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On the Brun–Titchmarsh theorem

by

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1. Introduction. We let $\pi(x;q,a)$ denote the number of primes less than or equal to x which are congruent to $a \pmod q$, for some real $x>0$ and positive coprime integers a,q. It is a classical theorem of Walfisz [24] based on the work of Siegel that, for any fixed $N>0$, uniformly for $q\le(\log x)^N$ and $(a,q)=1$, as $x\to\infty$ we have

$$\pi(x;q,a)\sim\frac{x}{\phi(q)\log x}. \tag{1.1}$$

It is generally believed that this asymptotic holds in a much wider range of q. If we assume the Generalised Riemann Hypothesis (GRH), then the asymptotic (1.1) holds uniformly in the larger range $q\le x^{1/2-\delta}$ for any fixed $\delta>0$. Montgomery [17] has conjectured that (1.1) holds uniformly in the even larger range $q\le x^{1-\delta}$. Friedlander, Granville, Hildebrand and Maier [5] have shown for any A, (1.1) cannot hold for all $q\ge x/(\log x)^A$.

Any improvement in the range of q for which the asymptotic holds would exclude the possibility of the existence of zeros of Dirichlet L-functions in certain regions, but unfortunately such a result seems beyond our current techniques. Without this type of improvement, however, we cannot hope to prove results stronger than

$$o\left(\frac{x}{\phi(q)\log x}\right)\le\pi(x;q,a)\le\frac{2x}{\phi(q)\log x} \tag{1.2}$$

when $\log x/\log q$ is bounded.

Linnik [15], [16] gave a non-trivial lower bound for $\pi(x;q,a)$ for a wider range of q. He showed that there is a constant $L>0$ such that, whenever $x>q^L$ and q is sufficiently large, there is at least one prime in the arithmetic progression $\{n\le x: n\equiv a \pmod q\}$ for any a with $(a,q)=1$. Pan [20] showed that one can take $L\le 10{,}000$. This has subsequently been improved

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by many authors including (in chronological order) Chen [1], Jutila [13], Chen [2], Jutila [14], Chen [3], Graham [9], Wang [25], Chen and Liu [4], and Heath-Brown [11]. The best known result is due to Xylouris [26], which shows that we can take $L = 5.2$.

Titchmarsh [23] used Brun's sieve to show that for $q < x$ we have the upper bound

$$\pi(x; q, a) \ll \frac{x}{\phi(q)\log(x/q)}. \tag{1.3}$$

The implied constant can be made explicit, and has been estimated by various authors. The strongest result of this type which holds for all ranges of q is due to Montgomery and Vaughan [18], who used the large sieve to obtain the following result.

Theorem 1.1 (Brun–Titchmarsh theorem). *For $x > q$ we have*

$$\pi(x; q, a) \le \frac{2}{1 - \log q/\log x}\,\frac{x}{\phi(q)\log x}.$$

The constant $2/(1-\log q/\log x)$ of the Brun–Titchmarsh theorem should be compared with the constant $1 + o(1)$ which Montgomery conjectures for $x > q^{1+\epsilon}$.

Since it appears unlikely that we can prove an upper bound with a constant less than 2 with the current techniques, any improvements are likely to reduce the factor $1/(1-\log q/\log x)$. Several authors including Motohashi [19], Goldfeld [7], Iwaniec [12] and Friedlander and Iwaniec [6] have made improvements of this type for different ranges of q. If we put

$$\theta = \frac{\log q}{\log x}, \tag{1.4}$$

then we have

$$\pi(x; q, a) \le \frac{(C + o(1))x}{\phi(q)\log x}, \tag{1.5}$$

where

$$C = \begin{cases} (2 - ((1-\theta)/4)^6)/(1-\theta), & 2/3 \le \theta, \\ 8/(6-7\theta), & 9/20 \le \theta \le 2/3, \\ 16/(8-3\theta), & \theta \le 9/20. \end{cases} \tag{1.6}$$

This improves the Brun–Titchmarsh bound of $C = 2/(1-\theta)$ slightly throughout the entire range of q. We note that in all cases we still have $C > 2$ for $\theta > 0$.

It has been known as a folklore amongst specialists that for θ less than some fixed constant we should be able to take $C = 2$. In this paper we establish this, and give a quantitative bound for the range when this happens.

We show that provided q is sufficiently large we can take

$$C = 2 \quad \text{if } \theta \le 1/8.$$

2. Notation. We will let p represent a generic prime. We will consider the arithmetic progression where all terms are $\le x$ and are congruent to a (mod q). We will assume that q is larger than some fixed constant throughout, and so may not explicitly say that we are assuming q to be sufficiently large for a given statement to hold. χ will refer to a Dirichlet character modulo q and χ_0 to the principal character.

For the purposes of this paper we shall define an *η-Siegel zero* to be a real zero ρ of a Dirichlet L-function $L(s,\chi)$ which lies in the region

$$1 - \frac{\eta}{\log q} \le \Re(\rho) \le 1.$$

3. Main result. We improve on the Brun–Titchmarsh constant for some range of q. Instead of using sieve methods to count primes in arithmetic progressions we will use the analytic techniques developed in the estimation of Linnik's constant.

In Linnik's theorem one counts primes with a smooth weight, and estimating this requires estimating corresponding weighted sums over the zeros of Dirichlet L-functions. In the most successful work on Linnik's theorem only zeros of the form $\rho = 1 + O(1/\log q)$ make a significant contribution. In this paper we wish to count primes weighted by the characteristic function of the interval $[0, x]$, however, and this means we must consider all zeros $\rho = \beta + i\gamma$ with $\gamma \ll 1$ in the corresponding weighted sums over zeros. Thus the zero density estimates of Heath-Brown [11] are insufficient, and we need to extend them to this larger range.

THEOREM 3.1. *There exists an effectively computable constant q_1 such that for $q \ge q_1$ and $x \ge q^8$ we have*

$$\pi(x; q, a) < \frac{2\,\mathrm{Li}(x)}{\phi(q)}.$$

We note that without excluding the possible existence of η-Siegel zeros for some $\eta > 0$ this is the strongest possible bound which we can hope to prove for $\log x/\log q$ bounded.

We also obtain lower bounds which are essentially the strongest possible for $\log x/\log q$ bounded without excluding the existence of an η-Siegel zero.

THEOREM 3.2. *There exists an effectively computable constant q_2 such that for $q \ge q_2$ and $x \ge q^8$ we have*

$$\frac{\log q}{q^{1/2}} \frac{x}{\phi(q) \log x} \ll \pi(x; q, a).$$

THEOREM 3.3. *Let $\epsilon > 0$. There exists an (ineffective) constant $q_3(\epsilon)$ such that for $q \geq q_3(\epsilon)$ and $x \geq q^8$ we have*

$$\frac{q^{-\epsilon}x}{\phi(q)\log x} \ll \pi(x;q,a).$$

THEOREM 3.4. *Assume that there exists a constant $\eta > 0$ such that there are no η-Siegel zeros. Then there exists an effectively computable constant q_4 such that for $q \geq q_4$ and $x \geq q^8$ we have*

$$\frac{x}{\phi(q)\log x} \ll \pi(x;q,a) < \frac{2x}{\phi(q)\log x}.$$

Thus the number of primes in an arithmetic progression is close to the expected order predicted by GRH, provided $\log x/\log q \geq 8$ and q is sufficiently large. If there are no zeros exceptionally close to 1 then the number of primes has the same order as the asymptotic predicted by GRH.

In order to establish Theorems 3.1–3.4 we prove the following proposition.

PROPOSITION 3.5. *There are fixed constants $\epsilon > 0$ and $\eta > 0$ such that:*

- *There exists an effectively computable constant q_5 such that if there is an η-Siegel zero $\rho_1 = 1 - \lambda_1/\log q$ to modulus $q \geq q_5$ then for $x \geq q^7$ we have*
$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{(1-\lambda_1)x}{\phi(q)}.$$
- *There exists an effectively computable constant q_6 such that if there are no η-Siegel zeros to modulus $q \geq q_6$ then for $x \geq q^{7.999}$ we have*
$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{(1-\epsilon)x}{\phi(q)}.$$

We now establish the theorems assuming the proposition.

By partial summation we have, for any constant $7 \leq A < 8$,

$$\begin{aligned}(3.1)\qquad \pi(x;q,a) &= \frac{\theta(x;q,a)}{\log x} + \int_2^x \frac{\theta(t;q,a)}{t\log^2 t}\,dt \\ &= \frac{\theta(x;q,a)}{\log x} + \int_{q^A}^x \frac{\theta(t;q,a)}{t\log^2 t}\,dt + \int_{q^2}^{q^A} \frac{\theta(t;q,a)}{t\log^2 t}\,dt + \int_2^{q^2} \frac{\theta(t;q,a)}{t\log^2 t}\,dt.\end{aligned}$$

The Brun–Titchmarsh theorem for $q^2 \leq t \leq q^A$ shows

$$\theta(t;q,a) \leq (\log t)\pi(t;q,a) \ll \frac{t}{\phi(q)}, \tag{3.2}$$

and trivially for $t \leq q^2$ we have

$$\theta(t;q,a) \leq t\log t. \tag{3.3}$$

We also note that

$$\theta(x;q,a)=\psi(x;q,a)+O(x^{1/2}).$$

Thus uniformly for $x\ge q^8$ and $7\le A\le 8$ we obtain

$$\pi(x;q,a)=\frac{\psi(x;q,a)}{\log x}+\int_{q^A}^{x}\frac{\psi(t;q,a)}{t\log^2 t}\,dt+O\biggl(x^{1/2}+\frac{q^A}{\phi(q)}\biggr). \tag{3.4}$$

This gives

$$\begin{aligned}\biggl|\pi(x;q,a)-\frac{\mathrm{Li}(x)}{\phi(q)}\biggr| &\le \frac{1}{\log x}\biggl|\psi(x;q,a)-\frac{x}{\phi(q)}\biggr|+\int_{q^A}^{x}\frac{|\psi(t;q,a)-t/\phi(q)|}{t\log^2 t}\,dt\\ &\quad+O\biggl(x^{1/2}+\frac{q^A}{\phi(q)}\biggr).\end{aligned} \tag{3.5}$$

If there is an η-Siegel zero (where η is the constant from Proposition 3.5) then we choose $A=7$ and by Proposition 3.5 uniformly for $q\ge q_6$ and $x\ge q^8$ we have

$$\begin{aligned}\biggl|\pi(x;q,a)-\frac{\mathrm{Li}(x)}{\phi(q)}\biggr| &\le \frac{(1-\lambda_1)x}{\phi(q)\log x}+\int_{q^7}^{x}\frac{1-\lambda_1}{\phi(q)\log^2 t}\,dt+O\biggl(x^{1/2}+\frac{q^7}{\phi(q)}\biggr)\\ &\le \frac{(1-\lambda_1)\,\mathrm{Li}(x)}{\phi(q)}+O\biggl(\frac{x}{q\phi(q)}\biggr).\end{aligned} \tag{3.6}$$

By Pintz [21, Theorem 3] we see that $\lambda_1\gg(\log q)/q^{1/2}$ (with the implied constant effectively computable). Thus for q sufficiently large the error term in (3.6) is at most

$$\frac{\lambda_1\,\mathrm{Li}(x)}{2\phi(q)}. \tag{3.7}$$

Hence for q sufficiently large and $x\ge q^8$ we have

$$\frac{x\log q}{q^{1/2}\phi(q)\log x}\ll\frac{\lambda_1\,\mathrm{Li}(x)}{2\phi(q)}\le\pi(x;q,a)\le\frac{(2-\lambda_1/2)\,\mathrm{Li}(x)}{\phi(q)}\le\frac{2\,\mathrm{Li}(x)}{\phi(q)}, \tag{3.8}$$

with all constants effectively computable.

By Siegel's theorem [22], given any $\epsilon>0$ there is a constant $C(\epsilon)$ such that if $q\ge C(\epsilon)$ then $\lambda_1\ge 2q^{-\epsilon}$. Here $C(\epsilon)$ is not effectively computable. In this case, we have

$$\frac{xq^{-\epsilon}}{\phi(q)\log x}\le\frac{\lambda_1\,\mathrm{Li}(x)}{2\phi(q)\log x}<\pi(x;q,a). \tag{3.9}$$

If there is no η-Siegel zero then we instead choose $A = 7.999$. By Proposition 3.5 and (3.5) there exist $\epsilon > 0$ and q_5 such that uniformly for $x \ge q^8$ and for $q \ge q_5$ we have

$$\left|\pi(x;q,a) - \frac{\mathrm{Li}(x)}{\phi(q)}\right| \le \frac{(1-\epsilon)x}{\phi(q)\log x} + \int_{q^{7.999}}^{x} \frac{1-\epsilon}{\phi(q)\log^2 t}\,dt + O\left(x^{1/2} + \frac{q^{7.999}}{\phi(q)\log x}\right) = \frac{(1-\epsilon)\,\mathrm{Li}(x)}{\phi(q)} + O\left(\frac{x^{1-1/10\,000}}{\phi(q)\log x}\right). \tag{3.10}$$

Thus for q sufficiently large and $q^8 \le x$ we get

$$\frac{x}{\phi(q)\log x} \ll \pi(x;q,a) < \frac{2x}{\phi(q)\log x}. \tag{3.11}$$

Theorems 3.1–3.4 now follow immediately from (3.8), (3.9) and (3.11).

4. Case 1: Siegel zeros. We first consider the case when there are zeros very close to 1. For this section we assume that η-Siegel zeros exist for some small constant $\eta > 0$.

In order to establish Proposition 3.5 we will make use of the analytic techniques developed in the estimation of Linnik's constant. In particular, there are three main results which we use:

Proposition 4.1 (Zero-free region). *There is a constant $c_1 > 0$ such that, for q sufficiently large,*

$$\prod_{\chi\,(\mathrm{mod}\,q)} L(\sigma+it,\chi)$$

has at most one zero in the region

$$1 - \frac{c_1}{\log(q(2+|t|))} \le \sigma.$$

Such a zero, if it exists, is real and simple, and the corresponding character must be a non-principal real character.

Proposition 4.2 (Deuring–Heilbronn phenomenon). *There is a constant $c_2 > 0$ such that, if the exceptional zero $\rho_1 = 1 - \lambda_1/\log q$ from Proposition 4.1 exists, then for q sufficiently large, the function*

$$\prod_{\chi\,(\mathrm{mod}\,q)} L(\sigma+it,\chi)$$

has no other zeros in the region

$$1 - \frac{c_2\log(\lambda_1^{-1})}{\log(q(2+|t|))} \le \sigma \le 1.$$

PROPOSITION 4.3 (Log-free zero density estimate). *For $T \geq 1$ there are constants $c_3 > 0$ and $C_3 > 0$ such that*

$$\sum_{\chi \,(\mathrm{mod}\, q)} N(\sigma, T, \chi) \leq C_3 (qT)^{c_3(1-\sigma)}.$$

Here

$$N(\sigma, T, \chi) = \#\{\rho : L(\rho, \chi) = 0,\ \Re(\rho) \geq \sigma,\ |\Im(\rho)| \leq T\}.$$

We recall that for the purposes of this article we are defining an η-Siegel zero to be a real zero ρ of some Dirichlet L-function in the region

$$1 - \frac{\eta}{\log q} \leq \rho \leq 1 \tag{4.1}$$

for a fixed small positive constant η.

We will choose $\eta \leq c_1/2$, so by Proposition 4.1 an η-Siegel zero, if it exists, must be simple, and the corresponding character must be a real character. Moreover, there can be at most one such zero. We label this exceptional zero $\rho_1 = 1 - \lambda_1/\log q$ with corresponding character χ_1. Thus we have $\lambda_1 \leq \eta$. We will also make use of the fact that $\lambda_1 \gg_\epsilon q^{-1/2-\epsilon}$ (with the implied constant effectively computable), which follows from Dirichlet's class number formula.

We note that by [10] and [11, equation (1.4)] we can take

$$c_2 = 2/3 - 1/1000, \qquad c_3 = 12/5 + 1/1000, \tag{4.2}$$

provided $\eta \leq c_4$, some suitably small absolute constant.

We wish to prove

$$\left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| \leq \frac{(1-\lambda_1)x}{\phi(q)}. \tag{4.3}$$

We have

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)} \Big(\sum_{n \leq x} \Lambda(n)\chi(n) \Big). \tag{4.4}$$

We use the explicit formula

$$\sum_{n \leq x} \Lambda(n)\chi(n) = \varepsilon_1(\chi)x - \varepsilon_2(\chi)\frac{x^{\rho_1}}{\rho_1} - \sum_{\rho} \frac{x^\rho}{\rho} + O\Big(\frac{x(\log x)^2}{T} \Big), \tag{4.5}$$

where

$$\varepsilon_1(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

$$\varepsilon_2(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is a character corresponding to the possible} \\ & \text{exceptional zero } \rho_1 \text{ of } \prod_\chi L(s,\chi), \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

and the sum $\sum_\rho$ is over all non-exceptional non-trivial zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ in the region $\{0<\beta<1,\,|\gamma|<T\}$.

We choose $T=q(\log x)^3/\lambda_1$ so that the last term is $o(\lambda_1 x/\phi(q))$.

Recalling that $\rho_1=1-\lambda_1/\log q$ we have

$$\frac{x^{\rho_1}}{\rho_1}=x\exp\Bigl(-\lambda_1\frac{\log x}{\log q}\Bigr)+o(\lambda_1 x). \tag{4.8}$$

Substituting (4.5) and (4.8) into (4.4) we obtain

$$\begin{aligned}\Bigl|\psi(x;q,a)-\frac{x}{\phi(q)}\Bigr|& \\ \le \frac{x}{\phi(q)}\exp\Bigl(-\lambda_1\frac{\log x}{\log q}\Bigr)&+\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\sum_\rho\Bigl|\frac{x^\rho}{\rho}\Bigr|+o\Bigl(\frac{\lambda_1 x}{\phi(q)}\Bigr).\end{aligned} \tag{4.9}$$

We now bound the sum

$$\sum_{\chi\,(\mathrm{mod}\,q)}\sum_\rho\Bigl|\frac{x^\rho}{\rho}\Bigr|. \tag{4.10}$$

We first consider the case when $\log x>q^{1/3000}$.

Since $\lambda_1\gg q^{-1/2-1/100}$ we have $T\ll q^{3/2+1/100}(\log x)^3\ll(\log x)^{4600}$. By Proposition 4.1 (and recalling $|\rho|\gg\lambda_1/\log q$ for all ρ) each zero in the sum (4.10) contributes at most

$$\Bigl|\frac{x^\rho}{\rho}\Bigr|\le x\exp\Bigl(-c\frac{\log x}{\log\log x}\Bigr) \tag{4.11}$$

for some constant $c>0$. By Proposition 4.3 the total number of zeros in the sum is

$$\ll(qT)^{12/5+1/1000}\ll(\log x)^{20000}. \tag{4.12}$$

Therefore

$$\sum_{\chi\,(\mathrm{mod}\,q)}\sum_\rho\Bigl|\frac{x^\rho}{\rho}\Bigr|\ll x(\log x)^{20000}\exp\Bigl(-c\frac{\log x}{\log\log x}\Bigr)=o(\lambda_1 x). \tag{4.13}$$

Thus we see that for x sufficiently large and $\log x>q^{1/3000}$, the right hand side of (4.9) is

$$\frac{x}{\phi(q)}\Bigl(\exp\Bigl(-\lambda_1\frac{\log x}{\log q}\Bigr)+o(\lambda_1)\Bigr)\le\frac{(1-\lambda_1)x}{\phi(q)}, \tag{4.14}$$

as required.

We now consider the case when $\log x\le q^{1/3000}$. In this case, since $\lambda_1\gg q^{-1/2-1/1000}$, we have $T\ll q^{3/2+2/1000}$.

We begin by considering the contribution to the sum (4.10) from zeros in the rectangle

$$1-\frac{m+1}{\log q}\le \Re(\rho)\le 1-\frac{m}{\log q},\qquad n\le |\Im(\rho)|\le 2n, \tag{4.15}$$

where $1\le n\le T$ and $m\le 0.4\log q$. By Proposition 4.2 with $c_2=2/3-1/1000$ there are no zeros in the rectangle unless

$$m\ge \left(\frac{2}{3}-\frac{1}{1000}\right)\frac{\log q}{\log(q(2+T))}\log\lambda_1^{-1}\ge 0.266\log\lambda_1^{-1}. \tag{4.16}$$

Recalling that $m\le 0.4\log q$, by Proposition 4.3 with $c_3=12/5+1/1000$ there are

$$\ll n^{0.97}\exp(2.41m) \tag{4.17}$$

zeros in the rectangle.

If (4.16) holds then we see that each zero contributes

$$\begin{aligned}\left|\frac{x^\rho}{\rho}\right| &\le \frac{x}{n}\exp\left(-m\frac{\log x}{\log q}\right) \\ &= \frac{x}{n}\exp\left(-m\left(\frac{\log x}{\log q}-\frac{1}{0.266}\right)\right)\exp\left(-\frac{m}{0.266}\right) \\ &\le \frac{\lambda_1 x}{n}\exp\left(-m\left(\frac{\log x}{\log q}-3.76\right)\right).\end{aligned} \tag{4.18}$$

Thus the zeros in the rectangle give a total contribution of

$$\ll \frac{\lambda_1 x}{n^{0.03}}\exp\left(-m\left(\frac{\log x}{\log q}-6.17\right)\right). \tag{4.19}$$

From summing this bound over $n=2^j$ with $j\in\mathbb{N}$, we see that provided $q^{6.18}\le x$, the contribution to the sum (4.10) from all non-exceptional zeros in the region

$$0.6\le\Re(\rho)\le 1,\qquad 1\le|\Im(\rho)|\le T \tag{4.20}$$

is at most

$$C\lambda_1 x\exp(-c\log\lambda_1^{-1})\le C\lambda_1 x\exp(c\log\eta) \tag{4.21}$$

for some constants $C,c>0$. Since $\lambda_1\le\eta$ we see that for η sufficiently small (depending only on C,c) this is at most $\lambda_1 x$.

Similarly we consider the contribution to the sum (4.10) from the zeros in the region

$$1-\frac{m+1}{\log q}\le \Re(\rho)\le 1-\frac{m}{\log q},\qquad |\Im(\rho)|\le 1, \tag{4.22}$$

with $m\le 0.4\log q$. As above, each zero contributes

$$\ll \lambda_1 x\exp\left(-m\left(\frac{\log x}{\log q}-3.76\right)\right). \tag{4.23}$$

The number of zeros in the rectangle is

$$\ll \exp(2.41m). \tag{4.24}$$

Thus again the contribution of all zeros from the rectangles is at most

$$C\lambda_1 x \exp(-c\log\lambda_1^{-1}) \le C\lambda_1 x\exp(c\log\eta) \tag{4.25}$$

for some positive constants C, c. Hence for η sufficiently small this contribution is at most $\lambda_1 x$.

Finally we consider zeros in the rectangles

$$0 \le \Re(\rho) \le 0.6, \qquad |\Im(\rho)| \le \sqrt{T}, \tag{4.26}$$

$$0 \le \Re(\rho) \le 0.6, \qquad \sqrt{T} \le |\Im(\rho)| \le T. \tag{4.27}$$

By symmetry of zeros around the line $\Re(s) = 1/2$ we have $\Re(\rho) \gg \lambda_1/\log q$ for all such ρ. Thus, since $\lambda_1 \gg q^{-1/2-1/100}$ and $x > q$, each zero satisfying (4.26) contributes

$$\left|\frac{x^\rho}{\rho}\right| \ll x^{0.6}, \tag{4.28}$$

and every zero satisfying (4.27) contributes

$$\left|\frac{x^\rho}{\rho}\right| \le \frac{x^{0.6}}{\sqrt{T}}. \tag{4.29}$$

For q sufficiently large there are

$$\ll (q\sqrt{T})^{1+1/1000} \le q^{1.76} \tag{4.30}$$

zeros satisfying (4.26), and

$$\ll (qT)^{1+1/1000} \le q^{1.76}\sqrt{T} \tag{4.31}$$

zeros satisfying (4.27). Thus the combined contribution is

$$\ll x^{0.6}q^{1.76} \ll \lambda_1 x\frac{q^{2.27}}{x^{0.4}}. \tag{4.32}$$

We see this is at most $\lambda_1 x$ for $q^6 \le x$ and q sufficiently large.

Since we have now covered all possible zeros in our sum, we see that for η sufficiently small and $q^{6.18} \le x$ we have

$$\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\rho} \left|\frac{x^\rho}{\rho}\right| \le 3\lambda_1 x. \tag{4.33}$$

Substituting this into (4.9) we get

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| \le \frac{x}{\phi(q)}\left(\exp\left(-\lambda_1\frac{\log x}{\log q}\right) + 4\lambda_1\right). \tag{4.34}$$

We note that if $q^7 \le x$ and $\eta < 1/10$ then

$$\exp\left(-\lambda_1\frac{\log x}{\log q}\right) + 4\lambda_1 < 1 - \lambda_1, \tag{4.35}$$

since $1 - e^{-7t} - 5t$ is zero and increasing at 0, has a unique turning point and is positive at $1/10$.

Thus we have shown that for η sufficiently small, $q^7 \le x$ and $\log x \le q^{1/3000}$ we have

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{(1-\lambda_1)x}{\phi(q)} \tag{4.36}$$

as required.

5. Case 2: no Siegel zeros. We now consider the case where there are no η-Siegel zeros for some small fixed constant $\eta > 0$. In this case we have $\lambda_\rho \ge \eta$ for all zeros ρ with $|\Im(\rho)| \le q^2$. Following the method in the previous section and using this zero-free region, we can establish Proposition 3.5 if $\log x/\log q$ is sufficiently large. To obtain an explicit lower bound for the range of $\log x/\log q$ in which this holds, however, would require us to estimate the constant C_3 in Proposition 4.3, and would likely produce a very large bound if done directly.

We will follow the work done on the estimation of Linnik's constant to obtain an explicit lower bound for $\log x/\log q$ for which the result holds. This section follows closely the method of Heath-Brown in [11, Section 13].

We define the following quantities which we shall use for the rest of the paper:

$$M := \frac{\log x}{\log q}, \tag{5.1}$$

$$\mathcal{L} := \log q, \tag{5.2}$$

$$\phi_\chi := \begin{cases} 1/4 & \text{for } q \text{ cube-free or } \operatorname{ord}(\chi) \le \log q, \\ 1/3 & \text{otherwise,} \end{cases} \tag{5.3}$$

$$\mathscr{Z}(\chi) := \{\rho : L(\rho,\chi) = 0\}. \tag{5.4}$$

5.1. Weighted sum over primes. We wish to investigate

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \,(\mathrm{mod}\, q)}} \Lambda(n). \tag{5.5}$$

We fix a small positive constant $\epsilon > 0$ and let

$$f(t) = \begin{cases} 0, & t \le 1/2, \\ \frac{\log x}{\epsilon}(t-1/2), & 1/2 \le t \le 1/2 + \epsilon/\log x, \\ 1, & 1/2 + \epsilon/\log x \le t \le 1, \\ 1 - \frac{\log x}{\epsilon}(t-1), & 1 \le t \le 1 + \epsilon/\log x, \\ 0, & t \ge 1 + \epsilon/\log x. \end{cases} \tag{5.6}$$

The Brun–Titchmarsh theorem for primes in short intervals (see [18], for example) states that

$$\pi(x;q,a)-\pi(x-y;q,a)\le \frac{2y}{\phi(q)\log{(y/q)}}. \tag{5.7}$$

We replace the sum

$$\sum_{\substack{n\le x\\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) \tag{5.8}$$

with the weighted sum

$$\sum_{\substack{n=1\\ n\equiv a \,(\mathrm{mod}\, q)}}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right). \tag{5.9}$$

By the Brun-Titchmarsh theorem for primes in short intervals and for ϵ sufficiently small, the error introduced by making this change is

$$\begin{aligned}\le &\sum_{\substack{x\le n\le xe^{\epsilon}\\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) + \sum_{n\le e^{\epsilon}x^{1/2}} \Lambda(n)\\ &\le (\log(xe^{\epsilon}))(\pi(xe^{\epsilon};q,a)-\pi(x;q,a)) + e^{\epsilon}(\log x)x^{1/2} \le \frac{4\epsilon x}{\phi(q)}.\end{aligned} \tag{5.10}$$

Thus in order to prove

$$\left|\psi(x;q,a)-\frac{x}{\phi(q)}\right| \le \frac{(1-\epsilon)x}{\phi(q)}, \tag{5.11}$$

it is sufficient to prove that

$$\left|\sum_{\substack{n=1\\ n\equiv a \,(\mathrm{mod}\, q)}}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right) - \frac{x}{\phi(q)}\right| < \frac{(1-5\epsilon)x}{\phi(q)}. \tag{5.12}$$

We note also

$$\begin{aligned}&\sum_{\substack{n=1\\ n\equiv a \,(\mathrm{mod}\ \ q)}}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right)\\ &\qquad = \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)} \left(\sum_{n=1}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right)\chi(n)\right).\end{aligned} \tag{5.13}$$

We now replace χ in the inner sum with the primitive character χ^* which induces it. This introduces an error

$$\ll \frac{1}{\phi(q)} \sum_{\chi} \sum_{p|q} \sum_{x^{1/2}\le p^e \le xe^{\epsilon}} \log p \ll \sum_{p|q} \log x \ll q^{\epsilon}\log x \le \epsilon x \tag{5.14}$$

(recalling that $x > q$).

Therefore it is sufficient to prove that

$$\left|\sum_{\chi}\overline{\chi}(a)\sum_{n=1}^{\infty}\Lambda(n)\chi^*(n)f\left(\frac{\log n}{\log x}\right)-x\right|\le(1-6\epsilon)x. \tag{5.15}$$

5.2. Sum over zeros. We let F be the Laplace transform of f. Hence

$$\begin{aligned}F(s)&=\int_0^{\infty}\exp(-st)f(t)\,dt\\&=e^{-s}\frac{1-\exp(s/2)}{-s}\,\frac{1-\exp((\epsilon/\log x)s)}{(-\epsilon/\log x)s}\exp\left(-\frac{\epsilon}{\log x}s\right).\end{aligned}\tag{5.16}$$

From the Laplace inversion formula we have

$$f\left(\frac{\log n}{\log x}\right)=\frac{\log x}{2\pi i}\int_{2-i\infty}^{2+i\infty}n^{-s}F(-s\log x)\,ds. \tag{5.17}$$

Therefore for $\chi\neq\chi_0$ we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\Lambda(n)\chi^*(n)f\left(\frac{\log n}{\log x}\right)&\\&=\frac{\log x}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(-\frac{L'}{L}(s,\chi^*)\right)(F(-s\log x))\,ds\\&=\frac{\log x}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\left(-\frac{L'}{L}(s,\chi^*)\right)(F(-s\log x))\,ds\\&\quad-\log x\sum_{\rho}F(-\rho\log x)\end{aligned}\tag{5.18}$$

where $\sum_{\rho}$ indicates a sum over all non-trivial zeros of $L(s,\chi)$.

On $\Re s=-1/2$ we have

$$\frac{L'}{L}(s,\chi^*)\ll\log(q(1+|s|)),\qquad F(-s\log x)\ll x^{-1/4}|s|^{-2}(\log x)^{-1}. \tag{5.19}$$

Hence, recalling that $q\le x$,

$$\frac{\log x}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty}\left(-\frac{L'}{L}(s,\chi^*)\right)(F(-s\log x))\,ds=O(x^{-1/4}\log x). \tag{5.20}$$

Thus

$$\sum_{\chi\neq\chi_0}\left|\sum_{n=1}^{\infty}\Lambda(n)\chi^*(n)f\left(\frac{\log n}{\log x}\right)\right| \qquad (5.21)$$
$$\leq \log x\sum_{\chi\neq\chi_0}\sum_{\rho}|F(-\rho\log x)|+O(qx^{-1/4}\log x)$$
$$\leq \log x\sum_{\chi\neq\chi_0}\sum_{\rho}|F(-\rho\log x)|+\epsilon x.$$

We now consider the case $\chi=\chi_0$. We note that χ_0^* is identically 1. Hence by the prime number theorem we have

$$\left|\sum_{n=1}^{\infty}\Lambda(n)\chi_0^*(n)f\left(\frac{\log n}{\log x}\right)-x\right|\leq 3\epsilon x. \qquad (5.22)$$

Thus putting together (5.21) and (5.22) we obtain

$$\left|\sum_{\chi}\overline{\chi}(a)\sum_{n=1}^{\infty}\Lambda(n)\chi^*(n)f\left(\frac{\log n}{\log x}\right)-x\right| \qquad (5.23)$$
$$\leq\left|\sum_{n=1}^{\infty}\Lambda(n)\chi_0^*(n)f\left(\frac{\log n}{\log x}\right)-x\right|+\sum_{\chi\neq\chi_0}\left|\sum_{n=1}^{\infty}\Lambda(n)\chi^*(n)f\left(\frac{\log n}{\log x}\right)\right|$$
$$\leq 4\epsilon x+\log x\sum_{\chi\neq\chi_0}\sum_{\rho}|F(-\rho\log x)|.$$

In particular it is sufficient to prove that

$$\log x\sum_{\chi\neq\chi_0}\sum_{\rho}|F(-\rho\log x)|\leq(1-10\epsilon)x. \qquad (5.24)$$

We now consider the contribution from the other characters where $\chi\neq\chi_0$. We begin by considering all zeros $\rho=\beta+i\gamma$ of all L-functions $L(s,\chi)$ (with $\chi\neq\chi_0$) in the rectangle

$$1-\frac{m+1}{\log q}\leq\beta\leq 1-\frac{m}{\log q},\qquad n\leq|\gamma|\leq 2n \qquad (5.25)$$

for $n\geq 1$.

We use the well-known zero density estimate

$$\sum_{\chi}N(\sigma,\chi,T)\ll q^{3(1-\sigma)}(1+T^{3/2}). \qquad (5.26)$$

Thus there are

$$\ll e^{3m}(1+n^{3/2}) \qquad (5.27)$$

such zeros in the rectangle.

Each zero contributes

$$\log x|F(-\rho\log x)| \ll x\frac{\exp(-m\log x/\log q)}{\epsilon n^2} \tag{5.28}$$

to the right hand side of (5.23).

Thus, provided $M > 3$, there is a constant R (depending only on ϵ) such that the contribution of all zeros in the rectangles with $\max(m,n) \geq R$ is

$$\leq \epsilon x. \tag{5.29}$$

Similarly we consider zeros in the rectangle

$$\max\left(\frac{1}{2}, 1-\frac{m+1}{\log q}\right) \leq \beta \leq 1-\frac{m}{\log q}, \qquad |\gamma| \leq 1. \tag{5.30}$$

There are

$$\ll e^{3m} \tag{5.31}$$

such zeros, and each zero contributes

$$\ll x\frac{\exp(-m\log x/\log q)}{\epsilon}. \tag{5.32}$$

Therefore again provided $M > 3$, the contribution from all zeros in the rectangles with $m \geq R$ is $\leq \epsilon x$.

We now consider the final rectangle

$$0 \leq \beta \leq 1/2, \qquad |\gamma| \leq 1. \tag{5.33}$$

All zeros must have $\beta \geq q^{-1/2-1/100}$ for q sufficiently large (by symmetry of zeros about the critical line and the non-existence of Siegel zeros which are within $q^{-1/2-1/100}$ of 1).

There are

$$\ll q^{3/2} \tag{5.34}$$

zeros in this rectangle, and each zero contributes

$$\ll \frac{x^{1/2}q^{2/100}}{\epsilon}. \tag{5.35}$$

Therefore the contribution from these zeros is

$$\ll \frac{x^{1/2}q^{3/2+1/50}}{\epsilon} \leq \epsilon x. \tag{5.36}$$

Thus at a cost of $3\epsilon x$ we only need to consider the contribution of zeros ρ satisfying

$$|1-\Re(\rho)| \ll_\epsilon \frac{1}{\log q}, \qquad \Im(\rho) \ll_\epsilon 1. \tag{5.37}$$

For such ρ, and for ϵ sufficiently small and q sufficiently large, we have

$$\left|\frac{1-x^{-\rho/2}}{\rho}e^{\epsilon\rho}\right| \leq 1+3\epsilon. \tag{5.38}$$

Also, for any $z \in \mathbb{C}$ with $\Re(z) \geq 0$ we have

$$\left|\frac{1-e^{-z}}{z}\right| \leq 1. \tag{5.39}$$

Thus, putting $\Re(\rho) = 1 - \lambda_\rho/\log q$, and recalling that $q^{7.999} \leq x$, we obtain

$$\begin{aligned}\log x|F(-\rho\log x)| &= x\exp(-(1-\rho)\log x)\left|\frac{1-x^{-\rho/2}}{\rho}\,\frac{1-e^{-\epsilon\rho}}{\epsilon\rho}e^{\epsilon\rho}\right| \\ &\leq x\exp\left(-\lambda_\rho\frac{\log x}{\log q}\right)(1+3\epsilon) = x\exp(-M\lambda_\rho)(1+3\epsilon).\end{aligned} \tag{5.40}$$

As before, we have put

$$M = \frac{\log x}{\log q}. \tag{5.41}$$

Thus we have shown that

$$\begin{aligned}\left|\psi(x;q,a) - \frac{x}{\psi(q)}\right| \\ \leq 12\epsilon\frac{x}{\phi(q)} + (1+3\epsilon)\frac{x}{\phi(q)}\sum_{\chi\neq\chi_0}\sum_{\rho}{}^{*}\exp(-M\lambda_\rho),\end{aligned} \tag{5.42}$$

where $\sum^*$ represents a sum over all zeros of $L(s,\chi)$ in

$$\mathcal{R} = \left\{z : 1 - \frac{R}{\log q} \leq \Re(z) \leq 1,\ \Im(z) \leq R\right\}, \tag{5.43}$$

with R a constant (independent of x and q).

6. Zero density estimates.

We wish to estimate the sum

$$\sum_{\chi\neq\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)}\exp(-M\lambda_\rho), \tag{6.1}$$

where

$$\mathcal{Z}(\chi) := \{\rho : L(\rho,\chi) = 0\}. \tag{6.2}$$

We do this by obtaining a zero density estimate for zeros in $\mathcal{R}$ by means of different weighted sums over zeros of $L(s,\chi)$. We note that by the log-free zero density estimate given in Proposition 4.3 this sum is finite for any $M \in \mathbb{R}$. We specifically wish to show that the sum is < 1 when $M = 7.999$.

Similar sums have been looked at in the estimation of Linnik's constant. We will broadly follow the approach of Heath-Brown in [11], but most of the estimates must be extended to cover a region where $\Im(\rho) \ll 1$ instead of $\Im(\rho) \ll \mathcal{L}^{-1}$.

We split $\mathcal{R}$ vertically into smaller rectangles each with height $1/\mathcal{L}$. We put

$$\mathcal{R}_m := \left\{ z : 1 - \frac{R}{\mathcal{L}} \le \Re(z) \le 1,\ \frac{m-1/2}{\mathcal{L}} \le |\Im(z)| \le \frac{m+1/2}{\mathcal{L}} \right\}. \tag{6.3}$$

We label our non-principal characters modulo q as $\chi^{(1)}, \chi^{(2)}, \ldots$ in some order. For each character $\chi^{(j)}$, and for each rectangle $\mathcal{R}_m$ for which $L(s,\chi^{(j)})$ has a zero in $\mathcal{R}_m$, we pick a zero of $L(s,\chi^{(j)})$ with greatest real part, which we label $\rho^{(j,m)}$.

We introduce the notation

$$\rho^{(j,m)} = \beta^{(j,m)} + i\gamma^{(j,m)}, \qquad 1-\beta^{(j,m)} = \frac{\lambda^{(j,m)}}{\log q}, \qquad \gamma^{(j,m)} = \frac{\nu^{(j,m)}}{\log q}. \tag{6.4}$$

We also specifically label special zeros ρ_1, ρ_1' and ρ_2. We let ρ_1 be a zero of $\prod_\chi L(s,\chi)$ which is in $\mathcal{R}$ and has largest real part. We let χ_1 be the corresponding character. We let ρ_2 be a zero of $\prod_{\chi\ne\chi_1,\overline{\chi}_1} L(s,\chi)$ which is in $\mathcal{R}$ and has largest real part. We let ρ_1' be a zero of $L(s,\chi_1)$ which is in $\mathcal{R}$ and is not ρ_1 or $\overline{\rho}_1$ but otherwise has largest real part. If ρ_1 is not a simple zero we simply have $\rho_1' = \rho_1$.

For simplicity we argue as if ρ_1, ρ_1', ρ_2 all exist. Our argument is simpler and stronger if any of these do not exist.

We now wish to estimate separately a weighted sum over rectangles and a weighted sum over zeros in any such rectangle. Specifically we wish to prove the following three lemmas:

Lemma 6.1. *For any $\delta > 0$ any $m \in \mathbb{Z}$ and any constant $K > 0$ we have, for $q > q_0(\delta)$,*

$$\sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi^{(j)})} B_1(\lambda_\rho) \le C_1(\lambda^{(j,m)})$$

where

$$B_1(\lambda) = \frac{(1-\exp(-K\lambda))^2}{\lambda^2+1/4},$$
$$C_1(\lambda) = \frac{\phi_\chi(1-\exp(-2K\lambda))}{2\lambda} + \frac{2K\lambda - 1 + \exp(-2K\lambda)}{2\lambda^2} + \delta.$$

Lemma 6.2. *Let $(\chi^{(i)})_{i\in I}$ be a set of characters modulo q. Then for any $\delta > 0$ and $q > q_0(\delta)$ we have*

$$\sum_{m\in\mathbb{Z},\, i\in\mathbb{I}} B_2(\lambda^{(i,m)}) \le C_2$$

where

$$B_2(\lambda) = \left(\frac{e^{2\lambda x_1} + e^{2\lambda x_0}}{x_1 - x_0} + \frac{e^{2\lambda u_1} + e^{2\lambda u_0}}{u_1 - u_0}\right)^{-1},$$
$$C_2 = \frac{x_1 + x_0 - v - u_1}{2w(v - u_1)}(1 + G_2) + \delta,$$
$$G_2 \text{ will be defined in } (6.55),$$

and x_1, x_0, v, u_1, u_0, w *are all constants* > 0 *satisfying*

$$x_1 > x_0, \quad x_0 > v + w + 1/3, \quad v > u_1, \quad u_1 > u_0, \quad u_0 > 2w + 1/3.$$

In particular,

$$\sum_{j,m}\left(\frac{e^{3.243\ldots\lambda^{(j,m)}} + e^{2.823\ldots\lambda^{(j,m)}}}{0.21} + \frac{e^{1.238\ldots\lambda^{(j,m)}} + e^{1.126\ldots\lambda^{(j,m)}}}{0.056}\right)^{-1} \le 11.826\ldots.$$

LEMMA 6.3. *Let* $g : [0, \infty) \to \mathbb{R}$ *be a non-negative continuous function, supported on* $[0, x_0)$ *for some* $x_0 > 0$, *which is twice differentiable on* $(0, x_0)$ *and has a bounded second derivative on* $(0, x_0)$. *Moreover, assume the Laplace transform* G *of* g *satisfies* $\Re(G(z)) \ge 0$ *for* $\Re(z) \ge 0$. *Let* $0 \le \lambda_{11} \le \lambda_1$ *and* $0 \le \lambda \le 2$ *be such that*

$$G(\lambda - \lambda_{11}) > g(0)/6 \quad \text{and} \quad (G(\lambda - \lambda_{11}) - g(0)/6)^2 > G(-\lambda_{11})g(0)/6.$$

Then for any $\delta > 0$ *and* $q > q_0(\delta, g)$ *we have*

$$\sum_{\substack{j,m \\ \lambda^{(j,m)} \le \lambda}} 1 \le \frac{G(-\lambda_{11})G_3}{(G(\lambda - \lambda_{11}) - g(0)/6)^2 - G(-\lambda_{11})g(0)/6} + \delta.$$

Here G_3 *will be defined in* (6.90).

We will now proceed to prove each of these lemmas in turn.

We note here that we can easily ensure the L given in [11, Lemma 6.1] satisfies $R \le L \le \frac{1}{10}\mathcal{L}$ rather than just $L \le \frac{1}{10}\mathcal{L}$ by following exactly the same argument but with this restriction. This means that all the results of Heath-Brown [11] and Xylouris [26] which are concerned with zeros in the region

$$1 - \frac{\log\log\mathcal{L}}{3\mathcal{L}} \le \sigma \le 1, \quad |t| \le L \tag{6.5}$$

also apply to the zeros which we consider in $\mathcal{R}$.

6.1. First zero density estimate. We now consider zeros within one of the rectangles $\mathcal{R}_m$. We follow almost identically the argument of Heath-Brown in [11, Lemma 13.3].

We put

$$h_1(t) = \begin{cases} \sinh((K-t)\lambda), & 0 \le t \le K, \\ 0, & t \ge K, \end{cases} \tag{6.6}$$

$$H_1(z) = \int_0^\infty e^{-zt} h_1(t)\,dt = \frac{1}{2}\left(\frac{e^{K\lambda}}{\lambda+z} + \frac{e^{-K\lambda}}{\lambda-z} - \frac{2\lambda e^{-Kz}}{\lambda^2 - z^2}\right), \tag{6.7}$$

$$H_2(z) = \left(\frac{1-e^{-Kz}}{z}\right)^2, \tag{6.8}$$

for some constants $K \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, which will be declared later.

We note that

$$\Re(H_1(it)) = \frac{\lambda e^{K\lambda}}{2}\left|\frac{1-e^{-K(\lambda+it)}}{\lambda+it}\right|^2 = \frac{\lambda e^{K\lambda}}{2}\Big|H_2(\lambda+it)\Big|. \tag{6.9}$$

Since $H_1(z)$ and $H_2(\lambda+z)$ tend uniformly to zero in $\Re(z) \ge 0$ as $|z| \to \infty$, and $\Re(H_1(z)) = \lambda e^{K\lambda}|H_2(\lambda+z)|/2$ when $\Re(z) = 0$, by [11, Lemma 4.1] we have

$$\Re(H_1(z)) \ge \frac{\lambda e^{K\lambda}}{2}|H_2(\lambda+z)| \tag{6.10}$$

whenever $\Re(z) \ge 0$.

We fix a character $\chi = \chi^{(j)} \ne \chi_0$ and take $\lambda = \lambda^{(j,m)}$. Therefore $L(s,\chi)$ has no zeros in the region $\{\sigma > 1 - \lambda/\mathcal{L}\} \cap \mathcal{R}_m$.

Thus

$$\begin{aligned} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi)} &|H_2((1-\rho+im/\mathcal{L})\mathcal{L})| \\ &\le \frac{2e^{-K\lambda}}{\lambda} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi)} \Re(H_1((s-\rho)\mathcal{L})), \end{aligned} \tag{6.11}$$

where $s = 1 - \lambda/\mathcal{L} + im/\mathcal{L}$.

By [11, Lemma 5.2] and [11, Lemma 5.3] we have (recalling that $|m| \ll \mathcal{L}$ so $|\Im(s)| \le \mathcal{L}$ for q sufficiently large), for any given $\delta > 0$ and $q > q(\delta)$,

$$\begin{aligned} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi)} &\Re(H_1((s-\rho)\mathcal{L})) \\ &\le \frac{h_1(0)\phi_\chi}{2} + \mathcal{L}^{-1}\left|\sum_{n=1}^\infty \Lambda(n)\Re\left(\frac{\chi(n)}{n^s}\right)h_1(\mathcal{L}^{-1}\log n)\right| + \delta \\ &\le \frac{h_1(0)\phi_\chi}{2} + \mathcal{L}^{-1}\sum_{n=1}^\infty \Lambda(n)\frac{\chi_0(n)}{n^{\Re(s)}}h_1(\mathcal{L}^{-1}\log n) + \delta \\ &\le \frac{h_1(0)\phi_\chi}{2} + |H_1((\Re(s)-1)\mathcal{L})| + 2\delta. \end{aligned} \tag{6.12}$$

This gives

$$(6.13)\qquad \sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi)}\left|\frac{1-e^{-K\lambda_\rho-iK(m-\gamma_\rho\mathcal{L})}}{\lambda_\rho+i(m-\gamma_\rho\mathcal{L})}\right|^2 \le \frac{\phi_\chi(1-e^{-2K\lambda})}{2\lambda}+\frac{2K\lambda-1+e^{-2K\lambda}}{2\lambda^2}+2\delta.$$

Since $\rho\in\mathcal{R}_m$ we have $|m-\gamma_\rho\mathcal{L}|\le 1/2$. Thus, recalling that $\chi=\chi^{(j)}$ and $\lambda=\lambda^{(j,m)}$, we obtain

$$(6.14)\qquad \sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi^{(j)})}\frac{(1-e^{-K\lambda_\rho})^2}{\lambda_\rho^2+1/4} \le \frac{\phi_\chi(1-e^{-2K\lambda^{(j,m)}})}{2\lambda^{(j,m)}}+\frac{2K\lambda^{(j,m)}-1+e^{-2K\lambda^{(j,m)}}}{2(\lambda^{(j,m)})^2}+2\delta.$$

Hence Lemma 6.1 holds.

6.2. Second zero density estimate. We now prove Lemma 6.2. The proof uses ideas originally due to Graham [9]. We follow the method of [11, Section 11], but extend the result to a weighted sum over zeros rather than just characters. We do this by using integrated exponential weights instead of exponential weights, an idea originally due to Jutila [14].

We adopt similar notation to that of [11, Section 11]. We put

$$(6.15)\qquad U_0=q^{u_0},\quad U_1=q^{u_1},\quad X_0=q^{x_0},\quad X_1=q^{x_1},\quad V=q^{v},\quad W=q^{w}$$

with constant exponents $0<w<u_0<u_1<v<x_0<x_1$ to be declared later. We put

$$(6.16)\qquad U=q^{u},\qquad X=q^{x}$$

with $u_0\le u\le u_1$ and $x_0\le x\le x_1$ parameters which we will integrate over.

We define

$$(6.17)\qquad \psi_d=\begin{cases}\mu(d), & 1\le d\le U_1,\\ \mu(d)\dfrac{\log(V/d)}{\log(V/U_1)}, & U_1\le d\le V,\\ 0, & d\ge V,\end{cases}$$

$$(6.18)\qquad \theta_d=\begin{cases}\mu(d)\dfrac{\log(W/d)}{\log W}, & 1\le d\le W,\\ 0, & d\ge W.\end{cases}$$

We wish to study the sum

$$(6.19)\qquad J(\rho^{(j,m)},\chi):=w_{j,m}\sum_{n=1}^{\infty}\Big(\sum_{d|n}\psi_d\Big)\Big(\sum_{d|n}\theta_d\Big)\chi(n)n^{-\rho^{(j,m)}}j(n),$$

where

$$j(n)=\frac{\int_{x_0}^{x_1}\int_{u_0}^{u_1}(e^{-n/X}-e^{-n\mathcal{L}^2/U})\,du\,dx}{(u_1-u_0)(x_1-x_0)} \tag{6.20}$$

and $w_{j,m}$ are some non-negative weights.

We start with the following weighted-sum result.

Lemma 6.4. *For $x_0 > w+v+\phi_{\chi^{(j)}}$ we have*

$$w_{j,m}^2 \le (1+O(\mathcal{L}^{-1}))|J(\rho^{(j,m)},\chi^{(j)})|^2.$$

Proof. The argument of [11, pp. 317–318] shows that

$$1+O(\mathcal{L}^{-1})=\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\psi_d\Big)\Big(\sum_{d\mid n}\theta_d\Big)\chi^{(j)}(n)n^{-\rho^{(j,m)}}(e^{-n/X}-e^{-n\mathcal{L}^2/U}) \tag{6.21}$$

for $x_0 > w+v+\phi_{\chi^{(j)}}$. We note that in [11] the definition of ψ_d is slightly different (it uses constants labelled U and V rather than U_1 and V as in our case), but this does not affect the argument in any way since $U_1 \ge U$.

Multiplying the above expression by weights $w_{j,m}$ and integrating over $x\in[x_0,x_1]$ and $u\in[u_0,u_1]$ gives

$$w_{j,m}=(1+O(\mathcal{L}^{-1}))J(\rho^{(j,m)},\chi^{(j)}). \tag{6.22}$$

Squaring both sides of the above expression leads to the result. ∎

We sum the expression of Lemma 6.4 over all zeros $\rho^{(j,m)}$. We let $\sum_{j,m}$ denote this sum.

Thus

$$\sum_{j,m} w_{j,m}^2 \le (1+O(\mathcal{L}^{-1}))\sum_{j,m}|J(\rho^{(j,m)},\chi^{(j)})|^2. \tag{6.23}$$

We now use the well-known duality principle, which we will state here for convenience.

Lemma 6.5 (Duality principle). *If*

$$\sum_n\Big|\sum_{j,m}a_{n,j,m}C_{j,m}\Big|^2 \le B\sum_{j,m}|C_{j,m}|^2$$

for all choices of the coefficients $C_{j,m}$, then

$$\sum_{j,m}\Big|\sum_n a_{n,j,m}b_n\Big|^2 \le B\sum_n |b_n|^2$$

for any choice of b_n.

We wish to use Lemma 6.5 with

$$a_{n,j,m} = w_{j,m}\chi^{(j)}(n)n^{1/2-\rho^{(j,m)}}\Big(\sum_{d|n}\theta_d\Big)j(n)^{1/2}, \tag{6.24}$$

$$b_n = \Big(\sum_{d|n}\psi_d\Big)n^{-1/2}j(n)^{1/2} \tag{6.25}$$

to bound this sum. We note that

$$\sum_{n=1}^{\infty} a_{n,j,m}b_n = J(\rho^{(j,m)}, \chi^{(j)}). \tag{6.26}$$

First we evaluate $\sum b_n^2$.

Lemma 6.6. *For $x_0 > v$ we have*

$$\sum_{n=1}^{\infty}|b_n|^2 = (1+O(\mathcal{L}^{-1}\log\mathcal{L}))\frac{x_1+x_0-u_1-v}{2(v-u_1)}.$$

Proof. The argument leading to equation (11.14) of [11, p. 319] shows (recalling that our definition of ψ_d used parameters U_1 and V rather than U and V) that provided $x > v$ we have

$$\sum_{n=1}^{\infty}\Big(\sum_{d|n}\psi_d\Big)^2 n^{-1}(e^{-n/X} - e^{-n\mathcal{L}^2/U}) = (1+O(\mathcal{L}^{-1}\log\mathcal{L}))\frac{2x-u_1-v}{2(v-u_1)}. \tag{6.27}$$

Since $x \geq x_0 > v$, this holds in our case.

Therefore, integrating with respect to $x \in [x_0, x_1]$ and $u \in [u_0, u_1]$ and dividing through by $(x_1-x_0)(u_1-u_0)$ gives

$$\sum_{n=1}^{\infty}\Big(\sum_{d|n}\psi_d\Big)^2 n^{-1}\frac{\int_{x_0}^{x_1}\int_{u_0}^{u_1}(e^{-n/X} - e^{-n\mathcal{L}^2/U})\,du\,dx}{(x_1-x_0)(u_1-u_0)} = (1+O(\mathcal{L}^{-1}\log\mathcal{L}))\frac{x_1+x_0-u_1-v}{2(v-u_1)}. \tag{6.28}$$

Hence the result holds. ∎

Therefore in order to use Lemma 6.5 we want to find a bound B such that

$$\sum_{n=1}^{\infty}\Big|\sum_{j,m}a_{n,j,m}C_{j,m}\Big|^2 \leq B\sum_{j,m}|C_{j,m}|^2 \tag{6.29}$$

for any possible choice of $C_{j,m}$.

Expanding the left hand side, terms are of the form

$$\begin{aligned}(6.30)\quad &\sum_{n=1}^{\infty} a_{n,j_1,m_1} a_{n,j_2,m_2} C_{j_1,m_1} \overline{C}_{j_2,m_2}\\ &\qquad = C_{j_1,m_1}\overline{C}_{j_2,m_2} w_{j_1,m_1} w_{j_2,m_2}\\ &\qquad\quad \times \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\theta_d\Big)^2 \chi^{(j_1)}(n)\overline{\chi}^{(j_2)}(n) n^{1-\rho^{(j_1,m_1)}-\overline{\rho}^{(j_2,m_2)}} j(n).\end{aligned}$$

To ease notation we let

$$(6.31)\qquad \rho_{(1)} = \rho^{(j_1,m_1)}, \qquad \rho_{(2)} = \rho^{(j_2,m_2)},$$

and correspondingly define $\chi_{(1)}, \chi_{(2)}, \beta_{(1)}, \beta_{(2)}, \lambda_{(1)}, \lambda_{(2)}, \gamma_{(1)}, \gamma_{(2)}$.

We first deal with the terms when $\chi_{(1)} \neq \chi_{(2)}$.

We put

$$(6.32)\qquad J_2(s,\chi) = \sum_{w_1,w_2\le W} \theta_{w_1}\theta_{w_2}\chi([w_1,w_2])[w_1,w_2]^{-s}.$$

(Here $[a,b]$ denotes the least common multiple of a and b.) By the inverse Laplace transform of the exponential function we have

$$\begin{aligned}(6.33)\quad &\sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\theta_d\Big)^2 \chi_{(1)}(n)\overline{\chi}_{(2)}(n) n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}(e^{-n/X} - e^{-n\mathcal{L}^2/U})\\ &= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})(X^s - (U\mathcal{L}^{-2})^s)\\ &\quad \times \Gamma(s) J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})\,ds\\ &= \frac{1}{2\pi i}\int_{2-\beta_{(1)}-\beta_{(2)}-1/k-i\infty}^{2-\beta_{(1)}-\beta_{(2)}-1/k+i\infty} L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})(X^s - (U\mathcal{L}^{-2})^s)\\ &\quad \times \Gamma(s) J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})\,ds.\end{aligned}$$

where $k > 10$ is a fixed constant (to be declared later).

On $\Re(s) = 2 - \beta_{(1)} - \beta_{(2)} - 1/k$ with $\chi \neq \chi_0$ we have

$$(6.34)\qquad L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi) \ll_k q^{\phi_\chi/k+1/k^2}(1+|t|),$$

$$(6.35)\qquad \Gamma(s) \ll e^{-|t|},$$

$$\begin{aligned}(6.36)\qquad J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi) &\ll \sum [w_1,w_2]^{-1+1/k}\\ &\ll \sum_{n\le W^2} n^{-1+1/k} d(n)^2 \ll W^{2/k}\mathcal{L}^3.\end{aligned}$$

Thus, letting $\chi = \chi_{(1)}\overline{\chi}_{(2)}$, we obtain

$$\begin{aligned}(6.37)\qquad \frac{1}{2\pi i}\int\limits_{1-\beta_{(1)}-\beta_{(2)}-1/k-i\infty}^{1-\beta_{(1)}-\beta_{(2)}-1/k+i\infty} &L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1,\chi)\Gamma(s)(X^s-(U\mathcal{L}^{-2})^s)\\ &\times J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1,\chi)\,ds\\ &\ll (q^{\phi_\chi}W^2U^{-1}\mathcal{L}^3)^{1/k}q^{1/k^2}\mathcal{L}^2(U\mathcal{L}^{-2})^{2-\beta_{(1)}-\beta_{(2)}}\\ &\ll (q^{\phi_\chi}W^2U^{-1})^{1/k}q^{2/k^2}.\end{aligned}$$

(Recall $1-\beta_{(1)}$ and $1-\beta_{(2)}$ are $o(1)$.)

This is $O(\mathcal{L}^{-1})$ provided that k is chosen sufficiently large and (keeping in mind $\phi_\chi \le 1/3$ for all χ) provided we have

$$(6.38)\qquad u_0 > 2w + 1/3.$$

The terms with $\chi_{(1)} \ne \chi_{(2)}$ therefore contribute

$$(6.39)\qquad \ll \mathcal{L}^{-1}\Big(\sum_{j,m}|C_{j,m}|w_{j,m}\Big)^2 \ll \mathcal{L}^{-1}\Big(\sum_{j,m}w_{j,m}^2\Big)\sum_{j,m}|C_{j,m}|^2.$$

We now consider the terms with $\chi_{(1)} = \chi_{(2)}$. Such terms are of the form

$$(6.40)\qquad C_{(1)}\overline{C}_{(2)}(w_{(1)}w_{(2)})\sum_{n=1}^{\infty}\Big(\sum_{d|n}\theta_d\Big)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n).$$

LEMMA 6.7. *For $x > v$ we have*

$$\begin{aligned}&\Big|\sum_{n=1}^{\infty}\Big(\sum_{d|n}\theta_d\Big)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n)\Big|\\ &\quad\le \bigg|\frac{(1+O(\mathcal{L}^{-1}\log\mathcal{L}))}{w\mathcal{L}^2(2-\rho_{(1)}-\overline{\rho}_{(2)})^2}\bigg|\\ &\quad\times\bigg|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\bigg|+O(\mathcal{L}^{-1}).\end{aligned}$$

Proof. We observe that

$$\begin{aligned}(6.41)\qquad \sum_{n=1}^{\infty}\Big(\sum_{d|n}\theta_d\Big)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}(e^{-n/X}-e^{-n\mathcal{L}^2/U})&\\ = \sum_{d_1,d_2}\theta_{d_1}\theta_{d_2}\chi_0([d_1,d_2])[d_1,d_2]^{1-\rho_{(1)}-\overline{\rho}_{(2)}}&\\ \times\sum_{k=1}^{\infty}k^{1-\rho_{(1)}-\overline{\rho}_{(2)}}\chi_0(k)(e^{-k[d_1,d_2]/X}-e^{-k[d_1,d_2]\mathcal{L}^2/U}).&\end{aligned}$$

By the inverse Laplace transform of the exponential function again we have

$$(6.42)\qquad \sum_{k=1}^{\infty}\chi_0(k)k^{1-\rho_{(1)}-\overline{\rho}_{(2)}}(e^{-k[d_1,d_2]/X}-e^{-k[d_1,d_2]\mathcal{L}^2/U}) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1,\chi_0)\Gamma(s) \times\Big(\Big(\frac{X}{[d_1,d_2]}\Big)^s-\Big(\frac{U}{\mathcal{L}^2[d_1,d_2]}\Big)^s\Big)\,ds.$$

We again move the line of integration to

$$\Re(s)=2-\beta_{(1)}-\beta_{(2)}-1/k,$$

and by exactly the same reasoning, we find that the integral over this contour is negligible when $u_0>2w$. We encounter a pole at $s=2-\rho_{(1)}-\overline{\rho}_{(2)}$, however, which contributes

$$(6.43)\qquad \frac{\phi(q)}{q}\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)}) \times\Big(\Big(\frac{X}{[d_1,d_2]}\Big)^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-\Big(\frac{U}{\mathcal{L}^2[d_1,d_2]}\Big)^{2-\rho_{(1)}-\overline{\rho}_{(2)}}\Big).$$

Thus

$$(6.44)\qquad \sum_{n=1}^{\infty}\Big(\sum_{d\mid n}\theta_d\Big)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}(e^{-n/X}-e^{-n\mathcal{L}^2/U}) = \frac{\phi(q)}{q}\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})(X^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-(U\mathcal{L}^{-2})^{2-\rho_{(1)}-\overline{\rho}_{(2)}}) \times\sum_{d_1,d_2}\frac{\theta_{d_1}\theta_{d_2}\chi_0([d_1,d_2])}{[d_1,d_2]}+O(\mathcal{L}^{-1}).$$

We now perform the integrations with respect to x and u. We have

$$(6.45)\qquad \frac{1}{(x_1-x_0)(u_1-u_0)}\int_{x_0}^{x_1}\int_{u_0}^{u_1}(X^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-(U\mathcal{L}^{-2})^{2-\rho_{(1)}-\overline{\rho}_{(2)}})\,du\,dx = \Big(\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\Big) \times\frac{1}{\mathcal{L}(2-\rho_{(1)}-\overline{\rho}_{(2)})}.$$

Thus

$$(6.46)\quad \Big|\sum_{n=1}^{\infty}\Big(\sum_{d|n}\theta_d\Big)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n)\Big|$$
$$\le \frac{\phi(q)}{\mathcal{L}q}\left|\frac{\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})}{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right|\times\left|\sum_{d_1,d_2}\frac{\theta_{d_1}\theta_{d_2}\chi_0([d_1,d_2])}{[d_1,d_2]}\right|$$
$$\times\left|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\right|+O(\mathcal{L}^{-1}).$$

We now estimate the sum over d_1, d_2:

$$(6.47)\quad \Big|\sum_{d_1,d_2}\theta_{d_1}\theta_{d_2}[d_1,d_2]^{-1}\chi_0([d_1,d_2])\Big|$$
$$=\frac{1}{N}\Bigg|\sum_{d_1,d_2\le W}\theta_{d_1}\theta_{d_2}\Bigg(\frac{q}{\phi(q)}\sum_{\substack{[d_1,d_2]|n\\ n\le N\\ (n,q)=1}}1+O(q)\Bigg)\Bigg|$$
$$=\frac{q}{\phi(q)N}\sum_{\substack{n\le N\\ (n,q)=1}}\Big(\sum_{d|n}\theta_d\Big)^2+O(qW^2N^{-1})$$
$$\le\frac{q}{\phi(q)N}\sum_{n\le N}\Big(\sum_{d|n}\theta_d\Big)^2+O(qW^2N^{-1}).$$

Graham [8] has shown that for $N > q^2W^2$ we have

$$(6.48)\quad N^{-1}\sum_{n\le N}\Big(\sum_{d|n}\theta_d\Big)^2=\frac{1+O(\mathcal{L}^{-1})}{\log W}.$$

Hence, for $N > q^2W^2$,

$$(6.49)\quad \Big|\sum_{d_1,d_2}\theta_{d_1}\theta_{d_2}[d_1,d_2]^{-1}\chi_0([d_1,d_2])\Big|\le\frac{q}{\phi(q)N}\sum_{n\le N}\Big(\sum_{d|n}\theta_d\Big)^2+O(q^{-1})$$
$$=\frac{(1+O(\mathcal{L}^{-1}))q}{\phi(q)\log W}=(1+O(\mathcal{L}^{-1}))\frac{q}{\phi(q)w\mathcal{L}}.$$

Thus

$$(6.50)\quad \Big|\sum_{n=1}^{\infty}\Big(\sum_{d|n}\theta_d\Big)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n)\Big|$$
$$\le\frac{1+O(\mathcal{L}^{-1})}{\mathcal{L}^2w}\left|\frac{\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})}{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right|$$
$$\times\left|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\right|+O(\mathcal{L}^{-1}).$$

We recall the Weierstrass product expansion of $\Gamma(s)$:

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}. \tag{6.51}$$

When $s = 2 - \rho_{(1)} - \overline{\rho}_{(2)}$, since $2 - \beta_{(1)} - \beta_{(2)} = O(\mathcal{L}^{-1} \log \mathcal{L})$, we therefore have

$$\begin{aligned} |\Gamma(s)| &\le \frac{e^{-\gamma \Re(s)}}{|s|} \prod_{n=1}^{\infty} \left|1 + \frac{s}{n}\right|^{-1} e^{\Re(s)/n} \\ &\le \frac{1 + O(\mathcal{L}^{-1} \log \mathcal{L})}{|s|} \prod_{n=1}^{\infty} \left(1 + \frac{\Re(s)}{n}\right)^{-1} e^{\Re(s)/n} \\ &\le \frac{1 + O(\mathcal{L}^{-1} \log \mathcal{L})}{|2 - \rho_{(1)} - \overline{\rho}_{(2)}|} \prod_{n=1}^{\infty} \left(1 + O\left(\frac{\Re(s)}{n^2}\right)\right) \\ &\le \frac{1 + O(\mathcal{L}^{-1} \log \mathcal{L})}{|2 - \rho_{(1)} - \overline{\rho}_{(2)}|}. \ \blacksquare \end{aligned} \tag{6.52}$$

To simplify notation we put

$$j_2(a,b) := \frac{1}{\mathcal{L}^2 (2 - a - b)^2} \left| \frac{X_1^{2-a-b} - X_0^{2-a-b}}{x_1 - x_0} - \frac{U_1^{2-a-b} - U_0^{2-a-b}}{u_1 - u_0} \right|. \tag{6.53}$$

Thus the sum over all the terms of the form (6.40) with $\chi_{(1)} = \chi_{(2)}$ is

$$\begin{aligned} &\le \frac{1 + O(\mathcal{L}^{-1} \log \mathcal{L})}{w} \sum_{\substack{\rho_{(1)}, \rho_{(2)} \\ \chi_{(1)} = \chi_{(2)}}} |C_{(1)} C_{(2)} w_{(1)} w_{(2)} j_2(\rho_{(1)}, \rho_{(2)})| \\ &\quad + O\Big(\mathcal{L}^{-1} \sum_{\substack{\rho_{(1)}, \rho_{(2)} \\ \chi_{(1)} = \chi_{(2)}}} |C_{(1)} C_{(2)} w_{(1)} w_{(2)})|\Big). \end{aligned} \tag{6.54}$$

We put

$$G_2 = \max_{\rho_{(1)}} \sum_{\substack{\rho_{(2)} \\ \chi_{(2)} = \chi_{(1)}}} |w_{(1)} w_{(2)} j_2(\rho_{(1)}, \rho_{(2)})|. \tag{6.55}$$

Since $2|C_{(1)} C_{(2)}| \le |C_{(1)}|^2 + |C_{(2)}|^2$, we have

$$\sum_{\substack{\rho_{(1)}, \rho_{(2)} \\ \chi_{(1)} = \chi_{(2)}}} |C_{(1)} C_{(2)} w_{(1)} w_{(2)} j_2(\rho_{(1)}, \rho_{(2)})| \le G_2 \sum_{\rho_{(1)}} |C_{(1)}|^2. \tag{6.56}$$

Combining (6.39) and (6.56) gives

$$\sum_{n=1}^{\infty}\Big|\sum_{j,m}a_{n,j,m}C_{j,m}\Big|^2 \le \Big(\frac{G_2}{w}(1+O(\mathcal{L}^{-1}\log\mathcal{L}))+O\Big(\mathcal{L}^{-1}\sum_{j,m}w_{j,m}^2\Big)\Big)\sum_{j,m}|C_{j,m}|^2 \tag{6.57}$$

for any choice of the coefficients $C_{j,m}$.

Therefore by (6.23), Lemma 6.5 and Lemma 6.6 we have

$$\sum_{j,m}w_{j,m}^2 \le (1+O(\mathcal{L}^{-1}\log\mathcal{L}))\Big(\frac{G_2}{w}+O\Big(\mathcal{L}^{-1}\sum_{j,m}w_{j,m}^2\Big)\Big) \times\frac{x_1+x_0-u_1-v}{2(v-u_1)}, \tag{6.58}$$

which gives

$$\sum_{j,m}w_{j,m}^2 \le (1+O(\mathcal{L}^{-1}\log\mathcal{L}))\frac{x_1+x_0-u_1-v}{2w(v-u_1)}G_2. \tag{6.59}$$

We are therefore left to choose suitable weights $w_{j,m}$, bound G_2 and choose suitable constants w, u_0, u_1, v, x_0, x_1.

We note that, using Cauchy's inequality, we have

$$\begin{aligned}&\left|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\right|\\ &\quad\le\frac{e^{(\lambda_{(1)}+\lambda_{(2)})x_1}+e^{(\lambda_{(1)}+\lambda_{(2)})x_0}}{x_1-x_0}+\frac{e^{(\lambda_{(1)}+\lambda_{(2)})u_1}+e^{(\lambda_{(1)}+\lambda_{(2)})u_0}}{u_1-u_0}\\ &\quad\le\left(\frac{e^{2\lambda_{(1)}x_1}+e^{2\lambda_{(1)}x_0}}{x_1-x_0}+\frac{e^{2\lambda_{(1)}u_1}+e^{2\lambda_{(1)}u_0}}{u_1-u_0}\right)^{1/2}\\ &\qquad\times\left(\frac{e^{2\lambda_{(2)}x_1}+e^{2\lambda_{(2)}x_0}}{x_1-x_0}+\frac{e^{2\lambda_{(2)}u_1}+e^{2\lambda_{(2)}u_0}}{u_1-u_0}\right)^{1/2}.\end{aligned} \tag{6.60}$$

Also

$$\begin{aligned}\sum_{\rho_{(2)}}|\mathcal{L}^{-2}(2-\rho_{(1)}-\rho_{(2)})^{-2}| &= \sum_{\rho_{(2)}}\frac{1}{(\lambda_{(1)}+\lambda_{(2)})^2+(v_{(1)}-v_{(2)})^2}\\ &\le 2\sum_{m=0}^{\infty}\frac{1}{(\lambda_{(1)}+\lambda_{(2)})^2+m^2},\end{aligned} \tag{6.61}$$

since $|\Im(\rho^{(j,m_1)})-\Im(\rho^{(j,m_2)})|\ge(|m_1-m_2|-1)/\mathcal{L}$ by our choice of the rectangles $\mathcal{R}_m$.

Motivated by these observations we choose

$$w_{j,m}=\left(\frac{e^{2\lambda^{(j,m)}x_1}+e^{2\lambda^{(j,m)}x_0}}{x_1-x_0}+\frac{e^{2\lambda^{(j,m)}u_1}+e^{2\lambda^{(j,m)}u_0}}{u_1-u_0}\right)^{-1/2}. \tag{6.62}$$

We assume from here on that we are only considering zeros $\rho^{(j,m)}$ with $\lambda^{(j,m)}\geq\lambda_{\min}$, for some fixed value of $\lambda_{\min}$.

We now wish to estimate G_2, and so bound $\sum_{\rho_{(2)}}|w_{(1)}w_{(2)}j_2(\rho_{(1)},\rho_{(2)})|$. We assume $\rho_{(1)}$ is in a rectangle $\mathcal{R}_{m_1}$ and then consider the contributions $G_{2,c}$ from zeros in rectangles $\mathcal{R}_{m_2}$ where $|m_1-m_2|=c\in\mathbb{Z}$ (since we have picked a fixed zero in each rectangle, there are at most two zeros corresponding to each choice of c).

We first consider $c=0$. In this case $\rho_{(2)}=\rho_{(1)}$ (and there is only one zero). This contributes at most

$$\begin{aligned}
G_{2,0}&\leq\sup_{\rho_{(1)}}|j(\rho_{(1)},\rho_{(1)})w_{(1)}^2| \\
&=\sup_{\rho_{(1)}}\left(\frac{X_1^{2-2\beta_{(1)}}-X_0^{2-2\beta_{(1)}}}{x_1-x_0}-\frac{U_1^{2-2\beta_{(1)}}-U_0^{2-2\beta_{(1)}}}{u_1-u_0}\right) \\
&\quad\times\left(\frac{X_1^{2-2\beta_{(1)}}+X_0^{2-2\beta_{(1)}}}{x_1-x_0}+\frac{U_1^{2-2\beta_{(1)}}+U_0^{2-2\beta_{(1)}}}{u_1-u_0}\right)^{-1}(2\lambda_{(1)})^{-2} \\
&=\sup_{\lambda_{(1)}\geq\lambda_{\min}}\left(\frac{e^{2x_1\lambda_{(1)}}-e^{2x_0\lambda_{(1)}}}{x_1-x_0}-\frac{e^{2u_1\lambda_{(1)}}-e^{2u_0\lambda_{(1)}}}{u_1-u_0}\right) \\
&\quad\times\left(\frac{e^{2x_1\lambda_{(1)}}+e^{2x_0\lambda_{(1)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(1)}}+e^{2u_0\lambda_{(1)}}}{u_1-u_0}\right)^{-1}(2\lambda_{(1)})^{-2}.
\end{aligned} \tag{6.63}$$

We now deal with the case $1\leq c\leq 6$. This means that

$$c-1\leq|\Im(\rho_{(1)})-\Im(\rho_{(2)})|\leq c+1,$$

and there are at most two zeros $\rho_{(2)}$. These zeros contribute at most

$$\begin{aligned}
&2\sup_{\substack{\lambda_{(1)},\lambda_{(2)}\geq\lambda_{\min}\\ c-1\leq t\leq c+1}}\left(\frac{e^{2x_1\lambda_{(1)}}+e^{2x_0\lambda_{(1)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(1)}}+e^{2u_0\lambda_{(1)}}}{u_1-u_0}\right)^{-1/2} \\
&\times\left(\frac{e^{2x_1\lambda_{(2)}}+e^{2x_0\lambda_{(2)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(2)}}+e^{2u_0\lambda_{(2)}}}{u_1-u_0}\right)^{-1/2}((\lambda_{(1)}+\lambda_{(2)})^2+t^2)^{-1} \\
&\times\left|\frac{e^{x_1(\lambda_{(1)}+\lambda_{(2)}+it)}-e^{x_0(\lambda_{(1)}+\lambda_{(2)}+it)}}{x_1-x_0}-\frac{e^{u_1(\lambda_{(1)}+\lambda_{(2)}+it)}-e^{u_0(\lambda_{(1)}+\lambda_{(2)}+it)}}{u_1-u_0}\right|.
\end{aligned} \tag{6.64}$$

As in (6.60), it follows from Cauchy's inequality that

$$\begin{aligned}(6.65)\quad &\left(\frac{e^{(\lambda_{(1)}+\lambda_{(2)})x_1}+e^{(\lambda_{(1)}+\lambda_{(2)})x_0}}{x_1-x_0}+\frac{e^{(\lambda_{(1)}+\lambda_{(2)})u_1}+e^{(\lambda_{(1)}+\lambda_{(2)})u_0}}{u_1-u_0}\right)^2\\ &\quad\le\left(\frac{e^{2x_1\lambda_{(2)}}+e^{2x_0\lambda_{(2)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(2)}}+e^{2u_0\lambda_{(2)}}}{u_1-u_0}\right)\\ &\quad\times\left(\frac{e^{2x_1\lambda_{(1)}}+e^{2x_0\lambda_{(1)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(1)}}+e^{2u_0\lambda_{(1)}}}{u_1-u_0}\right).\end{aligned}$$

Hence

$$\begin{aligned}(6.66)\quad G_{2,c}\le 2\sup_{\substack{\lambda\ge\lambda_{\min}\\ c-1\le t\le c+1}}&\left|\frac{e^{x_1(2\lambda+it)}-e^{x_0(2\lambda+it)}}{x_1-x_0}-\frac{e^{u_1(2\lambda+it)}-e^{u_0(2\lambda+it)}}{u_1-u_0}\right|\\ &\times\left(\frac{e^{2x_1\lambda}+e^{2x_0\lambda}}{x_1-x_0}+\frac{e^{2u_1\lambda}+e^{2u_0\lambda}}{u_1-u_0}\right)^{-1}(4\lambda^2+t^2)^{-1}.\end{aligned}$$

When $c\ge 7$ we use the simple estimate

$$(6.67)\quad G_{2,c}\le 2\sup_{\substack{\lambda_{(1)},\lambda_{(2)}\ge\lambda_{\min}\\ c-1\le t\le c+1}}((\lambda_{(1)}+\lambda_{(2)})^2+t^2)^{-1}\le\frac{2}{4\lambda_{\min}^2+(c-1)^2}.$$

For given constants x_1,x_0,u_1,u_0,w,v and $\lambda_{\min}$ we use *Mathematica*'s NMaximize function to calculate the bounds above for $G_{2,0}$ and $G_{2,c}$ for $1\le c\le 6$. We can estimate the bound given for $G_{2,c}$ when $7\le c\le 101$ exactly, and then for $c\ge 102$ we use an integral comparison to see that

$$\begin{aligned}(6.68)\quad \sum_{c\ge 102}G_{2,c}&\le\sum_{m\ge 101}\frac{2}{4\lambda_{\min}^2+m^2}\le\int_{100}^{\infty}\frac{2}{4\lambda_{\min}^2+t^2}\,dt\\ &\le\frac{\tan^{-1}(\lambda_{\min}/50)}{\lambda_{\min}}.\end{aligned}$$

We can then use this information to estimate G_2:

$$(6.69)\quad G_2\le G_{2,0}+\sum_{1\le c\le 6}G_{2,c}+\sum_{6\le m\le 100}\frac{2}{4\lambda_{\min}^2+m^2}+\frac{\tan^{-1}(\lambda_{\min}/50)}{\lambda_{\min}}.$$

As is the case in [11], it is optimal to choose $u_0=2w+1/3+\delta$ and $x_0=w+v+1/3+\delta$ with δ small. We will take $\delta=2/3000$ for our purposes. We are then left to choose suitable positive constants w, $u_1\ge u_0$, $v\ge u_1$ and $x_1\ge x_0$. We fix these now as

$$(6.70)\quad w=0.115,\qquad u_0=0.564,\qquad u_1=0.620,$$

$$(6.71)\quad v=0.964,\qquad x_0=1.413,\qquad x_1=1.623.$$

We consider $\lambda_{\min}=0.35$. For this value we calculate that

$$(6.72)\quad G_2\le 0.650.$$

Putting everything together we obtain

$$\sum_{\substack{j,m\\ \lambda^{(j,m)}\geq 0.35}} \left(\frac{e^{3.246\lambda^{(j,m)}}+e^{2.826\lambda^{(j,m)}}}{0.210}+\frac{e^{1.240\lambda^{(j,m)}}+e^{1.128\lambda^{(j,m)}}}{0.056}\right)^{-1} \leq 11.9288. \tag{6.73}$$

6.3. Third zero density estimate. We now prove Lemma 6.3. The proof uses the ideas from [11, Section 12] to obtain a stronger zero density estimate close to 1, but agan we extend this to our slightly larger region with $\Im(\rho)\ll 1$. Specifically we wish to estimate

$$N^*(\lambda):=\#\{\rho^{(j,m)}\in\mathcal{R}:\lambda^{(j,m)}\leq\lambda\} \tag{6.74}$$

in the range $0\leq\lambda\leq 2$. We note that from the log-free zero density bound, for $0\leq\lambda\leq 2$ we know that $N^*(\lambda)$ is uniformly bounded in q and λ.

Throughout this section we assume that we have a fixed non-negative constant λ_{11} such that $\lambda_{11}\leq\lambda_1$. We put $\beta_{11}=1-\lambda_{11}/\mathcal{L}$.

We adopt the notation of [11]. We put

$$K(s,\chi):=\sum_{n=1}^{\infty}\Lambda(n)\Re\left(\frac{\chi(n)}{n^s}\right)g(\mathcal{L}^{-1}\log n) \tag{6.75}$$

for some function g which satisfies:

Condition 1. $g:[0,\infty)\to\mathbb{R}$ is continuous, g is supported on $[0,x_0)$ for some $x_0>0$, g is twice differentiable on $(0,x_0)$ and g'' is bounded on $(0,x_0)$.

Condition 2. g is non-negative and its Laplace transform G satisfies $\Re(G(z))\geq 0$ for $\Re(z)\geq 0$.

We start with the following estimate:

Lemma 6.8. *Let g be a function satisfying Conditions* 1 *and* 2 *and let $\delta>0$. Then for $q>q_0(\delta,g)$ and $\lambda_1\geq\lambda_{11}$, if*

$$G(\lambda-\lambda_{11})>g(0)/6,\qquad (G(\lambda-\lambda_{11})-g(0)/6)^2>G(-\lambda_{11})g(0)/6 \tag{6.76}$$

then

$$N^*(\lambda)\leq\frac{G(-\lambda_{11})G_3}{(G(\lambda-\lambda_{11})-g(0)/6)^2-G(-\lambda_{11})g(0)/6}+\delta,$$

where G_3 will be defined in equation (6.90).

Proof. The first inequality of [11, Section 12] shows that for $q>q_0(g,\delta_1)$ we have

$$\mathcal{L}^{-1}K(\beta_{11}+i\gamma^{(j,m)},\chi^{(j)})\leq g(0)\phi_{\chi^{(j)}}/2+\delta_1-G(\lambda^{(j,m)}-\lambda_{11}). \tag{6.77}$$

Therefore, for any zero $\rho^{(j,m)}$ with $G(\lambda^{(j,m)}-\lambda_{11}) > g(0)\phi_{\chi^{(j)}}/2$ we obtain

$$0 < G(\lambda^{(j,m)}-\lambda_{11}) - g(0)\phi_{\chi^{(j)}}/2 \le -\mathcal{L}^{-1}K(\beta_{11}+i\gamma^{(j,m)},\chi^{(j)}) + \delta_1. \tag{6.78}$$

We note that $G(\lambda^{(j,m)}-\lambda_{11})$ is a decreasing function in $\lambda^{(j,m)}$ and recall that $\phi_\chi \le 1/3$ for all characters χ. Therefore, if

$$G(\lambda-\lambda_{11}) > g(0)/6, \tag{6.79}$$

then for any $\lambda^{(j,m)} \le \lambda$ we see that

$$\begin{aligned} 0 &\le G(\lambda-\lambda_{11}) - g(0)/6 \le G(\lambda^{(j,m)}-\lambda_{11}) - g(0)\phi_{\chi^{(j)}}/2 \\ &\le -\mathcal{L}^{-1}K(\beta_{11}+i\gamma^{(j,m)},\chi^{(j)}) + \delta_1. \end{aligned} \tag{6.80}$$

We sum over all j,m for which $\lambda^{(j,m)} \le \lambda$. Thus for $q > q_0(g,\delta_1)$ we have

$$\begin{aligned} &N^*(\lambda)(G(\lambda-\lambda_{11}) - g(0)/6) \\ &\le \sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} G(\lambda^{(j,m)}-\lambda_{11}) - g(0)/6 \\ &\le -\mathcal{L}^{-1} \sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} K(\beta_{11}+i\gamma^{(j,m)},\chi^{(j)}) + \sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} \delta_1 \\ &= -\mathcal{L}^{-1}\sum_{n=1}^{\infty} \Lambda(n)n^{-\beta_{11}} g(\mathcal{L}^{-1}\log n)\Re\Big(\sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} \chi^{(j)}(n)n^{-i\gamma^{(j,m)}}\Big) + \delta_2 \\ &\le \mathcal{L}^{-1}\sum_{n=1}^{\infty} \Lambda(n)n^{-\beta_{11}}\chi_0(n) g(\mathcal{L}^{-1}\log n)\Big|\sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} \chi^{(j)}(n)n^{-i\gamma^{(j,m)}}\Big| + \delta_2 \\ &\le \Sigma_1^{1/2}\Sigma_2^{1/2} + \delta_2, \end{aligned} \tag{6.81}$$

where

$$\delta_2 = \sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} \delta_1, \tag{6.82}$$

$$\Sigma_1 = \mathcal{L}^{-1}\sum_{n=1}^{\infty} \Lambda(n)n^{-\beta_{11}}\chi_0(n) g(\mathcal{L}^{-1}\log n), \tag{6.83}$$

$$\Sigma_2 = \mathcal{L}^{-1}\sum_{n=1}^{\infty} \Lambda(n)n^{-\beta_{11}} g(\mathcal{L}^{-1}\log n)\Big|\sum_{\substack{j,m \\ \lambda^{(j,m)}\le\lambda}} \chi^{(j)}(n)n^{-i\gamma^{(j,m)}}\Big|^2. \tag{6.84}$$

By [11, Lemma 5.3] for $q > q_0(g,\delta_1)$ we have

$$\Sigma_1 = \mathcal{L}^{-1}K(\beta_{11},\chi_0) \le G(-\lambda_{11}) + \delta_1. \tag{6.85}$$

We expand the square in Σ_2 and see that

$$(6.86)\quad \begin{aligned}\Sigma_2 &= \Re(\Sigma_2)\\ &= \mathcal{L}^{-1} \sum_{\substack{j_1,j_2,m_1,m_2\\ \lambda^{(j_1,m_1)},\lambda^{(j_2,m_2)}\le\lambda}} K(\beta_{11}+i(\gamma^{(j_1,m_1)}-\gamma^{(j_2,m_2)}),\chi^{(j_1)}\overline{\chi}^{(j_2)}).\end{aligned}$$

By [11, Lemma 5.3] the terms with $j_1=j_2$ contribute a total

$$(6.87)\quad \begin{aligned}&\mathcal{L}^{-1}\sum_{j_1,m_1,m_2} K(\beta_{11}+i(\gamma^{(j_1,m_1)}-\gamma^{(j_1,m_2)}),\chi_0)\\ &\qquad\le \sum_{j_1,m_1,m_2}(|\Re(G(-\lambda_{11}+i(v^{(j_1,m_1)}-v^{(j_1,m_2)})))|+\delta_1).\end{aligned}$$

By [11, Lemma 5.2] the terms with $j_1\ne j_2$ contribute

$$(6.88)\quad \begin{aligned}&\mathcal{L}^{-1}\sum_{j_1\ne j_2,m_1,m_2} K(\beta_{11}+i(\gamma^{(j_1,m_1)}-\gamma^{(j_2,m_2)}),\chi^{(j_1)}\overline{\chi}^{(j_2)})\\ &\qquad\le \sum_{j_1\ne j_2,m_1,m_2}(g(0)/6+\delta_1).\end{aligned}$$

Putting these together we get

$$(6.89)\quad \begin{aligned}\Sigma_2 \le &\sum_{\substack{j_1,m_1,m_2\\ \lambda^{(j_1,m_1)},\lambda^{(j_1,m_2)}\le\lambda}}(|\Re(G(-\lambda_{11}+i(\nu^{(j_1,m_1)}-\nu^{(j_1,m_2)})))|-g(0)/6)\\ &+N^*(\lambda)^2 g(0)/6+\delta_3.\end{aligned}$$

We define

$$(6.90)\quad G_3 := \sup_{j_1,m_1}\sum_{m_2}(|\Re(G(-\lambda_{11}+i(\gamma^{(j_1,m_1)}-\gamma^{(j_1,m_2)})))|-g(0)/6),$$

so

$$(6.91)\quad \Sigma_2 \le N^*(\lambda)^2 g(0)/6 + N^*(\lambda)G_3+\delta_3.$$

Combining (6.81), (6.85) and (6.91) we obtain

$$(6.92)\quad \begin{aligned}N^*(\lambda)^2(G(\lambda-\lambda_{11})-g(0)/6)^2 &\le \Sigma_1\Sigma_2+\delta_2\\ &\le (G(-\lambda_{11})+\delta_1)(N^*(\lambda)^2 g(0)/6+N^*(\lambda)G_3+\delta_3)+\delta_2.\end{aligned}$$

Since $N^*(\lambda)$ is bounded uniformly for $0\le\lambda\le 2$ by the log-free zero density estimate, all the sums and terms are finite. Hence, by a suitable choice of δ_1 we deduce for given $\delta>0$ and $q>q_0(g,\delta)$ that

$$(6.93)\quad N^*(\lambda)((G(\lambda-\lambda_{11})-g(0)/6)^2-G(-\lambda_{11})g(0)/6)^2 \le G(-\lambda_{11})G_3+\delta.$$

Therefore the lemma holds. ∎

We are now left to choose a suitable function g and evaluate this expression. As in the work of Heath-Brown [11] and Xylouris [26] we choose

$$g(t) := \begin{cases} \int_{t-\gamma}^{\gamma}(\gamma^2-x^2)(\gamma^2-(t-x)^2)\,dx \\ \qquad = -\frac{1}{30}t^5+\frac{2\gamma^2}{3}t^3-\frac{4\gamma^3}{3}t^2+\frac{16\gamma^5}{15}, & t\in[0,2\gamma), \\ 0, & t\geq 2\gamma, \end{cases} \tag{6.94}$$

for some constant $\gamma > 0$.

We see that g is the convolution of $\max(0,\gamma^2-x^2)$ with itself, and so satisfies Condition 2, that is, $\Re(z)\geq 0 \Rightarrow \Re(G(z))\geq 0$. We also see that g is twice differentiable on $(0,2\gamma)$ and its second derivative is continuous and bounded, and so also fulfills Condition 1.

We see the Laplace transform G is

$$\begin{aligned} G(z) &= \int_0^\infty e^{-zt}g(t)\,dt \\ &= \begin{cases} \frac{16\gamma^5}{15}z^{-1}-\frac{8\gamma^3}{3}z^{-3}+4\gamma^2(1+e^{-2\gamma z})z^{-4} \\ \qquad +4(-1+e^{-2\gamma z}+2\gamma z e^{-2\gamma z})z^{-6}, & z\neq 0, \\ \frac{8\gamma^6}{9}, & z=0. \end{cases} \end{aligned} \tag{6.95}$$

We bound G_3 in the same manner as we did in proving Lemma 6.2. We recall

$$G_3(\lambda) = \sup_{m_1,j_1}\sum_{m_2}(|\Re(G(-\lambda_{11}+i(v^{(j_1,m_1)}-v^{(j_2,m_2)})))|-g(0)/6). \tag{6.96}$$

As in the proof of Lemma 6.2 we consider the contribution $G_{3,c}$ of zeros from rectangles $\mathcal{R}_{m_2}$ with $|m_1-m_2|=c\in\mathbb{Z}$.

We first consider $G_{3,0}$. There is only one zero $\rho^{(j_1,m_2)}=\rho^{(j_1,m_1)}$, if it exists. Thus

$$G_{3,0}\leq G(-\lambda_{11})-g(0)/6. \tag{6.97}$$

For $G_{3,c}$ with $1\leq c\leq 5$ we see that there are at most two zeros both with $c-1\leq|v^{(j_1,m_1)}-v^{(j_1,m_2)}|\leq c+1$. These contribute

$$G_{3,c}\leq 2\max\Big(\sup_{c-1\leq t\leq c+1}|\Re(G(-\lambda_{11}+it))|-g(0)/6,0\Big). \tag{6.98}$$

We estimate them using *Mathematica*'s NMaximize function.

We use a simpler bound to estimate $G_{3,c}$ with $c\geq 6$. Letting $z=x+iy$ we have

$$\begin{aligned} |\Re(G(z))| \leq \left|\frac{16\gamma^5}{15}\Re(z^{-1})\right|+\left|8\frac{\gamma^3}{3}\Re(z^{-3})\right|+4\gamma^2|\Re((1+e^{-2\gamma z})z^{-4})| \\ +4|\Re((-1+e^{-2\gamma z}+2\gamma z e^{-2\gamma z})z^{-6})| \end{aligned} \tag{6.99}$$

$$\begin{aligned}&\le \frac{16\gamma^5 x}{15(x^2+y^2)} + \frac{8\gamma^3(|x|^3+3|x|y^2)}{3(x^2+y^2)^3} + \frac{4\gamma^2(1+e^{-2\gamma x})}{(x^2+y^2)^2}\\ &\quad + 4(1+e^{-2\gamma x}+2\gamma(x^2+y^2)^{1/2}e^{-2\gamma x})(x^2+y^2)^{-3}\\ &=: G_4(x,y).\end{aligned}$$

We see that $G_4(x,y)$ is decreasing in y, and so

$$\begin{aligned}G_{3,c} &\le 2\max\Big(\sup_{c-1\le|t|\le c+1} |\Re(G(-\lambda_{11}+it))| - g(0)/6, 0\Big)\\ &\le 2\max(G_4(-\lambda_{11}, c-1) - g(0)/6, 0).\end{aligned} \tag{6.100}$$

We estimate this directly. We note that if $G_4(-\lambda_{11}, c_1 - 1) \le g(0)/6$ then $G_{3,c} \le 0$ for all $c \ge c_1$.

Using these estimates we can then bound G_3 for any given value of our parameter γ and a given lower bound λ_{11} for λ_1.

We consider separately the cases $\lambda_1 \ge 0.35$, $\lambda_1 \ge 0.40$, $\lambda_1 \ge 0.44$, $\lambda_1 \ge 0.52$, $\lambda_1 \ge 0.60$, $\lambda_1 \ge 0.66$ and $\lambda_1 \ge 6/7$. In each case we choose the value of $\gamma \in \{1.00, 1.01, \dots, 1.60\}$ which gives the best bound whilst ensuring that conditions (6.76) still hold.

We give the results in the following table. We note that in comparison with [11, Table 13] these are worse by a factor of approximately 4, but are counting the number of rectangles containing a zero rather than just the number of characters.

Table 1. Third zero density estimate

λ	Bound for $N^*(\lambda)$						
	$\lambda_1 \ge 0.35$	$\lambda_1 \ge 0.40$	$\lambda_1 \ge 0.44$	$\lambda_1 \ge 0.52$	$\lambda_1 \ge 0.60$	$\lambda_1 \ge 0.66$	$\lambda_1 \ge 6/7$
0.74	30	29	28	27	26	26	–
0.75	31	30	29	28	27	26	–
0.76	32	31	30	29	28	27	–
0.77	33	32	31	30	29	28	–
0.78	34	33	32	31	29	29	–
0.79	35	34	33	32	30	29	–
0.80	36	35	34	32	31	30	–
0.81	37	36	35	33	32	31	–
0.82	38	37	36	34	33	32	–
0.83	40	38	37	35	34	33	–
0.84	41	39	38	37	35	34	–
0.85	42	41	40	38	36	35	–
0.86	44	42	41	39	37	36	–
0.87	45	44	42	40	38	37	34

Table 1 (cont.)

λ	Bound for $N^*(\lambda)$						
	$\lambda_1 \geq$ 0.35	$\lambda_1 \geq$ 0.40	$\lambda_1 \geq$ 0.44	$\lambda_1 \geq$ 0.52	$\lambda_1 \geq$ 0.60	$\lambda_1 \geq$ 0.66	$\lambda_1 \geq$ 6/7
0.88	47	45	44	41	39	38	35
0.89	49	47	45	43	41	39	36
0.90	51	49	47	44	42	40	37
0.91	53	50	49	46	43	42	38
0.92	55	52	51	47	45	43	39
0.93	57	54	52	49	46	44	40
0.94	59	57	55	51	48	46	41
0.95	62	59	57	53	49	47	43
0.96	65	61	59	55	51	49	44
0.97	68	64	61	57	53	51	45
0.98	71	67	64	59	55	52	47
0.99	74	70	67	61	57	54	48
1.00	78	73	70	64	59	56	50
1.01	82	77	73	67	62	58	51
1.02	86	80	76	70	64	61	53
1.03	91	84	80	73	67	63	55
1.04	96	89	84	76	70	66	57
1.05	101	94	88	80	73	68	59
1.06	108	99	93	83	76	71	61
1.07	114	105	98	88	79	74	63
1.08	122	111	104	92	83	78	65
1.09	131	118	110	97	87	81	68
1.10	141	127	117	103	91	85	71
1.11	152	136	125	109	96	89	73
1.12	164	146	134	115	101	94	76
1.13	179	157	143	122	107	98	80
1.14	197	171	155	130	113	104	83
1.15	218	186	167	139	120	110	87
1.16	243	205	182	150	128	116	91
1.17	274	226	199	161	136	123	95
1.18	313	253	220	175	146	131	100
1.19	365	286	244	190	156	140	105
1.20	435	328	274	208	169	149	110
1.21	536	383	312	229	183	160	116
1.22	695	458	361	255	199	173	123
1.23	981	568	426	286	218	187	130

Table 1 (cont.)

λ	Bound for $N^*(\lambda)$						
	$\lambda_1 \geq$ 0.35	$\lambda_1 \geq$ 0.40	$\lambda_1 \geq$ 0.44	$\lambda_1 \geq$ 0.52	$\lambda_1 \geq$ 0.60	$\lambda_1 \geq$ 0.66	$\lambda_1 \geq$ 6/7
1.24	1642	742	518	326	241	203	138
1.25	4835	1063	658	377	268	222	146
1.26	∞	1844	895	446	301	245	156
1.27		6602	1382	543	343	272	167
1.28		∞	2967	690	397	305	179
1.29			∞	940	470	347	193
1.30				1457	573	400	208
1.31				3156	729	471	226
1.32				∞	995	569	247
1.33					1549	716	272
1.34					3398	958	302
1.35					∞	1433	338
1.36						2782	382
1.37						35205	438
1.38						∞	513
1.39							614
1.40							763
1.41							998
1.42							1430
1.43							2480
1.44							8791
1.45							∞

7. Proof of Proposition 3.5. We wish to estimate

$$\sum_{\chi\neq\chi_0}\sum_{m\in\mathbb{Z}}\sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi)}\exp(-M\lambda_\rho).$$

We do this by Lemmas 6.1, 6.2 and 6.3.

We split the argument into two parts, when there is a zero close to 1 (in which case it must be a real zero from a real character) and when there are no zeros close to 1 (and so ρ_1 or χ_1 might be complex).

The work in this section follows that of [11, Sections 14 and 15].

7.1. A zero close to 1. We consider the case when $\eta \leq \lambda_1 \leq 0.35$. By [26, Table 11] we see that such a zero cannot exist if χ_1 or ρ_1 is complex, and hence ρ_1 must be a real zero corresponding to a real character. Moreover, ρ_1 is simple. Since χ_1 is real we have $\phi_{\chi_1} = 1/4$.

We first consider the contribution from characters $\chi^{(j)} \neq \chi_1$.

We note that

$$\frac{\exp(-M\lambda)}{B_1(\lambda)} = \left(\frac{\lambda}{\sinh(K\lambda/2)}\right)^2 \left(1+\frac{1}{4\lambda^2}\right) e^{-(M-K)\lambda}. \tag{7.1}$$

The first two terms in the product are decreasing in λ, and so for $M \geq K$ this is a decreasing function of λ. Therefore for all $\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})$, if $M \geq K$, we have

$$\exp(-M\lambda_\rho) \leq \frac{\exp(-M\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} B_1(\lambda_\rho). \tag{7.2}$$

Thus by Lemma 6.1 we get

$$\begin{aligned} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} \exp(-M\lambda_\rho) &\leq \frac{\exp(-M\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} B_1(\lambda_\rho) \\ &\leq \frac{\exp(-M\lambda^{(j,m)}) C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}. \end{aligned} \tag{7.3}$$

We note that

$$\frac{\exp(-2x_1\lambda)}{B_2(\lambda)} \quad \text{and} \quad C_1(\lambda)$$

are decreasing functions in λ. Thus for $M \geq 2x_1 + K$ we see that

$$\frac{\exp(-M\lambda) C_1(\lambda)}{B_1(\lambda) B_2(\lambda)} \tag{7.4}$$

is a decreasing function in λ. Since for $\chi^{(j)} \neq \chi_1$ we have $\lambda^{(j,m)} \geq \lambda_2$, this gives us

$$\begin{aligned} \sum_{\substack{j,m \\ \chi^{(j)} \neq \chi_1, \chi_0}} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} & \exp(-M\lambda_\rho) \\ &\leq \frac{\exp(-M\lambda_2) C_1(\lambda_2)}{B_2(\lambda_2) B_1(\lambda_2)} \sum_{\substack{j,m \\ \chi^{(j)} \neq \chi_1, \chi_0}} B_2(\lambda^{(j,m)}). \end{aligned} \tag{7.5}$$

We now consider the contribution from the character χ_1. We give the zero ρ_1 close to 1 special treatment, and so treat differently the rectangle $\mathcal{R}_0$ which contains ρ_1 ($\rho_1 \in \mathcal{R}_0$ since ρ_1 is real).

We begin by considering the contribution from rectangles $\mathcal{R}_m$ with $m \neq 0$. Using the same ideas as above we have

$$\sum_{m \neq 0} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \leq \frac{\exp(-M\lambda_1') C_1(\lambda_1')}{B_2(\lambda_1') B_1(\lambda_1')} \sum_{\substack{m \neq 0 \\ \chi^{(j)} = \chi_1}} B_2(\lambda^{(j,m)}). \tag{7.6}$$

We now turn to the rectangle $\mathcal{R}_0$. We have

$$\begin{aligned}(7.7)\qquad \sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) &\le \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \sum_{\substack{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi)\\ \rho\neq\rho_1}} B_1(\lambda_\rho)\\ &\le \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi)} B_1(\lambda_\rho)\\ &\le \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')C_1(\lambda_1)}{B_1(\lambda_1')}.\end{aligned}$$

We note that $B_2(\lambda)$ and $C_1(\lambda)$ are both decreasing in λ. Therefore

$$(7.8)\qquad \sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \le \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')C_1(0)}{B_1(\lambda_1')B_2(\lambda_1')}B_2(\lambda_1).$$

Combining this with (7.6) and using the fact the C_1 is decreasing we obtain

$$\begin{aligned}(7.9)\qquad \sum_{\substack{j,m\\ \chi^{(j)}=\chi_1}} \sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) &\le \frac{\exp(-M\lambda_1')C_1(0)}{B_1(\lambda_1')B_2(\lambda_1')} \sum_{\substack{j,m\\ \chi^{(j)}=\chi_1}} B_2(\lambda^{(j,m)}) + \exp(-M\lambda_1).\end{aligned}$$

Now combining (7.9) and (7.5) we get

$$\begin{aligned}(7.10)\qquad \sum_{\chi\neq\chi_0} \sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)} \exp(-M\lambda_\rho) &\le \exp(-M\lambda_1) + C_4(\lambda_1',\lambda_2)\sum_{j,m} B_2(\lambda^{(j,m)})\\ &\le \exp(-M\lambda_1) + C_4(\lambda_1',\lambda_2)C_2,\end{aligned}$$

where

$$(7.11)\qquad C_4(\lambda_1',\lambda_2) = \max\biggl(\frac{\exp(-M\lambda_2)C_1(\lambda_2)}{B_1(\lambda_2)B_2(\lambda_2)}, \frac{\exp(-M\lambda_1')C_1(0)}{B_1(\lambda_1')B_2(\lambda_1')}\biggr).$$

By [11, Lemmas 8.4 and 8.8] for any $\delta > 0$ and for all $q \ge q_0(\delta)$ we have

$$(7.12)\qquad \lambda_1', \lambda_2 \ge \left(\frac{12}{11} - \delta\right)\log(\lambda_1^{-1}).$$

Also by [11, Tables 4 and 7] for $\lambda_1 \le 0.35$ we see that

$$(7.13)\qquad \lambda_1' \ge 2.19, \qquad \lambda_2 \ge 1.42.$$

Thus, since $C_4(\lambda_1', \lambda_2)$ is decreasing in λ_1' and λ_2, we deduce for any constant B with $0 \le B \le M - K - 2x_1$ that

$$C_4(\lambda_1', \lambda_2) \le \exp\left(-\left(\frac{12}{11} - \delta\right) B \log(\lambda_1^{-1})\right) \times \max\left(\frac{\exp(-1.42(M-B))C_1(1.42)}{B_1(1.42)B_2(1.42)}, \frac{\exp(-2.19(M-B))C_1(0)}{B_1(2.19)B_2(2.19)}\right). \tag{7.14}$$

We choose

$$B = 1, \qquad \delta = 0.01, \qquad K = 0.66 \tag{7.15}$$

and as before

$$w = 0.115, \qquad u_0 = 0.564, \qquad u_1 = 0.620, \tag{7.16}$$

$$v = 0.964, \qquad x_0 = 1.413, \qquad x_1 = 1.623. \tag{7.17}$$

Given M we can now explicitly calculate the above quantities. For $M = 7.5$ we obtain

$$\sum_{\chi \ne \chi_0} \sum_{\rho \in \mathcal{R} \cap \mathcal{Z}(\chi)} \exp(-7.5\lambda_\rho) \le \exp(-7.5\lambda_1) + 2.38 \cdot \lambda_1^{1.08}. \tag{7.18}$$

We see that the right hand side is a function which is 1 when $\lambda_1 = 0$, and is decreasing at 0. Moreover, it is convex (has positive second derivative) on $(0, \infty)$ and so can have at most one turning point, which would be a minimum should it exist. Therefore the right hand side is always < 1 for $\lambda_1 \in [\eta, 0.35]$ if it is < 1 at 0.35.

Calculating this at 0.35 with $M = 7.5$ gives $0.8628\ldots$, and so this is < 1 for $\lambda_1 \in [\eta, 0.35]$ provided $M \ge 7.5$.

7.2. No zeros close to 1. We now consider the case when $\lambda_1 \ge 0.35$. As above, for characters $\chi^{(j)} \ne \chi_1, \overline{\chi}_1$ we have

$$\sum_{\rho \in \mathcal{R} \cap \mathcal{Z}(\chi^{(j)})} \exp(-M\lambda_\rho) \le \sum_m \frac{\exp(-M\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} B_1(\lambda_\rho) \le \sum_m \frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}. \tag{7.19}$$

We now consider the contributions for the character χ_1 (and $\overline{\chi}_1$ if χ_1 complex). We separate out the contribution of ρ_1 (and $\overline{\rho}_1$ if it exists). To do this we put

$$n_1(\chi_1) = \begin{cases} 2 & \text{for } \chi_1 \text{ complex}, \\ 1 & \text{otherwise}, \end{cases} \tag{7.20}$$

$$n_2(\chi_1) = \begin{cases} 2 & \text{for } \chi_1 \text{ real and } \rho_1 \text{ complex,} \\ 1 & \text{otherwise,} \end{cases} \tag{7.21}$$

$$n_3(\chi_1) = \begin{cases} 2 & \text{for } \chi_1 \text{ real and } \rho_1 \text{ complex and } \rho_1 \notin \mathcal{R}_0, \\ 1 & \text{otherwise.} \end{cases} \tag{7.22}$$

Then

$$\begin{aligned} &\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_1) \\ &\qquad = n_2(\chi_1)\exp(-M\lambda_\rho) + \sum_m \sum_{\substack{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi_1) \\ \rho\neq\rho_1,\overline{\rho}_1}} \exp(-M\lambda_\rho). \end{aligned} \tag{7.23}$$

We separate out the contribution from the rectangle $\mathcal{R}_{m_1}$ which contains ρ_1. If χ_1 is real and ρ_1 is complex then we also separate the rectangle $\mathcal{R}_{m_2}$ which contains $\overline{\rho}_1$ if this is different from $\mathcal{R}_{m_1}$. We note that all zeros in either of these rectangles have either $\lambda_\rho = \lambda_1$ or $\lambda_\rho \geq \lambda_1'$. The zeros in any other rectangle $\mathcal{R}_m$ have $\lambda_\rho \geq \lambda^{(j,m)}$. We then use Lemma 6.1 again. This gives

$$\begin{aligned} &\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \\ &= n_2(\chi_1)\exp(-M\lambda_1) + \sum_{\substack{\rho\in(\mathcal{R}_{m_1}\cup\mathcal{R}_{m_2})\cap\mathcal{Z}(\chi_1) \\ \rho\neq\rho_1,\overline{\rho}_1}} \exp(-M\lambda_\rho) \\ &\quad + \sum_{m\neq m_1,m_2} \sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \\ &\leq \sum_{m\neq m_1,m_2} \frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} \\ &\quad + \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \sum_{\rho\in(\mathcal{R}_{m_1}\cup\mathcal{R}_{m_2})\cap\mathcal{Z}(\chi_1)} B_1(\lambda_\rho) \\ &\quad + n_2(\chi_1)\left(\exp(-M\lambda_1) - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}B_1(\lambda_1)\right) \\ &\leq \sum_{m\neq m_1,m_2} \frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} + n_3(\chi_1)\frac{\exp(-M\lambda_1')C_1(\lambda_1)}{B_1(\lambda_1')} \\ &\quad + n_2(\chi_1)\left(\exp(-M\lambda_1) - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}B_1(\lambda_1)\right) \\ &= (n_2(\chi_1)B_1(\lambda_1) - n_3(\chi_1)C_1(\lambda_1))\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)} - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right) \\ &\quad + \sum_m \frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}. \end{aligned} \tag{7.24}$$

If χ_1 is complex we follow the same argument and obtain the same result for $\overline{\chi}_1$.

Putting together (7.19) and (7.24) we obtain

$$\sum_{\chi\neq\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)}\exp(-M\lambda_\rho)\leq\sum_{m,j}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}+A_1, \tag{7.25}$$

where

$$A_1=n_1(\chi_1)(n_2(\chi_1)B_1(\lambda_1)-n_3(\chi_1)C_1(\lambda_1))\times\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)}-\frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right). \tag{7.26}$$

We now use Lemmas 6.2 and 6.3 to estimate the sum on the right hand side of (7.25). We fix a constant Λ (to be declared later) and consider separately the terms with $\lambda^{(j,m)}>\Lambda$ and $\lambda^{(j,m)}\leq\Lambda$. We use Lemma 6.2 to estimate the first set of terms, and Lemma 6.3 to estimate the second set.

We first consider the terms with $\lambda^{(j,m)}>\Lambda$:

$$\sum_{\substack{j,m\\ \lambda^{(j,m)}>\Lambda}}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}=\sum_{j,m}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})B_2(\lambda^{(j,m)})}B_2(\lambda^{(j,m)}). \tag{7.27}$$

Again we note that

$$\frac{\exp(-K\lambda)}{B_1(\lambda)},\quad\frac{\exp(-2x_1\lambda)}{B_2(\lambda)},\quad\text{and}\quad C_1(\lambda)$$

are all decreasing functions of λ. Therefore, provided $M\geq K+2x_1$ we have

$$\begin{aligned}\sum_{\substack{j,m\\ \lambda^{(j,m)}>\Lambda}}&\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}\\
&\leq\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\sum_{\substack{j,m\\ \lambda^{(j,m)}>\Lambda}}B_2(\lambda^{(j,m)})\\
&=\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\sum_{j,m}B_2(\lambda^{(j,m)})-\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\sum_{\substack{j,m\\ \lambda^{(j,m)}\leq\Lambda}}B_2(\lambda^{(j,m)})\\
&\leq\frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)}-\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\sum_{\substack{j,m\\ \lambda^{(j,m)}\leq\Lambda}}B_2(\lambda^{(j,m)}).\end{aligned} \tag{7.28}$$

Hence

$$\sum_{m,j} \frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} \le \frac{\exp(-M\varLambda)C_1(\varLambda)C_2}{B_1(\varLambda)B_2(\varLambda)} \tag{7.29}$$
$$+ \sum_{\substack{j,m\\ \lambda^{(j,m)}\le\varLambda}} \left(\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})B_2(\lambda^{(j,m)})} - \frac{\exp(-M\varLambda)C_1(\varLambda)}{B_1(\varLambda)B_2(\varLambda)}\right)B_2(\lambda^{(j,m)}).$$

We therefore are left to evaluate

$$\sum_{\substack{j,m\\ \lambda^{(j,m)}\le\varLambda}} \left(\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})B_2(\lambda^{(j,m)})} - \frac{\exp(-M\varLambda)C_1(\varLambda)}{B_1(\varLambda)B_2(\varLambda)}\right)B_2(\lambda^{(j,m)}). \tag{7.30}$$

To ease notation we put

$$D(\lambda) = \left(\frac{\exp(-M\lambda)C_1(\lambda)}{B_1(\lambda)B_2(\lambda)} - \frac{\exp(-M\varLambda)C_1(\varLambda)}{B_1(\varLambda)B_2(\varLambda)}\right)B_2(\lambda). \tag{7.31}$$

We note that $D(\lambda)$ is a decreasing function of λ (and is non-negative for $\lambda \le \varLambda$).

We separate the terms for λ_1 and put $\lambda^* = \min(\lambda_1', \lambda_2)$. This gives

$$\sum_{\substack{j,m\\ \lambda^{(j,m)}\le\varLambda}} D(\lambda^{(j,m)}) = n_3(\chi_1)n_1(\chi_1)D(\lambda_1) + \sum_{\substack{j,m\\ \lambda^*\le\lambda^{(j,m)}\le\varLambda}} D(\lambda^{(j,m)}). \tag{7.32}$$

We further put $\varLambda_r = \varLambda - (0.01)r$ and define s such that $\varLambda_{s+1} \le \lambda^* < \varLambda_s$. We then split the sum into sums over the different ranges $\varLambda_{r+1} \le \lambda^{(j,m)} < \varLambda_r$:

$$\begin{aligned}\sum_{\substack{j,m\\ \lambda^*\le\lambda^{(j,m)}\le\varLambda}} D(\lambda^{(j,m)}) &\le \sum_{r=0}^{s-1} \sum_{\substack{j,m\\ \varLambda_{r+1}\le\lambda^{(j,m)}\le\varLambda_r}} D(\lambda^{(j,m)}) + \sum_{\substack{j,m\\ \lambda^*\le\lambda^{(j,m)}\le\varLambda_s}} D(\lambda^{(j,m)})\\ &\le (N^*(\varLambda_s) - n_1(\chi_1)n_3(\chi_1))D(\lambda^*)\\ &\quad + \sum_{r=0}^{s-1}(N^*(\varLambda_r) - N^*(\varLambda_{r+1}))D(\varLambda_{r+1}).\end{aligned} \tag{7.33}$$

Note that we have used the fact that $D(\lambda)$ is decreasing in λ.

By Abel's identity we have

$$(7.34)\qquad \sum_{\substack{j,m\\ \lambda^*\le\lambda^{(j,m)}\le\Lambda}} D(\lambda^{(j,m)}) \le -n_1(\chi_1)n_3(\chi_1)D(\lambda^*)+N^*(\Lambda_s)(D(\lambda^*)-D(\Lambda_s)) + \sum_{r=0}^{s-1}N^*(\Lambda_r)(D(\Lambda_{r+1})-D(\Lambda_r)),$$

since $D(\Lambda)=0$.

Since $D(\Lambda_{r+1})\ge D(\Lambda_r)$ and $D(\lambda^*)\ge D(\Lambda_s)$ we may replace $N^*(\lambda)$ with an upper bound, say $N_0^*(\lambda)$. This gives

$$(7.35)\qquad \sum_{\substack{j,m\\ \lambda^*\le\lambda^{(j,m)}\le\Lambda}} D(\lambda^{(j,m)}) \le -n_1(\chi_1)n_3(\chi_1)D(\lambda^*)+N_0^*(\Lambda_s)D(\lambda^*) + \sum_{r=0}^{s-1}(N_0^*(\Lambda_r)-N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}).$$

Hence

$$(7.36)\qquad \sum_{\substack{j,m\\ \lambda^{(j,m)}\le\Lambda}} D(\lambda^{(j,m)}) \le n_1(\chi_1)n_3(\chi_1)(D(\lambda_1)-D(\lambda^*))+N_0^*(\Lambda_s)D(\lambda^*) + \sum_{r=0}^{s-1}(N_0^*(\Lambda_r)-N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}).$$

Putting (7.25), (7.29) and (7.36) together we obtain

$$(7.37)\qquad \sum_{\chi\ne\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)}\exp(-M\lambda_\rho) \le \frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)}+N_0^*(\Lambda_s)D(\lambda^*)+A_1' + \sum_{r=0}^{s-1}(N_0^*(\Lambda_r)-N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}),$$

where

$$(7.38)\qquad A_1' = n_1(\chi_1)n_2(\chi_1)B_1(\lambda_1)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)}-\frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right) + n_1(\chi_1)n_3(\chi_1)(D(\lambda_1)-D(\lambda^*)) - n_1(\chi_1)n_3(\chi_1)C_1(\lambda_1)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)}-\frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right).$$

We now wish to bound this when we consider λ_1, λ_1' and λ_2 constrained in size. Specifically, we consider $\lambda_1\in[\lambda_{11},\lambda_{12}]$, $\lambda_2\ge\lambda_{21}$ and $\lambda_1'\ge\lambda_{11}'$.

By definition $N_0^*(\Lambda_s) \ge n_1(\chi_1)n_3(\chi_1)$, and so the coefficient of $D(\lambda^*)$ is > 0. Since D is a decreasing function, the right hand side of (7.37) is decreasing as a function of λ_2. The term $B_1(\lambda_1)$ occurs $n_2(\chi_1)/n_3(\chi_1)$ times in the sum

$$\sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi_1)} B_1(\lambda_\rho).$$

Since the sum is $\le C_1(\lambda_1)$, and all terms in the sum are positive, we have

$$n_2(\chi_1)B_1(\lambda_1) \le n_3(\chi_1)C_1(\chi_1). \tag{7.39}$$

Therefore, by expanding out A' we see that the right hand side of (7.37) is also decreasing as a function of λ_1'.

Therefore we may replace λ_1' and λ_2 with their lower bounds λ_{11}' and λ_{21} respectively.

Considering this bound as a function of λ_1 we find that the right hand side equals

$$\begin{aligned}n_1(\chi_1)n_3(\chi_1)&\left(\frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}C_1(\lambda_1) - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}B_2(\lambda_1)\right)\\&+ n_1(\chi_1)n_2(\chi_1)B_1(\lambda_1)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)} - \frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}\right) + C,\end{aligned} \tag{7.40}$$

where C is independent of λ_1. We see this is

$$\begin{aligned}\le n_1(\chi_1)n_3(\chi_1)&\left(\frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}C_1(\lambda_{11}) - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}B_2(\lambda_{12})\right)\\&+ 2B_1(\lambda_{11})\left(\frac{\exp(-M\lambda_{11})}{B_1(\lambda_{11})} - \frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}\right) + C.\end{aligned} \tag{7.41}$$

Therefore

$$\begin{aligned}\sum_{\chi\ne\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)} &\exp(-M\lambda_\rho)\\&\le \frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)} + N_0^*(\Lambda_s)D(\lambda^*) + A_1''\\&\quad+ \sum_{r=0}^{s-1}(N_0^*(\Lambda_r) - N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}),\end{aligned} \tag{7.42}$$

where

$$\begin{aligned}A_1'' = 2B_1(\lambda_{11})&\left(\frac{\exp(-M\lambda_{11})}{B_1(\lambda_{11})} - \frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}\right)\\&+ n_4\left(\frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}C_1(\lambda_{11}) - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}B_2(\lambda_{12}) - D(\lambda^*)\right),\end{aligned} \tag{7.43}$$

and n_4 is chosen to be 1 or 2 so as to give the largest value for A_1''.

We now proceed to estimate (7.42) for various ranges of λ_1 which cover the region $\lambda_1 \geq 0.35$. We consider

$$M = 7.8. \tag{7.44}$$

For each range of λ_1 we use the lower bounds for λ_1' and λ_2 as given by [26, Tables 2, 3, 7] and [11, Table 4 and 7]. We use the upper bounds for N_0^* as calculated in Table 1.

We give these bounds on λ_1' and λ_2, our choices of Λ and the calculation of the right hand side of (7.42) in Table 2.

Table 2. Calculation of the RHS of (7.42) for different ranges of λ_1

λ_{11}	λ_{12}	λ_{21}	λ_{11}'	Λ	Total RHS of (7.42)
0.35	0.40	1.29	2.10	1.29	0.8579...
0.40	0.44	1.18	2.03	1.27	0.9821...
0.44	0.46	1.08	1.66	1.28	0.9213...
0.46	0.48	1.08	1.53	1.28	0.9120...
0.48	0.50	1.08	1.47	1.28	0.9041...
0.50	0.52	1.00	1.40	1.28	0.9304...
0.52	0.54	1.00	1.34	1.31	0.8049...
0.54	0.56	0.92	1.28	1.31	0.8427...
0.56	0.58	0.92	1.23	1.31	0.8385...
0.58	0.60	0.92	1.18	1.31	0.8349...
0.60	0.62	0.85	1.13	1.34	0.7782...
0.62	0.64	0.85	1.09	1.34	0.7756...
0.64	0.66	0.79	1.04	1.34	0.8363...
0.66	0.68	0.79	1.00	1.36	0.7652...
0.68	0.70	0.79	0.96	1.36	0.7636...
0.70	0.72	0.745	0.93	1.36	0.8241...
0.72	0.74	0.745	0.91	1.36	0.8229...
0.74	0.76	0.745	0.89	1.36	0.8219...
0.76	0.78	0.76	0.86	1.36	0.7988...
0.78	0.80	0.78	0.84	1.36	0.7708...
0.80	0.82	0.80	0.83	1.36	0.7463...
0.82	0.86	0.82	0.827	1.36	0.7243...
0.86	∞	0.86	0.86	1.44	0.5110...

We see that for each range of λ_1 we obtain an upper bound for (7.42) which is < 0.99. Since the expression is decreasing in M, this holds for all $M \geq 7.8$. We have therefore established Proposition 3.5 by taking $\epsilon = 10^{-3}$.

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