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On sums of two kth powers: a mean-square asymptotics over short intervals

by

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1. Introduction. For $k \geq 2$ a fixed integer, define the arithmetic function $r_k(n)$ as the number of ways to write $n \in \mathbb{N}^*$ as a sum of two kth powers of absolute values of integers, i.e.,

$$r_k(n) = \#\{(u_1, u_2) \in \mathbb{Z}^2 : |u_1|^k + |u_2|^k = n\}.$$

To describe its average behaviour, one is interested in asymptotic results about the Dirichlet summatory function

$$R_k(u) = \sum_{1 \leq n \leq u^k} r_k(n),$$

where u is a large real variable (1).

For $k = 2$, the classic Gaussian circle problem, a detailed historical exposition can be found in the monograph of Krätzel [10]. The sharpest published results to date (2) read

$$R_2(u) = \pi u^2 + P_2(u), \tag{1.1}$$

$$P_2(u) = O(u^{46/73} (\log u)^{315/146}), \tag{1.2}$$

and (3)

$$P_2(u) = \Omega_-(u^{1/2} (\log u)^{1/4} (\log\log u)^{(\log 2)/4} \times \exp(-c\sqrt{\log\log\log u})) \quad (c > 0), \tag{1.3}$$

$$P_2(u) = \Omega_+(u^{1/2} \exp(c' (\log\log u)^{1/4} (\log\log\log u)^{-3/4})) \quad (c' > 0). \tag{1.4}$$

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(1) Note that, in part of the relevant literature, $t = u^2$ is used as the basic variable.

(2) Actually, M. Huxley has meanwhile improved further this upper bound, essentially replacing the exponent $46/73 = 0.6301\ldots$ by $131/208 = 0.6298\ldots$ The author is indebted to Professor Huxley for sending him a copy of his unpublished manuscript.

(3) We recall that $F_1(u) = \Omega_*(F_2(u))$ means that $\limsup_{u\to\infty} (*F_1(u)/F_2(u)) > 0$ where $*$ is either $+$ or $-$, and $F_2(u)$ is positive for u sufficiently large.

While (1.2) is due to Huxley [5], [7], (1.3) has been established by Hafner [4], and (1.4) by Corrádi & Kátai [2]. Most experts conjecture that

$$\inf\{\theta \in \mathbb{R} : P_2(u) \ll_\theta u^\theta\} = 1/2. \tag{1.5}$$

This hypothesis is supported by the mean-square asymptotics

$$\int_0^T (P_2(u))^2\,du = C_2 T^2 + O(T(\log T)^2), \qquad C_2 = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(r_2(n))^2}{n^{3/2}}, \tag{1.6}$$

which in this precise form is due to Kátai [8].

The results (1.3), (1.4), (1.6) were obtained by means of the fact that the generating function (Dirichlet series) of $r_2(n)$ is the Epstein zeta-function of the quadratic form $u_1^2+u_2^2$, which satisfies a well known functional equation and thus opens the possibility of an approach via complex integration.

For the general case $k \ge 3$, quite different methods must be employed. Investigations in this direction have first been undertaken by van der Corput [18] and Krätzel [9]. In Krätzel's textbook [10], an enlightening exposition of the history of the problem including all results until 1988 can be found. It turns out that

$$R_k(u) = \frac{2\Gamma^2(1/k)}{k\Gamma(2/k)}u^2 + B_k\Phi_k(u)u^{1-1/k} + P_k(u) \tag{1.7}$$

where

$$B_k = 2^{3-1/k}\pi^{-1-1/k}k^{1/k}\Gamma\left(1+\frac{1}{k}\right),$$

$$\Phi_k(u) = \sum_{n=1}^{\infty} n^{-1-1/k}\sin\left(2\pi nu - \frac{\pi}{2k}\right),$$

and the new remainder term $P_k(u)$ can essentially be bounded by (1.2), i.e.,

$$P_k(u) = O(u^{46/73}(\log u)^{315/146}). \tag{1.8}$$

This was proved by Kuba [11], on the basis of Huxley's method [5], [7].

For lower bounds, it was shown by the second named author [15] that, for any fixed $k \ge 3$,

$$P_k(u) = \Omega_-(u^{1/2}(\log u)^{1/4}), \tag{1.9}$$

and by Kühleitner, Nowak, Schoißengeier & Wooley [13] that

$$P_3(u) = \Omega_+(u^{1/2}(\log\log u)^{1/4}). \tag{1.10}$$

The analogy between these results and those for the case $k = 2$ might suggest extending the classic conjecture (1.5) to arbitrary $k \ge 2$. In fact, this is true again in mean-square: According to Nowak [14],

$$\frac{1}{T}\int_0^T (P_k(u))^2\,du \ll T \tag{1.11}$$

for any fixed $k \geq 3$ and T large. Kühleitner [12] refined this result, proving an asymptotic formula

$$\frac{1}{T}\int_0^T (P_k(u))^2\,du = C_k\,T + O(T^{1-\varepsilon_0(k)}), \tag{1.12}$$

with explicitly given $\varepsilon_0(k) > 0$ and

$$C_k := \frac{4}{\pi^2(k-1)} \sum_{\substack{(h_1,m_1,h_2,m_2)\in\mathbb{Z}_+^4\\ |(h_1,m_1)|_q=|(h_2,m_2)|_q}} (h_1m_1h_2m_2)^{-1+q/2}|(h_1,m_1)|_q^{1-2q}. \tag{1.13}$$

Here $q = k/(k-1)$ and $|(h,m)|_q = (|h|^q + |m|^q)^{1/q}$ denotes the q-norm in $\mathbb{R}^2$.

Inspired by a work of Huxley [6] on the lattice point discrepancy of a convex disc, the second named author recently [16] proved a localized form of (1.11), with only a logarithmic loss of accuracy, namely

$$\int_{T-1/2}^{T+1/2} (P_k(u))^2\,du \ll T(\log T)^2. \tag{1.14}$$

In view of (1.9), this result seems pretty close to what might be possible. Nevertheless, our aim in the present article is to shed some more light on this short-interval behaviour of this remainder term. It will turn out that the bound in (1.14) (even refined by a factor $\log T$) remains valid for an interval up to a length of order $\log T$. In fact, it will be shown that, for any fixed $c_1 > 0$,

$$\int_{T-c_1\log T}^{T+c_1\log T} (P_k(u))^2\,du \ll T\log T. \tag{1.15}$$

Furthermore, we shall see that, as soon as the interval becomes a little longer, we can observe essentially the same asymptotic behaviour as stated in (1.12).

THEOREM. *Let $k \geq 3$ be a fixed integer, T a large real variable, and $T \mapsto \Lambda = \Lambda(T)$ an increasing function such that $\Lambda(T) \leq \frac{1}{2}T$ throughout and*

$$\lim_{T\to\infty} \frac{\log T}{\Lambda(T)} = 0. \tag{1.16}$$

Then, as $T \to \infty$,

$$\int_{T-\Lambda}^{T+\Lambda} (P_k(u))^2\,du \sim 4C_k\,\Lambda T, \tag{1.17}$$

the constant C_k being defined in (1.13).

2. Two pivotal lemmas

LEMMA 1 (Transition from fractional parts to trigonometric sums according to Vaaler [17]; see also Graham & Kolesnik [3], p. 116). *For arbitrary* $w \in \mathbb{R}$ *and* $H \in \mathbb{N}^*$, *let*

$$\psi(w) = w - [w] - \frac{1}{2}, \qquad \psi_H^*(w) = -\frac{1}{\pi}\sum_{h=1}^{H} \frac{\sin(2\pi h w)}{h}\tau\left(\frac{h}{H+1}\right),$$

where

$$\tau(\xi) = \pi\xi(1-\xi)\cot(\pi\xi) + \xi \qquad \textit{for } 0 < \xi < 1.$$

Then

$$|\psi(w) - \psi_H^*(w)| \le \frac{1}{H+1}\sum_{h=1}^{H}\left(1 - \frac{h}{H+1}\right)\cos(2\pi h w) + \frac{1}{2H+2}.$$

LEMMA 2. *Let* $k \ge 3$ *be a positive integer, and* $q = k/(k-1)$. *Then, for* $M \to \infty$,

$$S(M) := \sum_{\substack{(h_1,m_1,h_2,m_2)\in\mathbb{Z}_+^4 \\ |(h_1,m_1)|_q = |(h_2,m_2)|_q \ge M}} (h_1h_2m_1m_2)^{-1+q/2}|(h_1,m_1)|_q^{1-2q} \ll M^{-1/2}.$$

Proof. For positive integers h_1, h_2, m_1, m_2 the condition $|(h_1,m_1)|_q = |(h_2,m_2)|_q$ is satisfied if and only if either $(h_1,m_1)=(h_2,m_2)$ or h_1,h_2,m_1,m_2 all have the same maximal $(k-1)$-free divisor r, say, i.e.,

$$h_1 = a^{k-1}r, \qquad m_1 = b^{k-1}r, \qquad h_2 = c^{k-1}r, \qquad m_2 = d^{k-1}r,$$

with $a,b,c,d \in \mathbb{N}^*$ satisfying $a^k + b^k = c^k + d^k$. This follows from the fact that, by a classic theorem of Besicovitch [1], the $(k-1)$th roots of distinct $(k-1)$-free positive integers are linearly independent over the rationals. Therefore, the sum in question is

$$\ll R_1(M) + R_2(M)$$

with

$$R_1(M) = \sum_{\substack{h_1=1 \\ m_1 \gg M}}^{\infty} (h_1m_1)^{-2+q}(h_1m_1)^{-q+1/2},$$

$$R_2(M) = \sum_{\substack{a\le b,\ c\le d \\ b^{k-1}r,\ d^{k-1}r \gg M}} (abcd)^{(k-1)(-1+q/2)} \times r^{-3}(|(a^{k-1},b^{k-1})|_q|(c^{k-1},d^{k-1})|_q)^{-q+1/2},$$

since

$$|(h_1,m_1)|_q = r|(a^{k-1},b^{k-1})|_q \quad \text{and} \quad |(h_2,m_2)|_q = r|(c^{k-1},d^{k-1})|_q.$$

Clearly,

$$R_1(M) \ll \sum_{m_1 \gg M} m_1^{-3/2} \ll M^{-1/2}.$$

We estimate the contribution of $R_2(M)$ in the cases $k = 3, 4$, resp. $k \geq 5$ in two different ways. In the first case we use

$$\frac{1}{|(u^{k-1}, v^{k-1})|_q^{q-1/2}} \ll (uv)^{-\frac{1}{2}(k-1)(q-1/2)},$$

to conclude that

$$\begin{aligned} R_2(M) &\ll \sum_{b^{k-1}r, d^{k-1}r \gg M} \sum_{a,c=1}^{\infty} (abcd)^{(k-1)(-1+q/2)} r^{-3} (abcd)^{-\frac{1}{2}(k-1)(q-1/2)} \\ &\ll \sum_{r=1}^{\infty} r^{-3} \Big(\sum_{b \gg (M/r)^{1/(k-1)}} b^{-\frac{3}{4}(k-1)} \Big)^2 \ll \sum_{r=1}^{\infty} r^{-3} \Big(\frac{M}{r} \Big)^{2/(k-1)-3/2} \\ &\ll M^{-1/2}. \end{aligned}$$

In the case $k \geq 5$ we use the fact that

$$\sum_{a,c=1}^{\infty} (ac)^{(k-1)(-1+q/2)} \ll 1,$$

to infer

$$\begin{aligned} R_2(M) &\ll \sum_{b^{k-1}r, d^{k-1}r \gg M} (bd)^{(k-1)(-1+q/2-q+1/2)} r^{-3} \\ &\ll \sum_{r=1}^{\infty} r^{-3} \Big(\sum_{b \gg (M/r)^{1/(k-1)}} b^{-k+1/2} \Big)^2 \\ &\ll \sum_{r=1}^{\infty} r^{-3} \Big(\frac{M}{r} \Big)^{(3-2k)/(k-1)} \ll M^{-7/4}. \end{aligned}$$

3. Proof of the Theorem. Throughout what follows, let T and M be large real parameters, independent of each other. All constants implied in the symbols O, $\ll$, or $\asymp$ do not depend on M and T, but may depend on k. Suppose that $u \in [T - \Lambda, T + \Lambda] \subseteq \left[\frac{1}{2}T, \frac{3}{2}T\right]$, thus $u \asymp T$ as $T \to \infty$.

For any complex-valued function $f : u \mapsto f(u)$ which is square-integrable on $[T - \Lambda, T + \Lambda]$, we shall write

$$\mathcal{Q}(f) = \mathcal{Q}_{T,\Lambda}(f) := \int_{T-\Lambda}^{T+\Lambda} |f(u)|^2 \, du. \tag{3.1}$$

By Cauchy's inequality, for arbitrary $f_1, f_2 \in L^2[T-\Lambda, T+\Lambda]$,

$$\mathcal{Q}(f_1+f_2) \le 2(\mathcal{Q}(f_1)+\mathcal{Q}(f_2)), \tag{3.2}$$

which will be used frequently in what follows.

We start from formulae (3.57), (3.58) (and the asymptotic expansion below) of Krätzel [10], p. 148. In our notation, this reads

$$P_k(u) = -8 \sum_{2^{-1/k}u<n\le u} \psi((u^k-n^k)^{1/k}) + O(1), \tag{3.3}$$

with $\psi(w) = w - [w] - 1/2$ throughout. We define q by $1/k + 1/q = 1$, i.e., $q = k/(k-1)$, and thus $1 < q \le 3/2$. We break up the range of summation into subintervals [4] $\mathcal{N}_j(u) =]N_j, N_{j+1}]$, where $N_j = u\,(1+2^{-jq})^{-1/k}$, $j = 0, 1, \ldots, J$, with J minimal such that $u - N_J < 1$ for all $u \in [T-\Lambda, T+\Lambda]$. It is clear that $J \ll \log T$. Furthermore, the length of any $\mathcal{N}_j(u)$ is equal to $N_{j+1} - N_j \asymp 2^{-jq}T$. By means of Lemma 1, ψ will be approximated by ψ_H^*, with $H := [T]$. Thus overall $P_k(u)$ is approximated by

$$P_k^*(u) := -8 \sum_{j=0}^{J} \sum_{n\in\mathcal{N}_j(u)} \psi_H^*((u^k-n^k)^{1/k}). \tag{3.4}$$

By the definition in Lemma 1,

$$\sum_{n\in\mathcal{N}_j(u)} \psi_H^*((u^k-n^k)^{1/k}) = -\frac{1}{\pi} \sum_{1\le h\le T} \frac{1}{h}\tau\left(\frac{h}{H+1}\right) \sum_{n\in\mathcal{N}_j(u)} \sin(2\pi h(u^k-n^k)^{1/k}). \tag{3.5}$$

The innermost sum on the right hand side is now subject to a van der Corput transformation ("B-step"). See Kühleitner [12], Lemmas 2 and 3, for details. In particular, we use formula (3.5) from [12] which reads (with u instead of $\sqrt{t}$, and $e(z) = e^{2\pi i z}$ as usual)

$$\begin{aligned}\sum_{n\in\mathcal{N}_j(u)} e(-h(u^k-n^k)^{1/k}) &= \frac{e(1/8)}{\sqrt{k-1}} h u^{1/2} \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} (hm)^{-1+q/2} |(h,m)|_q^{-q+1/2} e(-u|(h,m)|_q) \\ &\quad + O(j+\log(1+h)),\end{aligned} \tag{3.6}$$

[4] The idea of this special choice of subdivision points is that $\frac{d}{dw}((u^k-w^k)^{1/k})$ assumes integer values at $w = N_j$. This is useful for the van der Corput transformation of the exponential sums involved.

where $\mathcal{M}_j(h) = [2^j h, 2^{j+1}h]$, $|(h,m)|_q = (|h|^q + |m|^q)^{1/q}$ denotes the q-norm in $\mathbb{R}^2$, and $\sum''$ means that the terms corresponding to $m = 2^j h$ and $m = 2^{j+1}h$ get a factor $1/2$.

Using the imaginary part of (3.6) in (3.5), we obtain

$$\text{(3.7)}\quad \sum_{n\in\mathcal{N}_j(u)} \psi_H^*((u^k - n^k)^{1/k}) = \frac{u^{1/2}}{\sqrt{k-1}} \sum_{1\le h\le T} \gamma_0(h,T) \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} \frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}} \sin(\pi/4 - 2\pi u|(h,m)|_q) + O(\log T)$$

with

$$\gamma_0(h,T) := \frac{1}{\pi}\tau\biggl(\frac{h}{[T]+1}\biggr).$$

In fact, the main contribution to our mean-square asymptotics will come from a truncation of the double sum here, namely ([5])

$$\text{(3.8)}\quad \Sigma_j(M,u) := \frac{u^{1/2}}{\sqrt{k-1}} \sum_{1\le h\le T} \gamma_0(h,T) \times \sideset{}{''}\sum_{\substack{m\in\mathcal{M}_j(h)\\ |(h,m)|_q\le M}} \frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}} \sin(\pi/4 - 2\pi u|(h,m)|_q).$$

What about the errors we commit by these approximations? First of all, evidently,

$$\text{(3.9)}\quad \sum_{n\in\mathcal{N}_j(u)} \psi_H^*((u^k - n^k)^{1/k}) - \Sigma_j(M,u) \ll T^{1/2}\biggl| \sum_{1\le h\le T} \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)} \gamma_1(h,m,T) \frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}} e(u|(h,m)|_q)\biggr| + \log T$$

with

$$\gamma_1(h,m,T) := \begin{cases} \gamma_0(h,T) & \text{if } |(h,m)|_q > M,\\ 0 & \text{else.}\end{cases}$$

Furthermore, by Lemma 1,

$$\sum_{n\in\mathcal{N}_j(u)} (\psi((u^k - n^k)^{1/k}) - \psi_H^*((u^k - n^k)^{1/k})) \ll \sum_{1\le h\le T} \frac{1 - h/([T]+1)}{[T]+1} \sum_{n\in\mathcal{N}_j(u)} \cos(2\pi h(u^k - n^k)^{1/k}) + 2^{-jq}.$$

([5]) Recall that M is another large parameter independent of T.

Applying again (3.6) to the cosine sum here, we see that this is

$$(3.10)\qquad \ll T^{1/2}\Big|\sum_{1\le h\le T}\ \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)}\gamma_2(h,m,T)\frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}}e(u|(h,m)|_q)\Big|+2^{-jq}$$

with

$$\gamma_2(h,m,T):=\frac{(1-h/([T]+1))h}{[T]+1}.$$

The great similarity of the main parts of the expressions (3.9) and (3.10) enables us to estimate their mean-square by essentially the same calculation. Let

$$\mathcal{R}_j(u):=\sum_{1\le h\le T}\ \sideset{}{''}\sum_{m\in\mathcal{M}_j(h)}\gamma(h,m,T)\,\frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}}\,e(u|(h,m)|_q)$$

where γ is either of γ_1,γ_2.

We want to bound $\mathcal{Q}(\mathcal{R}_j)$. To this end, we employ an ingenious trick due to Huxley [6] which involves the Fejér kernel

$$\varphi(w):=\left(\frac{\sin(\pi w)}{\pi w}\right)^2.$$

By Jordan's inequality, $\varphi(w)\ge 4/\pi^2$ for $|w|\le 1/2$, and the Fourier transform has the simple shape

$$\widehat{\varphi}(y)=\int_{\mathbb{R}}\varphi(w)e(yw)\,dw=\max(0,1-|y|).$$

Therefore,

$$(3.11)\qquad \mathcal{Q}(\mathcal{R}_j)$$

$$=2\Lambda\int_{-1/2}^{1/2}|\mathcal{R}_j(T+2\Lambda w)|^2\,dw\le\frac{\pi^2}{2}\Lambda\int_{\mathbb{R}}\varphi(w)|\mathcal{R}_j(T+2\Lambda w)|^2\,dw$$

$$=\frac{\pi^2}{2}\Lambda\sum_{1\le h_1,h_2\le T}\ \sideset{}{''}\sum_{\substack{m_1\in\mathcal{M}_j(h_1)\\ m_2\in\mathcal{M}_j(h_2)}}\frac{\gamma(h_1,m_1,T)\gamma(h_2,m_2,T)(h_1m_1h_2m_2)^{-1+q/2}}{(|(h_1,m_1)|_q|(h_2,m_2)|_q)^{q-1/2}}$$

$$\times e(-T(|(h_1,m_1)|_q-|(h_2,m_2)|_q))$$

$$\times\int_{\mathbb{R}}\varphi(w)e(-2\Lambda w(|(h_1,m_1)|_q-|(h_2,m_2)|_q))\,dw$$

$$\ll\Lambda\sum_{1\le h_1,h_2\le T}\ \sideset{}{''}\sum_{\substack{m_1\in\mathcal{M}_j(h_1)\\ m_2\in\mathcal{M}_j(h_2)}}\frac{\gamma(h_1,m_1,T)\gamma(h_2,m_2,T)(h_1m_1h_2m_2)^{-1+q/2}}{(|(h_1,m_1)|_q|(h_2,m_2)|_q)^{q-1/2}}$$

$$\times\max(0,1-2\Lambda|\,|(h_1,m_1)|_q-|(h_2,m_2)|_q|).$$

We recall that $m \in \mathcal{M}_j(h)$ implies that $|(h,m)|_q \asymp m \asymp 2^j h$. Furthermore, for a term of the last multiple sum to be nonzero it is necessary that $|\,|(h_1,m_1)|_q - |(h_2,m_2)|_q| < (2\Lambda)^{-1}$, hence $h_1 \asymp h_2$ and $m_1 \asymp m_2$. Therefore, the last expression in (3.11) is

$$\ll \Lambda 2^{-j(q+1)} \sum_{1\le h_1\le T} h_1^{-3} \sum_{m_1\in\mathcal{M}_j(h_1)} \gamma(h_1,m_1,T) \times \sum_{\substack{(h_2,m_2)\in\mathbb{Z}_+^2 \\ ||(h_1,m_1)|_q-|(h_2,m_2)|_q|<(2\Lambda)^{-1}}} \gamma(h_2,m_2,T). \tag{3.12}$$

We now have to distinguish if we are dealing with γ_1 or γ_2, recalling the respective definitions: For $\gamma_1(h,m,T)$, we know that this is bounded and vanishes for $|(h,m)|_q \le M$. Further, denote by $A_q^*(U)$ the number of lattice points $\mathbf{v} \in \mathbb{Z}^2$ with $|\mathbf{v}|_q \le U$; then it is known that

$$A_q^*(U) = \frac{2\Gamma^2(1/q)}{q\Gamma(2/q)}\, U^2 + O(U^{2/3}) \tag{3.13}$$

for any fixed q with $1 < q < 2$. This asymptotic formula is contained in Theorem 3.6 of Krätzel [10], p. 116. From this it is immediate that, for any fixed (h_1,m_1),

$$\sum_{\substack{(h_2,m_2)\in\mathbb{Z}^2 \\ |\,|(h_1,m_1)|_q-|(h_2,m_2)|_q|<(2\Lambda)^{-1}}} 1 \ll \frac{|(h_1,m_1)|_q}{\Lambda} + |(h_1,m_1)|_q^{2/3}. \tag{3.14}$$

Thus, for $\gamma = \gamma_1$, the expression in (3.12) is

$$\begin{aligned} &\ll \Lambda 2^{-jq} \sum_{\substack{1\le h_1\le T \\ 2^j h_1 \gg M}} h_1^{-2}\left(\frac{2^j h_1}{\Lambda} + (2^j h_1)^{2/3}\right) \\ &\ll 2^{-j(q-1)} \log T + \Lambda M^{-1/6} 2^{-j(q-5/6)} \sum_{1\le h_1\le T} h_1^{-7/6} \\ &\ll 2^{-j(q-1)}(\log T + \Lambda M^{-1/6}). \end{aligned} \tag{3.15}$$

For $\gamma = \gamma_2$, we may use that $\gamma_2(h,m,T) \ll h\,T^{-1}$. Thus (3.12) is now, again by (3.14),

$$\begin{aligned} &\ll \Lambda 2^{-j(q+1)}\, T^{-2} \sum_{1\le h_1\le T} h_1^{-1} \sum_{m_1\in\mathcal{M}_j(h_1)} \left(\frac{|(h_1,m_1)|_q}{\Lambda} + |(h_1,m_1)|_q^{2/3}\right) \\ &\ll \Lambda 2^{-j(q+1)} T^{-2} \sum_{1\le h_1\le T} \left(\frac{2^{2j} h_1}{\Lambda} + 2^{5j/3} h_1^{2/3}\right) \\ &\ll 2^{-j(q-1)}(1 + \Lambda T^{-1/3}). \end{aligned} \tag{3.16}$$

Let us summarize what we have proved so far: The remainder term in question can be represented as

$$P_k(u) = \sum_{j=0}^{J} (-8\Sigma_j(M,u) + \Delta_j(M,u)), \tag{3.17}$$

where $\Sigma_j(M,u)$ has been defined in (3.8) and $\Delta_j(M,u)$ satisfies (in view of (3.9), (3.10), (3.15), (3.16))

$$\mathcal{Q}(\Delta_j(M,u)) \ll 2^{-j(q-1)}(T\log T + \Lambda T M^{-1/6} + \Lambda T^{2/3}) + \Lambda(\log T)^2.$$

To proceed further, let δ be a positive constant, less than $\frac{1}{2}(q-1)$ and small compared to $(\log T)/J$. Then, by Cauchy's inequality,

$$\begin{aligned}\mathcal{Q}\Big(\sum_{j=0}^{J} \Delta_j(M,u)\Big) &\le \int_{T-\Lambda}^{T+\Lambda} \sum_{j=0}^{J} 2^{-j\delta} \sum_{j=0}^{J} 2^{j\delta} |\Delta_j(M,u)|^2 \, du \\ &\ll \sum_{j=0}^{J} 2^{j\delta} \mathcal{Q}(\Delta_j(M,u)) \ll T\log T + \Lambda T M^{-1/6} + \Lambda T^{2/3}.\end{aligned}$$

Adding up the main terms $\Sigma_j(M,u)$, we arrive at:

PROPOSITION. *Uniformly in* $T - \Lambda \le u \le T + \Lambda$,

$$P_k(u) = \Sigma(M,u) + \Delta(M,u),$$

with

$$\mathcal{Q}(\Delta(M,u)) \ll T\log T + \Lambda T M^{-1/6} + \Lambda T^{2/3}$$

and

$$\begin{aligned}\Sigma(M,u) := \frac{-8u^{1/2}}{\pi\sqrt{k-1}} \sum_{1\le h\le T} \tau\Big(\frac{h}{[T]+1}\Big) \\ \times \sideset{}{'}\sum_{\substack{|(h,m)|_q \le M \\ h\le m\le h2^{J+1}}} \frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}} \cos(\pi/4 + 2\pi u|(h,m)|_q),\end{aligned}$$

where $\sum'$ *means that the terms corresponding to* $m = h$ *and* $m = h2^{J+1}$ *get a factor* $1/2$.

Next we infer from the definition in Lemma 1 that $\tau(w) = 1 + O(w^2)$. Therefore, defining

$$\begin{aligned}&\Sigma^{(0)}(M,u) \\ &\quad := \frac{-8u^{1/2}}{\pi\sqrt{k-1}} \sum_{1\le h\le T} \sideset{}{'}\sum_{\substack{|(h,m)|_q \le M \\ h\le m\le h2^{J+1}}} \frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}} \cos(\pi/4 + 2\pi u|(h,m)|_q),\end{aligned}$$

it is immediate that

$$\Sigma(M,u)=\Sigma^{(0)}(M,u)+O(K_1(M)T^{-3/2}),\tag{3.18}$$

where $K_1(M),K_2(M),\ldots$ will denote appropriate bounds depending on M (but not on T). If we keep M fixed and make T (and thus u) large, the summation conditions $h\le T$ and $m\le h2^{J+1}$ ultimately become meaningless, and $\Sigma^{(0)}(M,u)$ becomes equal to

$$\Sigma^{(1)}(M,u):=\frac{-4u^{1/2}}{\pi\sqrt{k-1}}\sum_{\substack{|(h,m)|_q\le M\\ h,m\in\mathbb{N}^*}}\frac{(hm)^{-1+q/2}}{|(h,m)|_q^{q-1/2}}\cos(\pi/4+2\pi u|(h,m)|_q).$$

We now square out $(\Sigma^{(1)}(M,u))^2$, using the elementary formula

$$\cos A\cos B=\frac{1}{2}(\cos(A-B)+\cos(A+B)),$$

and integrate over $u\in[T-\Lambda,T+\Lambda]$. The main contribution comes from the diagonal terms, i.e. those with $|(h_1,m_1)|_q=|(h_2,m_2)|_q$, and reads altogether

$$\frac{16}{\pi^2(k-1)}\Lambda T\sum_{\substack{|(h_1,m_1)|_q=|(h_2,m_2)|_q\le M\\ h_1,m_1,h_2,m_2\in\mathbb{N}^*}}\frac{(h_1m_1h_2m_2)^{-1+q/2}}{|(h_1,m_1)|_q^{2q-1}}.$$

By Lemma 2 and the definition of the constant C_k in (1.13), this is equal to

$$4\Lambda T(C_k+O(M^{-1/2})).$$

All the other terms are pretty small: In fact,

$$\int_{T-\Lambda}^{T+\Lambda}\begin{matrix}\cos\\ \sin\end{matrix}(2\pi u(|(h_1,m_1)|_q\pm|(h_2,m_2)|_q))\,u\,du$$

$$\ll\frac{T}{\big||(h_1,m_1)|_q\pm|(h_2,m_2)|_q\big|},$$

which contributes altogether $\ll K_2(M)T$ to $\mathcal{Q}(\Sigma^{(1)}(M,u))$. Going back to (3.18) and to the Proposition, and applying Cauchy's inequality one more time, we end up with

$$\begin{aligned}\mathcal{Q}(P_k)&=4C_k\Lambda T+O(K_3(M)T)+O(T(\Lambda\log T)^{1/2})\\&\quad+O(\Lambda TM^{-1/12})+O(\Lambda T^{5/6}).\end{aligned}\tag{3.19}$$

Therefore, for any fixed M,

$$\limsup_{T\to\infty}\left|\frac{1}{\Lambda T}\mathcal{Q}(P_k)-4C_k\right|\ll M^{-1/12},$$

if we recall our condition (1.16). Since M can be chosen arbitrarily large, the proof of our Theorem is thereby complete.

We finally establish (1.15). To this end, it suffices to choose $M=1/2$ in the above argument; then all sums over $0<|(h,m)|_q\le M$ are empty, and (3.19) yields what we claimed, since now $\Lambda\asymp\log T$.

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