_{T\le t\le (X\log X)^2} \frac{1}{t}F_t^\pi\Bigl(\frac{\log X}{2\pi}\Bigr).$$

Assuming (1.19), it follows that

$$\int_X^{2X} |A_2|^2\,dx \ll \frac{X^2}{T}\sup_{T\le t\le (X\log X)^2}\log(Q_\pi t) \ll \frac{X^2\log(Q_\pi X)}{T}.$$

The same estimate also holds for $\int_X^{2X} |A_3|^2\,dx$.

From these conclusions and (5.2), we get

$$\int_X^{2X} |A|^2\,dx \ll h^2 T\log(Q_\pi T) + \frac{X^2\log(Q_\pi X)}{T}. \tag{5.3}$$

Finally, inserting (5.3), (4.7) and (4.8) into (4.1), and taking $T = X/h(2X)$, we find that

$$\int_X^{2X} \bigl|\psi(x+h,\pi) - \psi(x,\pi)\bigr|^2\,dx \ll h(2X)X\log(Q_\pi X)$$
$$+ X\log^2(Q_\pi X) + \frac{(\log Q_\pi)^4}{X(\log X)^4}$$

for any increasing function $h(x)$ satisfying $1 \le h(x) \le x$. A splitting-up argument then gives the required inequality (1.20).

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