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Additive functions with respect to expansions over the set of Gaussian integers

by

I. Kátai (Budapest) and P. Liardet (Marseille)

1. Introduction

1.1. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, $\theta \in \mathbb{Z}[i]$ such that $t := |\theta|^2 \geq 2$, and $\mathcal{A} = \{a_0 = 0, a_1, \ldots, a_{t-1}\}$ ($\subseteq \mathbb{Z}[i]$) a complete residue system $\bmod \theta$. We call $\mathcal{A}$ the *set of digits*. Then, for each $\alpha \in \mathbb{Z}[i]$, there exists a unique $\alpha_1 \in \mathbb{Z}[i]$ and a unique $b_0 \in \mathcal{A}$ such that $\alpha = b_0 + \theta\alpha_1$. The function $J : \mathbb{Z}[i] \to \mathbb{Z}[i]$ is defined by $J(\alpha) = \alpha_1$.

Iterating J, we define the orbit

$$\alpha\ (= \alpha_0), \quad \alpha_1 = J(\alpha_0), \quad \alpha_2 = J(\alpha_1), \ \ldots \tag{1.1}$$

Let

$$L := \frac{1}{|\theta| - 1} \max_{a \in \mathcal{A}} |a|.$$

It is easy to show that

(a) if $|\alpha| > L$, then $|\alpha_1| < |\alpha|$,
(b) if $|\alpha| \leq L$, then $|\alpha_1| \leq L$.

Hence the orbit (1.1) is ultimately periodic for every $\alpha \in \mathbb{Z}[i]$. The proof of the two easy assertions stated above is given in the lecture notes [3].

An integer $\pi \in \mathbb{Z}[i]$ is said to be *periodic* if there is a positive integer k for which $J^k(\pi) = \pi$. Let $\mathcal{P}$ be the set of periodic points. From the assertions (a) and (b) we see that if $\pi \in \mathcal{P}$, then $|\pi| \leq L$.

Repeating the expansion defined above, we obtain

$$\alpha = b_0 + b_1\theta + \ldots + b_{k-1}\theta^{k-1} + \theta^k\alpha_k \qquad (k = 0, 1, \ldots), \tag{1.2}$$

where the sequence of the digits $b_0, \ldots, b_{k-1}$ is uniquely determined by α and θ. Let k be the smallest nonnegative integer for which $\alpha_k \in \mathcal{P}$. Then (1.2) with this k is called the *correct expansion* of α. By this convention,

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each α has a unique correct expansion. Let $l(\alpha) := k$ be the length of the representation. Then $l(\alpha) = 0$ if and only if $\alpha \in \mathcal{P}$.

A system $(\theta, \mathcal{A})$ is called a *number system* (or a *numeration system*) if $\mathcal{P} = \{0\}$. In that case each $\alpha \in \mathbb{Z}[i]$ has a finite expansion.

In [4] it was proved that $\mathcal{A} = \{0, 1, \ldots, t-1\}$ is an appropriate digit set for θ to generate a number system if and only if it has the form $\theta = -A + i$ or $\theta = -A - i$ with $A \geq 1$. G. Steidl [6] proved that for $\theta \in \mathbb{Z}[i]$ there is a suitable digit set $\mathcal{A}$ such that $(\theta, \mathcal{A})$ is a number system if and only if $t \geq 2$ and $1 - \theta$ is not a unit.

1.2. Assume that $\theta, \mathcal{A}$ are fixed.

Definition 1. A function $f : \mathbb{Z}[i] \to \mathbb{R}$ is *additive* (with respect to the expansion generated by θ and $\mathcal{A}$) if

(a) $f(\pi\theta^k) = 0$ for $\pi \in \mathcal{P}$ and $k = 0, 1, \ldots,$

(b) for every $\alpha \in \mathbb{Z}[i]$,

$$f(\alpha) = f(b_0) + f(b_1\theta) + \ldots + f(b_{k-1}\theta^{k-1}),$$

where $\alpha = b_0 + b_1\theta + \ldots + b_{k-1}\theta^{k-1} + \theta^k\pi$ is the correct expansion of α.

Let $\mathcal{E}_\theta$ be the class of additive functions in the above sense.

Definition 2. A function $g : \mathbb{Z}[i] \to \mathbb{C}$ is *multiplicative* (with respect to the expansion generated by θ and $\mathcal{A}$) if

(a) $g(\pi\theta^k) = 1$ for $\pi \in \mathcal{P}$ and $k = 0, 1, \ldots,$

(b) for every $\alpha \in \mathbb{Z}[i]$,

$$g(\alpha) = \prod_{j=0}^{k-1} g(b_j\theta^j).$$

Let $\mathcal{M}_\theta$ be the class of multiplicative functions in the above sense. Let $\overline{\mathcal{M}}_\theta \subseteq \mathcal{M}_\theta$ be the set of those g for which additionally $|g(\alpha)| = 1$ for all $\alpha \in \mathbb{Z}[i]$.

1.3. Since

$$\frac{|\alpha| - K}{|\theta|} \leq |J(\alpha)| \leq \frac{|\alpha| + K}{|\theta|}$$

where

$$K = \max_{a \in \mathcal{A}} |a|, \tag{1.3}$$

iterating we get the following

Lemma 1. *There exist suitable positive constants c_1, c_2 (depending on θ and K) such that*

$$-c_2 < l(\alpha) - \frac{\log|\alpha|}{\log|\theta|} < c_1 \quad \text{for every } \alpha \in \mathbb{Z}[i]\backslash\{0\}. \tag{1.4}$$

2. Formulation of the main results. Our purpose in this paper is to give necessary and sufficient conditions for the existence of the mean value of $g \in \overline{\mathcal{M}}_\theta$, where the summation is extended to a disc around zero with growing radius, or to some sectors of it.

We shall prove that the analogue of Delange's theorem for q-multiplicative functions [1] remains valid (see Theorem 1). As an application we give necessary and sufficient conditions for the existence of the limit distribution of $f \in \mathcal{E}_\theta$ (see Theorem 2). Finally we prove a theorem for the local distribution of the sum of digits function (see Theorem 4).

3. Lemmata. For an interval $I \subseteq [-1/2, 1/2)$ let C_I denote the annulus $\{z \mid z \in \mathbb{C}, 1/|\theta| < |z| < 1,\ (\arg z)/(2\pi) \in I\}$. For $g \in \overline{\mathcal{M}}_\theta$ let

$$S_I(x|g) := \sum_{\alpha \in xC_I} g(\alpha), \tag{3.1}$$

where x is a positive growing parameter and $xC_I = \{xz \mid z \in C_I\}$.

It is well known that $S_I(x|1)$ = number of Gaussian integers in xC_I is $\pi|I|x^2(1-1/t) + O(x)$ as $x \to \infty$, uniformly in I.

Let

$$N_x := \frac{\log x}{\log|\theta|}, \tag{3.2}$$

$$\Delta_j := \sum_{b \in \mathcal{A}} g(b\theta^j). \tag{3.3}$$

Lemma 2. *Assume that $g(b\theta^j) \to 1$ as $j \to \infty$, $b \in \mathcal{A}$. Then there is a monotonic sequence $R_N \to \infty$ of positive integers such that*

$$\max_{|\beta| \le |\theta|^{R_N}} |1 - g(\beta\theta^N)| \to 0 \qquad (N \to \infty).$$

Proof. Clear.

Let Γ_k be the set of those Gaussian integers which can be written as $b_0 + b_1\theta + \ldots + b_{k-1}\theta^{k-1}$, where the b_ν run over the set $\mathcal{A}$. Then Γ_k is a complete residue system $\bmod \theta^k$. For $\alpha \in \mathbb{Z}[i]$ let $s_k(\alpha)$ $(\in \Gamma_k)$ be defined by $\alpha \equiv s_k(\alpha) \pmod{\theta^k}$.

Lemma 3. *Assume that $g(b\theta^j) \to 1$ as $j \to \infty$, for all $b \in \mathcal{A}$. Then there exists an increasing sequence of integers $M_x < N_x$ such that $N_x - M_x \to \infty$, $M_x \to \infty$, for which*

$$g(\alpha) = (1 + o_x(1))g(s_{M_x}(\alpha)) \qquad (x \to \infty)$$

uniformly as $x/|\theta| \le |\alpha| \le x$; furthermore

$$S_I(x|g) = (1 + o_x(1))|I|\left(1 - \frac{1}{t}\right)\pi x^2 \prod_{j=0}^{M_x - 1} \frac{1}{t}\Delta_j + o(x^2). \tag{3.4}$$

Proof. The first assertion is a direct consequence of Lemma 2. Let $\alpha = s_{M_x}(\alpha) + \theta^{M_x}\alpha_{M_x}$. Then

$$|\alpha_{M_x}| \le \frac{x}{|\theta|^{M_x}} + \frac{|s_{M_x}(\alpha)|}{|\theta|^{M_x}} \le |\theta|^{N_x - M_x + 1} + \frac{K}{|\theta| - 1} < |\theta|^{N_x - M_x + c}$$

with some constant $c > 0$. By taking $M_x = N_x + c - R_{N_x}$ we find that $g(\alpha_{M_x}\theta^{M_x}) \to 1$ uniformly in the domain, thus the first assertion is proved.

Thus we have

$$S_I(x|g) = \sum_{\alpha \in xC_I} g(s_{M_x}(\alpha)) + o(1)x^2|I|.$$

To evaluate the sum on the right hand side, we write α as $\alpha = \beta + \theta^{M_x}\gamma$, where $\beta \in \Gamma_{M_x}$. If $\gamma \in \mathbb{Z}[i]$ occurs as a component of some α in xC_I, then

$$\frac{x}{|\theta|^{M_x+1}} - \frac{K}{|\theta| - 1} \le |\gamma| \le \frac{x}{|\theta|^{M_x}} + \frac{K}{|\theta| - 1}, \tag{3.5}$$

$$|\arg\gamma - M_x \arg\theta| < \frac{c|\theta|^{M_x}}{\alpha} < c \cdot |\theta|^{M_x - N_x}. \tag{3.6}$$

For all but $O(x/|\theta|^{M_x})$ of γ satisfying (3.5) and (3.6) all of the integers $\beta + \theta^{M_x}\gamma$, $\beta \in \Gamma_{M_x}$, belong to xC_I. Since the number of Gaussian integers in the domain defined by (3.5), (3.6) is

$$\pi|I|\left(\frac{x}{|\theta|^{M_x}}\right)^2 + O\left(\frac{x}{|\theta|^{M_x}}\right),$$

we have

$$S_I(x|g) = \pi|I|\left(\frac{x}{|\theta|^{M_x}}\right)^2\left(1 - \frac{1}{t}\right)\sum_{\beta \in \Gamma_k} g(\beta) + O(x|\theta|^{M_x}) + o(1)(x^2|I|).$$

Since $|\theta|^{M_x} \ll x|\theta|^{M_x - N_x}$ and $\sum_{\beta \in \Gamma_k} g(\beta) = \prod_{j=0}^{M_x - 1}(1/t)\Delta_j$, (3.4) immediately follows.

Lemma 4. *Let $g \in \overline{\mathcal{M}}_q$. Assume that there exists a constant $c > 0$, an infinite sequence $0 \le l_1 < l_2 < \ldots$ of integers and a suitable sequence of digits $b_1, b_2, \ldots \in \mathcal{A}$ such that $|1 - g(b_\nu\theta^{b_\nu})| \ge c$ $(\nu = 0, 1, \ldots)$. Then*

$$\frac{S_I(x|g)}{S_I(x|1)} \to 0 \qquad (x \to \infty)$$

uniformly for every interval I whose length is bounded below by a positive constant.

Proof. We argue as in the proof of Lemma 3. Let M_x be so chosen that $N_x - M_x \to \infty$ slowly. Let us write each $\alpha \in xC_I$ as $\beta + \theta^{M_x}\gamma$. Then

$$S_I(x|g) = \sum g(\theta^{M_x}\gamma)\Sigma_\gamma$$

where Σ_γ is the sum of $g(\beta)$ over those $\beta \in \Gamma_{M_x}$ for which $\beta + \theta^{M_x}\gamma \in xC_I$. Thus

$$S_I(x|g) \le \sum_\gamma |\Sigma_\gamma|.$$

We have $|\Sigma_\gamma| \le t^{M_x}$, and

$$\Sigma_\gamma = \prod_{j=0}^{M_x-1} \Delta_j$$

if $\beta + \theta^{M_x}\gamma \in xC_I$ for every $\beta \in \Gamma_{M_x}$. Hence we obtain

$$|S_I(x|g)| \le cx^2 \prod_{j=0}^{M_x} \frac{1}{t}|\Delta_j| + O(x|\theta|^{M_x}).$$

To finish the proof it is enough to observe that $(1/t)|\Delta_j| < 1 - \delta(c)$ with some positive constant $\delta(c)$ depending on c, if $j \in \{l_\nu\}_{\nu=1}^\infty$. This is a direct consequence of the following

LEMMA 5. *Let* $\omega_0, \ldots, \omega_{t-1}$ *be complex numbers of modulus* 1, $\omega_0 = 1$, *and* $\Delta := \omega_0 + \ldots + \omega_{t-1}$. *Then*

$$t^2 - |\Delta|^2 \ge \sum_{j=1}^{t-1} |1 - \omega_j|^2.$$

Proof. It is enough to observe that $2\,\mathrm{Re}(1 - \omega_j) = |1 - \omega_j|^2$. From the identity

$$t^2 - |\Delta|^2 = 2t \sum \mathrm{Re}(1 - \omega_j) - \Big|\sum (1 - \omega_j)\Big|^2,$$

and from the Hölder inequality

$$\Big|\sum_{j=1}^{t-1}(1 - \omega_j)\Big|^2 \le (t-1)\sum |1 - \omega_j|^2$$

the assertion immediately follows.

4. Consequences. We are ready to formulate our result.

THEOREM 1. *Let* $g \in \overline{\mathcal{M}}_\theta$.

(1) *If the series*

$$\sum_{j=0}^{\infty} \sum_{c \in \mathcal{A}} \mathrm{Re}(1 - g(c\theta^j)) \tag{4.1}$$

is divergent, *then*

$$\frac{S_I(x|g)}{S_I(x|1)} \to 0 \qquad \text{as } x \to \infty$$

uniformly on the intervals I whose length is bounded below by a positive constant. Consequently,

$$\frac{1}{\pi x^2|I|}\sum_{\substack{|\alpha|\le x\\ \arg\alpha\in I(\alpha)}} g(\alpha)\to 0. \tag{4.2}$$

(2) *If* (4.1) *is convergent, then*

$$\lim_{x\to\infty}\left|\frac{S_I(x|g)}{S_I(x|1)}\right| = \prod_{j=0}^{\infty}\frac{1}{t}|\Delta_j|,$$

and the right hand side is non-zero if and only if $\Delta_j \ne 0$ $(j=0,1,\dots)$.

(3) *The non-zero limit*

$$\lim_{x\to\infty}\frac{S_I(x|g)}{S_I(x|1)} \qquad (=m)$$

exists if and only if

$$\sum_{j=0}^{\infty}\sum_{c\in\mathcal{A}}(1-g(c\theta^j)) \tag{4.3}$$

is convergent, and $\Delta_j\ne 0$ $(j=0,1,\dots)$.

Proof. If (4.1) is divergent, then by Lemma 5, $\sum(1-(1/t)|\Delta_j|)=\infty$, and so $\prod_{j=0}^{M_x}(1/t)|\Delta_j|\to 0$; consequently, from Lemmas 3 and 4 we obtain the first assertion in (1). The fulfilment of (4.2) is obvious, since the left hand side equals to

$$S_I(x|g)+S_I\left(\frac{x}{|\theta|}\middle|g\right)+S_I\left(\frac{x}{|\theta|^2}\middle|g\right)+\dots$$

If (4.1) is convergent, then so is $\prod(1/t)|\Delta_j|$, and by (3.4) the second assertion follows. The proof of the last assertion is similar.

As a consequence we have

Theorem 2. *Let $f\in\mathcal{A}_\theta$, and assume that it has a limit distribution, i.e.*

$$\lim_{x\to\infty}\frac{1}{\pi x^2}\#\{\alpha \mid |\alpha|\le x,\ f(\alpha)<y\}=F(y)$$

exists, where F is a distribution function. Then both of the series

$$\sum_{j=0}^{\infty}\sum_{c\in\mathcal{A}} f(c\theta^j), \tag{4.4}$$

$$\sum_{j=0}^{\infty}\sum_{c\in\mathcal{A}} f^2(c\theta^j) \tag{4.5}$$

are convergent.

If (4.4), (4.5) *are convergent, then for each interval* $I \subseteq [-1/2, 1/2)$,

$$\lim_{x\to\infty} \frac{1}{\pi x^2 |I|} \#\{\alpha \mid |\alpha| \le x,\ f(\alpha) < y,\ \arg\alpha \in I\} = F(y).$$

The characteristic function of F *can be given by*

$$\varphi(\tau) = \prod_{j=0}^{\infty} \left\{ \frac{1}{t} \sum_{c\in\mathcal{A}} e^{i\tau f(c\theta^j)} \right\}.$$

Another corollary of Theorem 1 is

THEOREM 3. *Let* $f \in \mathcal{A}_\theta,\ f(c\theta^j) = O(1)$ *as* $j \to \infty,\ c \in \mathcal{A}$,

$$m_j = \frac{1}{t} \sum_{c\in\mathcal{A}} f(c\theta^j), \qquad \sigma_j^2 = \frac{1}{t} \sum_{c\in\mathcal{A}} (f(c\theta^j) - m_j)^2,$$

$$T_N^2 := \sum_{j=0}^{\infty} \sigma_j^2, \qquad E_N = \sum_{j=0}^{N} m_j.$$

Assume that $T_N \to \infty$. *Let* $I \subseteq [-1/2, 1/2)$ *be an interval. Then*

$$\frac{1}{\pi x^2 |I|} \#\left\{ \alpha \,\middle|\, |\alpha| < x,\ \frac{\arg\alpha}{2\pi} \in I,\ \frac{f(\alpha) - E_{N_x}}{T_{N_x}} < y \right\} = (1 + o_x(1))\varPhi(y),$$

where $\varPhi$ *is the Gaussian law.*

Theorems 2 and 3 can be derived from Theorem 1 by making use of the method of characteristic functions in probability theory.

5. The local distribution of the sum of digits and similar additive functions. Assume that $f \in \mathcal{E}_\theta$, the values of $f(c\theta^j)$ $(c \in \mathcal{A})$ are rational integers, and that $f(c\theta^j) = f(c)$ for every $j \ge 0,\ c \in \mathcal{A}$. Assume furthermore that the greatest common divisor of the values $\{f(c) \mid c \in \mathcal{A}\}$ is 1.

Let ξ_j $(j = 0, 1, \ldots)$ be identically distributed independent random variables with distribution

$$P(\xi_j = f(c)) = 1/t \qquad (c \in \mathcal{A}).$$

Let $\eta_N = \xi_0 + \ldots + \xi_{N-1}$ and

$$m = \frac{1}{t} \sum_{c\in\mathcal{A}} f(c), \qquad \sigma^2 = \frac{1}{t} \sum_{c\in\mathcal{A}} (f(c) - m)^2, \qquad \varphi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right).$$

According to Theorem 6 (Chapter VII) in the book of V. V. Petrov [5],

$$\left| P(\eta_N = k) - \frac{1}{\sqrt{N}} \varphi\left(\frac{k - mN}{\sigma\sqrt{N}}\right) \right| = O\left(\frac{1}{N}\right) \tag{5.1}$$

as $N \to \infty$, uniformly in k.

Since $\varphi(\omega_2)-\varphi(\omega_1)=\varphi'(\xi)(\omega_2-\omega_1)$ with some $\xi\in[\omega_1,\omega_2]$, and $\varphi'(\xi)=-\xi\varphi(\xi)$, from (5.1) we easily obtain

$$|P(\eta_N=k_1)-P(\eta_N=k_2)|\ll\frac{|k_2-k_1|}{N}\varphi(\xi)+O\left(\frac{1}{N}\right), \tag{5.2}$$

where ξ is located in the interval with endpoints $(k_i-mN)/(\sigma\sqrt{N})$ $(i=1,2)$.

We would like to count

$$R_k:=\#\{\alpha\mid\alpha\in xC_I,\ f(\alpha)=k\}. \tag{5.3}$$

Acting as in the proof of Lemmas 3 and 4, we write each α as $\beta+\theta^{M_x}\gamma,\ \beta\in\Gamma_{M_x}$. Let $\alpha\in xC_I,\ \alpha=\beta+\theta^{M_x}\gamma,\ f(\alpha)=k$. Let us drop α if there is some $\beta'\in\Gamma_{M_x}$ for which $\beta'+\theta^{M_x}\gamma\notin xC_I$. The cardinality of these integers is at most $O(x|\theta|^{M_x})$. Fixing a remaining γ, we count those $\beta\in\Gamma_{M_x}$ for which $f(\beta+\theta^{M_x}\gamma)=k$.

The size of these numbers is

$$t^{M_x}P(\eta_{M_x}=k-f(\theta^{M_x}\gamma)). \tag{5.4}$$

Since $|\gamma|\le|\theta|^{N_x-M_x+1}$, the value $f(\theta^{M_x}\gamma)=f(\gamma)$ is bounded by N_x-M_x+1.

From (5.2) we see that (5.4) equals

$$t^{M_x}P(\eta_{M_x}=k)+O(t^{M_x/2})+O(t^{M_x/2}(N_x-M_x+1)\varphi(\xi_\gamma)) \tag{5.5}$$

where ξ_γ is located in the interval with endpoints

$$\frac{k-mM_x}{\sigma\sqrt{M_x}},\qquad\frac{k-f(\gamma)-mM_x}{\sigma\sqrt{M_x}}.$$

Let us sum over the appropriate values of γ, i.e. over those for which $\beta+\theta^{M_x}\gamma\in xC_I$ for every $\beta\in\Gamma_{M_x}$. The number of appropriate Gaussian integers γ approximately equals the number of Gaussian integers in the annulus $x\theta^{-M_x}C_I$ with error bounded by the boundary, which is $O(x\cdot|\theta|^{-M_x})$, thus it is

$$\left(1-\frac{1}{t^2}\right)\pi|I|\frac{x^2}{t^{M_x}}+O\left(\frac{x}{|\theta|^{M_x}}\right).$$

Since φ is a bounded function, from (5.5) we deduce that

$$\begin{aligned}R_k=\left(1-\frac{1}{t^2}\right)\pi|I|x^2P(\eta_{M_x}=k)+O\left(\frac{x^2}{t^{M_x/2}}\right)\\+O((N_x-M_x+1)x^2t^{-M_x/2})+O(x).\end{aligned} \tag{5.6}$$

Let us choose now $M_x=N_x-[c\log N_x]$, with a positive constant c. Then the error terms on the right hand side of (5.6) are bounded by $O(x^{2(1-\delta)})$ with some constant $\delta>0$.

From (5.1) we easily get

$$P(\eta_{N_x}=k)=P(\eta_{M_x}=k)+O\biggl(\frac{(\log N_x)^{3/2}}{N_x}\biggr) \tag{5.7}$$

uniformly in k.

Let

$$\xi_1=\frac{k-mN_x}{\sigma\sqrt{N_x}},\qquad \xi_2=\frac{k-mM_x}{\sigma\sqrt{M_x}}.$$

If $|\xi_1|\ge\sqrt{\log N_x}$, then from (5.1) both of $P(\eta_{N_x}=k),\ P(\eta_{M_x}=k)$ are less than $O(1/N_x)$. If $|\xi_1|\le\sqrt{\log N_x}$, then

$$\xi_2=\xi_1+O\biggl(\frac{\log N_x}{\sqrt{N_x}}\biggr),\quad\text{and so}\quad \xi_2^2=\xi_1^2+O\biggl(\frac{(\log N_x)^{3/2}}{\sqrt{N_x}}\biggr),$$

whence

$$|e^{-\xi_2^2/2}-e^{-\xi_1^2/2}|\ll\frac{(\log N_x)^{3/2}}{\sqrt{N_x}}e^{-\xi_1^2/2},$$

and by (5.1),

$$|P(\xi_{N_x}=k)-P(\xi_{M_x}=k)| \ll\biggl|\frac{1}{\sqrt{N_x}}-\frac{1}{\sqrt{M_x}}\biggr|+\frac{(\log N_x)^{3/2}}{N_x}\ll\frac{(\log N_x)^{3/2}}{N_x}.$$

Thus

$$R_k=\biggl(1-\frac{1}{t^2}\biggr)\pi|I|x^2\biggl\{\frac{1}{\sqrt{N_x}}\varphi\biggl(\frac{k-mN_x}{\sigma\sqrt{N_x}}\biggr)+O\biggl(\frac{(\log N_x)^{3/2}}{N_x}\biggr)\biggr\}. \tag{5.8}$$

We formulate our result in the following

THEOREM 4. *Let $f\in\mathcal{E}_\theta$, $f(c\theta^j)=f(c)=$ rational integer for $c\in\mathcal{A}$, and assume that the greatest common divisor of $f(c)$ $(c\in\mathcal{A})$ is* 1. *Let*

$$m=\frac{1}{t}\sum f(c),\qquad \sigma^2=\frac{1}{t}\sum(f(c)-m)^2.$$

Then (5.8) *holds for R_k defined in* (5.3).

Let $N_I(x|k)$ be the number of Gaussian integers α satisfying $f(\alpha)=k$ in the sector $|\alpha|\le x$, $(\arg\alpha)/(2\pi)\in I$. Then

$$N_I(x|k)=\pi|I|x^2\biggl\{\frac{1}{\sqrt{N_x}}\varphi\biggl(\frac{k-mN_x}{\sigma\sqrt{N_x}}\biggr)+O\biggl(\frac{(\log N_x)^{3/2}}{N_x}\biggr)\biggr\}. \tag{5.9}$$

Proof. It remains to prove (5.9). This follows immediately if we use (5.8) by choosing $x, x/t, x/t^2$ and observing that $N_x, N_{x/t},\ldots$ are close to N_x.

REMARK. The sum of digits function with respect to number systems over $\mathbb{Z}[i]$ has been investigated earlier by Grabner and Liardet [2].

References

[1] H. Delange, *Sur les fonctions q-additives ou q-multiplicatives*, Acta Arith. 21 (1972), 285–298.

[2] P. J. Grabner and P. Liardet, *Harmonic properties of the sum-of-digits function for complex bases*, ibid. 91 (1999), 329–349.

[3] I. Kátai, *Generalized number systems and fractal geometry*, Pécs, Hungary, 1995 (manuscript).

[4] I. Kátai and J. Szabó, *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) 37 (1975), 225–260.

[5] V. V. Petrov, *Sums of Independent Random Variables*, Springer, Berlin, 1975.

[6] G. Steidl, *On symmetric representation of Gaussian integers*, BIT 29 (1989), 563–571.

Computer Algebra Department
Eötvös Loránd University
Pázmány Péter sétány I/D
H-1117 Budapest, Hungary
E-mail: katai@compalg.inf.elte.hu

Université de Provence, CMI
Château Gombert
39, Rue Joliot-Curie
13453 Marseille, Cedex 13, France
E-mail: liardet@gyptis.univ.-mrs.fr

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