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The evaluation of two-dimensional lattice sums via Ramanujan's theta functions

by

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1. Introduction. In general, elementary evaluations are rare for higher-dimensional lattice-type sums. They have been studied for many years in the mathematical physics community. The most famous higher-dimensional sum is Madelung's constant from crystallography. In this paper, we analyze various generalized two-dimensional lattice sums, one of which arose from the solution to a certain Poisson equation. We evaluate certain lattice sums in closed form using results from Ramanujan's theory of theta functions, continued fractions and class invariants. For instance,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{(8m)^2+(4n+1)^2}=-\frac{\sqrt{2}\,\pi}{16}\log\frac{(\sqrt{2}-1)-(\sqrt{\sqrt{2}+1}-\sqrt[4]{2})}{1+(\sqrt{2}-1)(\sqrt{\sqrt{2}+1}-\sqrt[4]{2})}.$$

In [5], Berndt, Lamb and Rogers evaluated in closed form the sum

$$F_{(a,b)}(q):=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+n}}{(xm)^2+(an+b)^2},\qquad q=e^{-\pi/x}, \tag{1.1}$$

for any positive rational value of x, and for certain values of $(a,b)\in\mathbb{N}^2$. They used the notation $F_{(a,b)}(x)$ instead of $F_{(a,b)}(q)$. We use $F_{(a,b)}(q)$ here so that we can state Theorem 3.3 more easily and clearly. The authors of [5] first proved the following theorem.

Theorem 1.1. *Suppose that a and b are integers with $a\geq 2$ and $(a,b)=1$, and that $\operatorname{Re} x>0$. Then*

$$F_{(a,b)}(q)=-\frac{2\pi}{ax}\sum_{j=0}^{a-1}\omega^{-(2j+1)b}\log\prod_{m=0}^{\infty}(1-\omega^{2j+1}q^{2m+1})(1-\omega^{-2j-1}q^{2m+1}), \tag{1.2}$$

where $\omega=e^{\pi i/a}$ and $q=e^{-\pi/x}$.

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Now, for any positive rational number x and positive integers a and b, in addition to (1.1), we consider two new types of sums:

$$G_{(a,b)}(q) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(xm)^2 + (an+b)^2}, \tag{1.3}$$

$$H_{(a,b)}(q) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(xm)^2 + (an+b)^2}, \tag{1.4}$$

where $q = e^{-\pi/x}$.

Before we establish the evaluations of $F_{(a,b)}(q)$, $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$, we first note that these double sums are conditionally convergent. However, it is possible to convert them into single sums which are absolutely and rapidly convergent by substituting the partial fractions decomposition for $\operatorname{csch}(z)$ [12, p. 28, Entry 1.217]:

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(am+b)^2 + c^2} = \frac{\pi}{ac}\left[\frac{\sinh\left(\frac{\pi c}{a}\right)\cos\left(\frac{\pi b}{a}\right)}{\cosh^2\left(\frac{\pi c}{a}\right) - \cos^2\left(\frac{\pi b}{a}\right)}\right].$$

While we have [5, eq. (2.1)] for $F_{(a,b)}(q)$, we are able to rewrite $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$ as the following absolutely and rapidly convergent sums which can be calculated to high numerical precision:

$$G_{(a,b)}(x) = \sum_{n=-\infty}^{\infty} \frac{\pi}{x} \frac{\operatorname{cosech}\left[\frac{\pi(an+b)}{x}\right]}{an+b}; \tag{1.5}$$

$$H_{(a,b)}(x) = \sum_{n=1}^{\infty} \frac{2\pi}{anx} \frac{\sinh\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi b}{a}\right)}{\cosh^2\left(\frac{\pi nx}{a}\right) - \cos^2\left(\frac{\pi b}{a}\right)} + \frac{\pi^2 \cot\left(\frac{\pi b}{a}\right)\csc\left(\frac{\pi b}{a}\right)}{a^2}. \tag{1.6}$$

Indeed, one just need sum from $(n, -50, 50)$ for (1.5) and $(n, 1, 100)$ for (1.6) to get many decimal places accurately.

In analogy with Theorem 1.1 for $F_{(a,b)}(q)$, we are able to prove results for $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$. Moreover, it can be shown that $G_{(a,b)}(q)$ can be recast in the theory of $F_{(a,b)}$, which is done in Section 3. In Section 6, we study the theory of $G_{(a,b)}(q)$ with the aid of the results on $F_{(a,b)}$ and derive many explicit examples afterwards.

The authors of [5] simplified Theorem 1.1 for $a \in \{3, 4, 5, 6\}$ and $b = 1$ using classical results for theta functions and q-series, and evaluated in closed form certain classes of double series. When $a > 6$, the situation is much more complicated; we study the case $(a, b) = (8, 1)$ in detail in Section 5 and derive an explicit example afterwards. The case $a = 12$ can be derived in a similar fashion. Similarly, we can also derive the theory of $H_{(a,b)}$, and more double series can be evaluated. Although the main theorems on all these three types of sums are similar and not difficult to prove, the examination of special cases of $H_{(a,b)}$ is quite different from those of $F_{(a,b)}$. In Section 4, we consider $H_{(a,b)}$ for $a \in \{3,4,5,6\}$ and $b = 1$. The resulting formulas are closely related to continued fractions including the famous Rogers–Ramanujan continued fraction, Ramanujan's cubic continued fraction, the Ramanujan–Göllnitz–Gordon continued fraction and continued fractions of order 12. We are able to produce many explicit examples from the values of these continued fractions. In these instances, we assume $b = 1$ without loss of generality, because other possible values of the lattice sums can be easily recovered from the case when $b = 1$.

Inspired by all the nice results for $F_{(a,b)}(q)$, $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$, we consider a generalization of these lattice sums that is defined by

$$J_{(a,b,s,t)}(q) := \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{\pi ims} e^{\pi int}}{(xm)^2 + (an+b)^2}, \tag{1.7}$$

where $q = e^{-\pi/x}$. In Section 3, we prove the main theorem for $J_{(a,b,s,t)}(q)$ in analogy with Theorem 1.1, and the main theorems for $G_{(a,b)}(q)$ and $H_{(a,b)}(q)$ follow easily. We then specialize to the case $(a, b) = (2, 1)$ in Section 7. For certain s and t, we are able to obtain very nice evaluations. Ramanujan's cubic continued fraction plays an important role in determining explicit examples. For instance,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{\frac{2}{3}\pi im} e^{\frac{2}{3}\pi in}}{2m^2 + (2n+1)^2} = \frac{\pi}{4\sqrt{2}} e^{-\pi i/3} \log 3.$$

Recently, Bailey, Borwein, Crandall and Zucker [2] studied a class of lattice sums arising from solutions to Poisson's equation. They determined some closed-form evaluations using Jacobi theta functions for the series $\psi_2(x, y)$ defined by

$$\psi_2(x,y) = \frac{1}{\pi^2} \sum_{m,n \text{ even}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2 + n^2}.$$

As graphically illustrated in [10], $\psi_2(x, y)$ is the 'natural' potential of the 2-dimensional jellium crystal, that is, the solution to the Poisson equation of the physical model of the jellium,

$$\nabla^2 \psi_2(\mathbf{r}) = 1 - \sum_{\mathbf{m} \in \mathbb{Z}^2} \delta^2(\mathbf{r} - \mathbf{m}),$$

where $\mathbf{r} = (x, y)$ and $\delta^2(\mathbf{r}) = \delta(x)\delta(y)$ is the Dirac delta function, with the integral of this δ^2 over $\mathbb{R}^2$ being unity.

We are thus motivated to consider the class of sums when $b = 0$ in $J_{(a,b,s,t)}(q)$, that is,

$$J_{(a,0,s,t)}(q) := \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an)^2}. \tag{1.8}$$

Clearly,

$$\mathrm{Re}\{J_{(1,0,2x,2y)}(e^{-\pi})\} = 4\pi^2\psi_2(x,y).$$

After proving the main theorem for $J_{(a,0,s,t)}(q)$ in Section 3, we examine $J_{(a,0,s,t)}(q)$ when $a \in \{1,2\}$ in Section 8. While the authors of [2] are mainly interested in applying numerical methods to first deduce the values of lattice sums, our rigorous determinations focus from the start on Ramanujan's theory of theta functions. We not only rigorously derive all the evaluations of $\psi_2(x,y)$ established in [2], but also produce further nice results such as

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{m^2+n^2} = \frac{\pi}{6}\log\frac{2-\sqrt{3}}{3\sqrt{3}}.$$

The explicit values of the two class invariants and Ramanujan's cubic continued fraction are frequently applied during the examinations.

2. Preliminary results. Let us begin by introducing the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

We now introduce Ramanujan's general theta function $f(a,b)$ and the famous Jacobi triple product identity for $f(a,b)$ [3, p. 35, Entry 19]. For $|ab|<1$,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

Following Ramanujan's notation for theta functions, define

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.2}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{2.3}$$

$$\chi(q) = (-q;q^2)_\infty, \tag{2.4}$$

$$f(-q) = (q;q)_\infty. \tag{2.5}$$

From Entry 24 in Chapter 16 of Ramanajan's Third Notebook [3, p. 39], we have

$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}, \tag{2.6}$$

and

$$\chi(q)\chi(-q) = \chi(-q^2). \tag{2.7}$$

If n is any positive rational number and $q = \exp(-\pi\sqrt{n})$, the two class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \quad \text{and} \quad g_n := -2^{-1/4}q^{-1/24}\chi(-q). \tag{2.8}$$

In the notation of Weber [16], $G_n = 2^{-1/4}\mathfrak{f}(\sqrt{-n})$ and $g_n = 2^{-1/4}\mathfrak{f}_1(\sqrt{-n})$. The term *invariant* is due to Weber. From the definitions, it follows easily that $G_n = G_{1/n}$ is equivalent to the identity [3, p. 43, Entry 27(v)]

$$e^{\alpha/24}\chi(e^{-\alpha}) = e^{\beta/24}\chi(e^{-\beta}), \tag{2.9}$$

where $\alpha\beta = \pi^2$.

There are four continued fractions that play important roles in this paper. First of all, let us recall the famous *Rogers–Ramanujan continued fraction* and its product representation:

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}+\cdots = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}, \qquad |q|<1.$$

The second one is *Ramanujan's cubic continued fraction*, which is defined by

$$G(q) := \frac{q^{1/3}}{1\ +}\,\frac{q+q^2}{1\ +}\,\frac{q^2+q^4}{1\ +}\,\frac{q^3+q^6}{1\ +}\cdots, \qquad |q|<1. \tag{2.10}$$

From Ramanujan's Lost Notebook [1, p. 94, eq. (3.3.1a) and p. 95, eq. (3.3.6)], we have

$$G(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty} = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)}. \tag{2.11}$$

Thirdly, the *Ramanujan–Göllnitz–Gordon continued fraction* is defined as

$$T(q) := \frac{q^{1/2}}{1+q+}\,\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^6}{1+q^7+}\cdots, \qquad |q|<1. \tag{2.12}$$

Ramanujan recorded a product representation of $T(q)$ on p. 229 of his Second Notebook [14], namely,

$$T(q) = q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}. \tag{2.13}$$

The last one is the continued fraction of order 12 defined by

(2.14)
$$K(q) := \frac{q(1-q)}{1-q^3} + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} + \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})} + \cdots, \qquad |q|<1.$$

This is a special case of one of the fascinating continued fraction identities recorded by Ramanujan in his Second Notebook [14], [3, p. 24, Entry 12]. Indeed, replacing q by q^3 and letting $a = q$ and $b = q^2$ in [3, Entry 12], we can obtain the product representation

$$K(q) = q\frac{f(-q;-q^{11})}{f(-q^5;-q^7)} = q\frac{(q,q^{12})_\infty(q^{11},q^{12})_\infty}{(q^5,q^{12})_\infty(q^7,q^{12})_\infty}. \tag{2.15}$$

The addition formula for theta functions [3, p. 48, Entry 31] is stated below.

Lemma 2.1. *Let* $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ *and* $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$. *Then, for each positive integer* n,

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.16}$$

We also need the following two lemmas [3, p. 36, Entry 20], [3, p. 45, Entry 29].

Lemma 2.2. *If* $\alpha\beta = \pi$, $\mathrm{Re}(\alpha^2) > 0$, *and* n *is any complex number, then*

$$\sqrt{\alpha}\, f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) = e^{n^2/4}\sqrt{\beta}\, f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}). \tag{2.17}$$

Lemma 2.3. *If* $ab = cd$, *then*

(i) $f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc)$,

(ii) $f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right)$.

As special cases of the above lemma [3, p. 51, Example (iv)], we have

$$\varphi(-q) + \phi(q^2) = 2\frac{f^2(q^3,q^5)}{\psi(q)}, \tag{2.18}$$

$$\varphi(-q) - \phi(q^2) = -2\frac{f^2(q,q^7)}{\psi(q)}. \tag{2.19}$$

3. Main theorem. We begin this section by proving the main theorem for $J_{(a,b,s,t)}(q)$ defined by (1.7).

Theorem 3.1. *Suppose that* a *and* b *are integers with* $a \geq 2$ *and* $(a,b) = 1$, s *and* t *are any real numbers with at least one not being an even number,* *and* $\operatorname{Re} x > 0$. *Then*

$$J_{(a,b,s,t)}(q) = -\frac{\pi}{ax}\sum_{j=0}^{a-1}\omega^{-(2j+t)b}\log\prod_{m=-\infty}^{\infty}(1-\omega^{2j+t}q^{|2m+s|})(1-\omega^{-(2j+t)}q^{|2m+s|}),$$

where $\omega = e^{\pi i/a}$ *and* $q = e^{-\pi/x}$.

Proof. Let N be a positive integer. Since the series $J_{(a,b,s,t)}$ is not absolutely convergent, we adopt the convention $\sum_n = \lim_{N\to\infty}\sum_{-N<n<N}$. Then we have

$$\sum_{-N<n<N}\sum_{m\in\mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an+b)^2} = \pi\sum_{-N<n<N}e^{\pi int}\int_0^\infty e^{-\pi(an+b)^2u}\Big(\sum_{m\in\mathbb{Z}}e^{\pi ims}e^{-\pi m^2x^2u}\Big)\,du. \tag{3.1}$$

Now we apply (2.17) with $\alpha = \sqrt{\pi}/(x\sqrt{u})$, $\beta = x\sqrt{u\pi}$, and $n = \pi s/(x\sqrt{u})$ to deduce that

$$\sum_{m\in\mathbb{Z}}e^{\pi ims-\pi m^2x^2u} = \frac{1}{x\sqrt{u}}\sum_{m\in\mathbb{Z}}e^{-\frac{\pi(m+s/2)^2}{x^2u}}. \tag{3.2}$$

Using (3.2) and inverting the order of summation and integration twice by absolute convergence in (3.1), we obtain

$$\begin{aligned}\sum_{-N<n<N}\sum_{m\in\mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an+b)^2} &= \frac{\pi}{x}\sum_{-N<n<N}e^{\pi int}\int_0^\infty e^{-\pi(an+b)^2u}\sum_{m\in\mathbb{Z}}e^{-\frac{\pi(m+s/2)^2}{x^2u}}\frac{du}{\sqrt{u}}\\ &= \frac{\pi}{x}\sum_{m\in\mathbb{Z}}\sum_{-N<n<N}e^{\pi int}\int_0^\infty e^{-\pi(an+b)^2u-\frac{\pi(m+s/2)^2}{x^2u}}\frac{du}{\sqrt{u}}.\end{aligned}$$

Applying the elementary formula [12, p. 384, eq. (3.471), no. 9], [7, p. 39],

$$\int_0^\infty e^{-\pi(A^2u+B^2/u)}\frac{du}{\sqrt{u}} = \frac{e^{-2\pi|A||B|}}{|A|}, \tag{3.3}$$

we have

$$\sum_{-N<n<N}\sum_{m\in\mathbb{Z}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(an+b)^2} = \frac{\pi}{x}\sum_{m\in\mathbb{Z}}\sum_{-N<n<N}\frac{e^{\pi int}q^{|2m+s||an+b|}}{|an+b|}. \tag{3.4}$$

Now we introduce a variable r and establish the following claim by comparing Taylor series coefficients in r and letting $N\to\infty$:

(3.5)
$$\sum_{-\infty<n<\infty} \frac{e^{\pi int} r^{|an+b|}}{|an+b|} = -\frac{1}{a}\sum_{j=0}^{a-1} \omega^{-(2j+t)b} \log(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r).$$

Note that if both s and t are even numbers, then we have $\log 0$ at $m = 0$ on the right-hand side of the above identity. Therefore we exclude this case in the assumption of the theorem to ensure the convergence of the series. Similarly to [5, eq. (2.5)], we use a crude error estimate to bound the terms where $n \geq N$ and $n \leq -N$ as follows:

(3.6)
$$\sum_{-N<n<N} \frac{e^{\pi int} r^{|an+b|}}{|an+b|} = -\frac{1}{a}\sum_{j=0}^{a-1} \omega^{-(2j+t)b} \log[(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r)] + O\left(\frac{r^N}{(1-r)N}\right).$$

To complete the proof, we substitute (3.6) into (3.4) and take the limit as $N \to \infty$. ∎

Note that $F_{(a,b)}(q) = J_{(a,b,1,1)}(q)$, $G_{(a,b)}(q) = J_{(a,b,1,0)}(q)$ and $H_{(a,b)}(q) = J_{(a,b,0,1)}(q)$. Thus we have the following corollary.

Corollary 3.2. *Suppose that a and b are integers with $a \geq 2$, $(a,b) = 1$, and assume that Re $x > 0$. Then*

$$G_{(a,b)}(q) = -\frac{2\pi}{ax}\sum_{j=0}^{a-1} \omega^{-2jb} \log \prod_{m=0}^{\infty} (1-\omega^{2j}q^{2m+1})(1-\omega^{-2j}q^{2m+1}), \tag{3.7}$$

$$H_{(a,b)}(q) = -\frac{\pi}{ax}\sum_{j=0}^{a-1} \omega^{-(2j+1)b} \log \prod_{m\in\mathbb{Z}} (1-\omega^{2j+1}q^{2|m|})(1-\omega^{-2j-1}q^{2|m|}), \tag{3.8}$$

where $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

The following theorem shows that $G_{(a,b)}(q)$ can be placed within the theory of $F_{(a,b)}$.

Theorem 3.3.

(i) *Suppose that a and b are integers with $a \geq 2$ and $(2a,b) = 1$, and that $\operatorname{Re} x > 0$. Then*

$$G_{(2a,b)}(q) = \tfrac{1}{2}F_{(a,b)}(q) + \tfrac{1}{2}G_{(a,b)}(q). \tag{3.9}$$

(ii) *If we further assume that a is any odd integer, then*

$$G_{(a,b)}(q) = \begin{cases} F_{(a,b)}(-q) & \text{if } b \text{ is even,} \\ -F_{(a,b)}(-q) & \text{if } b \text{ is odd.} \end{cases} \tag{3.10}$$

Proof. We begin by proving (3.10). Note that $\sin(2jb\pi/a)$ does not appear in the summation below since $G_{(a,b)}(q)$ is real-valued and the imaginary terms sum to 0. Now from (3.7), we have

$$G_{(a,b)}(q) = -\frac{2\pi}{ax}\sum_{j=0}^{a-1}\cos\left(\frac{2jb\pi}{a}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\left(\frac{2j\pi}{a}\right)q^{2m+1}+q^{4m+2}\right)$$

$$= -\frac{4\pi}{ax}\sum_{j=0}^{(a-1)/2}\cos\left(\frac{2jb\pi}{a}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\left(\frac{2j\pi}{a}\right)q^{2m+1}+q^{4m+2}\right)$$
$$+\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= \frac{4\pi}{ax}\sum_{j=0}^{(a-1)/2}\cos\left(\frac{(a-2jb)\pi}{a}\right)\log\prod_{m=0}^{\infty}\left(1+2\cos\left(\frac{(a-2j)\pi}{a}\right)q^{2m+1}+q^{4m+2}\right)$$
$$+\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= \frac{4\pi}{ax}\sum_{j=0}^{(a-1)/2}\cos\left(\frac{\left[a-2\left(\frac{a-1}{2}-j\right)b\right]\pi}{a}\right)$$
$$\times\log\prod_{m=0}^{\infty}\left(1+2\cos\left(\frac{\left[a-2\left(\frac{a-1}{2}-j\right)\right]\pi}{a}\right)q^{2m+1}+q^{4m+2}\right)$$
$$+\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= \frac{4\pi}{ax}\sum_{j=0}^{(a-1)/2}\cos\left(\frac{[a(1-b)+(2j+1)b]\pi}{a}\right)$$
$$\times\log\prod_{m=0}^{\infty}\left(1+2\cos\left(\frac{(2j+1)\pi}{a}\right)q^{2m+1}+q^{4m+2}\right)$$
$$+\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= -\frac{4\pi}{ax}\sum_{j=0}^{(a-1)/2}(-1)^b\cos\left(\frac{(2j+1)b\pi}{a}\right)$$
$$\times\log\prod_{m=0}^{\infty}\left(1+2\cos\left(\frac{(2j+1)\pi}{a}\right)q^{2m+1}+q^{4m+2}\right)$$
$$+\frac{2\pi}{ax}\log\prod_{m=0}^{\infty}(1-q^{2m+1})^2$$

$$= -(-1)^b \frac{4\pi}{ax} \sum_{j=0}^{(a-1)/2} \cos\Big(\frac{(2j+1)b\pi}{a}\Big) \log \prod_{m=0}^{\infty} \Big(1 + 2\cos\Big(\frac{(2j+1)\pi}{a}\Big) q^{2m+1} + q^{4m+2}\Big)$$
$$+ (-1)^b \frac{2\pi}{ax} \log \prod_{m=0}^{\infty} \cos\Big(\frac{\big(2\cdot\frac{a-1}{2}+1\big)\pi b}{a}\Big)\Big(1 + 2\cos\Big(\frac{\big(2\cdot\frac{a-1}{2}+1\big)\pi}{a}\Big) q^{2m+1} + q^{4m+2}\Big)$$
$$= (-1)^b F_{(a,b)}(-q).$$

This completes the proof of (3.10). It remains to prove (3.9) from (3.7):

$$G_{(2a,b)}(q)$$
$$= -\frac{\pi}{ax} \sum_{j=0}^{2a-1} e^{\frac{\pi ibj}{a}} \log \prod_{m=0}^{\infty} (1 - e^{\frac{\pi ij}{a}} q^{2m+1})(1 - e^{-\frac{\pi ij}{a}} q^{2m+1})$$
$$= -\frac{\pi}{ax} \Big\{ \sum_{j=0}^{a-1} e^{\frac{\pi ib(2j+1)}{a}} \log \prod_{m=0}^{\infty} (1 - e^{\frac{\pi i(2j+1)}{a}} q^{2m+1})(1 - e^{-\frac{\pi i(2j+1)}{a}} q^{2m+1})$$
$$+ \sum_{j=0}^{a-1} e^{\frac{\pi ib(2j)}{a}} \log \prod_{m=0}^{\infty} (1 - e^{\frac{\pi i(2j)}{a}} q^{2m+1})(1 - e^{-\frac{\pi i(2j)}{a}} q^{2m+1}) \Big\}.$$
$$= \tfrac{1}{2} F_{(a,b)}(q) + \tfrac{1}{2} G_{(a,b)}(q). \ \blacksquare$$

To finish this section, we prove the main theorem for $J(a, 0, s, t)$ defined by (1.8).

THEOREM 3.4. *Suppose that a is a positive integer, s and t are any real numbers with at least one not being an even number, and $\operatorname{Re} x > 0$. Then*

$$J_{(a,0,s,t)}(q) = -\frac{\pi}{ax} \sum_{j=0}^{a-1} \log \prod_{m=-\infty}^{\infty} (1 - \omega^{2j+t} q^{|2m+s|})(1 - \omega^{-(2j+t)} q^{|2m+s|}) + \sum_{m\neq 0} \frac{e^{\pi ims}}{(xm)^2},$$

where $\omega = e^{\pi i/a}$ and $q = e^{-\pi/x}$.

Proof. The proof is similar to the proof of Theorem 3.1. The main difference is that the index (m, n) cannot be $(0, 0)$. Therefore, we need to separate the sum when $n = 0$ at the very beginning, and thus we have

$$\sum_{\substack{-N<n<N \\ n\neq 0}} \sum_{m\in\mathbb{Z}} \frac{e^{\pi ims} e^{\pi int}}{(xm)^2 + (an)^2} = \frac{\pi}{x} \sum_{m\in\mathbb{Z}} \sum_{\substack{-N<n<N \\ N\neq 0}} \frac{e^{\pi int} q^{|2m+s||an|}}{|an|}. \tag{3.11}$$

Then we claim that for $|r| < 1$,

$$\sum_{n\neq 0} \frac{e^{\pi int} r^{|na|}}{|na|} = -\frac{1}{a} \sum_{j=0}^{a-1} \log[(1 - \omega^{2j+t} r)(1 - \omega^{-(2j+t)} r)]. \tag{3.12}$$

Indeed, by expanding the right side of (3.12) as Taylor series in r, we have

$$\begin{aligned}
-\frac{1}{a}\sum_{j=0}^{a-1}&\log[(1-\omega^{2j+t}r)(1-\omega^{-(2j+t)}r)]\\
&=\frac{1}{a}\sum_{j=0}^{a-1}\sum_{k=1}^{\infty}\Big(\frac{r^k}{k}e^{\pi ik(2j+t)/a}+\frac{r^k}{k}e^{-\pi ik(2j+t)/a}\Big)\\
&=\frac{1}{a}\sum_{k=1}^{\infty}\frac{r^k}{k}\Big(e^{\pi ikt/a}\sum_{j=0}^{a-1}e^{2\pi ijk/a}+e^{-\pi ikt/a}\sum_{j=0}^{a-1}e^{-2\pi ijk/a}\Big)\\
&=\frac{1}{a}\sum_{na\ge1}\frac{r^{|na|}}{|na|}e^{\pi int}\cdot a+\frac{1}{a}\sum_{na\ge1}\frac{r^{|na|}}{|na|}e^{-\pi int}\cdot a\\
&=\sum_{na\ge1}\frac{r^{|na|}}{|na|}e^{\pi int}+\sum_{na\le1}\frac{r^{|na|}}{|na|}e^{\pi int}=\sum_{n\ne0}\frac{e^{\pi int}r^{|na|}}{|na|}.
\end{aligned}$$

To finish the argument, we use the same idea as in the proof of Theorem 3.1. ∎

4. Examinations of $H_{(a,b)}$ for $a \in \{3,4,5,6\}$ and explicit examples.

Although the proofs of (3.8) and Theorem 1.1 are similar, the examinations of special cases of $H_{(a,b)}$ are quite different from those of $F_{(a,b)}$, and they are actually more difficult, because we have even powers of q instead of odd powers in the evaluation. In this section, we examine the cases where $a \in \{3,4,5,6\}$.

Let us prove a couple of lemmas before the examinations.

LEMMA 4.1. *For $|q|<1$, we have*

$$\prod_{m\ge1}(1-\sqrt{2}\,q^{2m}+q^{4m}) = \frac{qf(-q^{32})}{f(-q^2)}\sqrt{\frac{f(-q^4)}{f(-q^8)}}\Big(\frac{1}{\sqrt{T(q^4)}}-(\sqrt{2}+1)\sqrt{T(q^4)}\Big), \tag{4.1}$$

$$\prod_{m\ge1}(1+\sqrt{2}\,q^{2m}+q^{4m}) = \frac{qf(-q^{32})}{f(-q^2)}\sqrt{\frac{f(-q^4)}{f(-q^8)}}\Big(\frac{1}{\sqrt{T(q^4)}}+(\sqrt{2}-1)\sqrt{T(q^4)}\Big), \tag{4.2}$$

$$\prod_{m\ge1}\frac{1-\sqrt{2}\,q^{2m}+q^{4m}}{1+\sqrt{2}\,q^{2m}+q^{4m}} = \frac{1-(\sqrt{2}+1)T(q^4)}{1+(\sqrt{2}-1)T(q^4)}, \tag{4.3}$$

where $T(q)$ is the Ramanujan–Göllnitz–Gordon continued fraction (2.12).

Proof. The equality (4.3) can be easily derived from (4.1) and (4.2). Here we give the proof of (4.1) only, as the proof of (4.2) is similar. Letting $\omega = e^{\pi i/4}$ and using the Jacobi triple product identity (2.1) for Ramanujan's general theta function $f(a,b)$, we have

$$\begin{aligned}(4.4)\qquad \prod_{m\ge 1}(1-\sqrt{2}\,q^{2m}+q^{4m}) &= \prod_{m\ge 1}(1-(\omega+\omega^{-1})q^{2m}+q^{4m})\\ &= \prod_{m\ge 1}(1-\omega q^{2m})(1-\omega^{-1}q^{2m})\\ &= (\omega q^2;q^2)_\infty(\omega^{-1}q^2;q^2)_\infty\\ &= \frac{1}{(q^2;q^2)_\infty}\,\frac{f(-\omega,-\omega^{-1}q^2)}{1-\omega}.\end{aligned}$$

Applying the addition formula (2.16) with $n=4$, $a=-\omega$ and $b=-\omega^{-1}q^2$, we obtain

$$f(-\omega,-\omega^{-1}q^2) = (1-\omega)f(-q^{12},-q^{20}) + (\omega^2+\omega^3)q^2 f(-q^4,-q^{28}).$$

It follows that

$$(4.5)\qquad \frac{f(-\omega,-\omega^{-1}q^2)}{1-\omega} = f(-q^{12},-q^{20}) - (\sqrt{2}+1)q^2 f(-q^4,-q^{28}).$$

To complete the proof, we substitute (4.5) into (4.4), divide both the denominator and numerator by $\sqrt{f(-q^{12},-q^{20})f(-q^4,-q^{28})}$, use the product representation (2.13) of the Ramanujan–Göllnitz–Gordon continued fraction, and manipulate theta products to deduce that

$$f(-q^{12};-q^{20})f(-q^4;-q^{28}) = f^2(-q^{32})\frac{f(-q^4)}{f(-q^8)}. \quad\blacksquare$$

Lemma 4.2. *For $|q|<1$, we have*

$$\begin{aligned}(4.6)\qquad &\prod_{m\ge 1}(1-\sqrt{3}\,q^{2m}+q^{4m})\\ &\quad= \frac{f(-q^{30},-q^{42})}{f(-q^2)}(1+(\sqrt{3}+1)J(q^6)+(2+\sqrt{3})K(q^6)),\end{aligned}$$

$$\begin{aligned}(4.7)\qquad &\prod_{m\ge 1}(1+\sqrt{3}\,q^{2m}+q^{4m})\\ &\quad= \frac{f(-q^{30},-q^{42})}{f(-q^2)}(1+(\sqrt{3}-1)J(q^6)+(2-\sqrt{3})K(q^6)),\end{aligned}$$

$$(4.8)\qquad \prod_{m\ge 1}\frac{1-\sqrt{3}\,q^{2m}+q^{4m}}{1+\sqrt{3}\,q^{2m}+q^{4m}} = \frac{1-(\sqrt{3}+1)J(q^6)+(2+\sqrt{3})K(q^6)}{1+(\sqrt{3}-1)J(q^6)+(2-\sqrt{3})K(q^6)},$$

where $J(q) := q^{1/3}\frac{f(-q^3,-q^9)}{f(-q^5,-q^7)}$, *and* $K(q) = q\frac{f(-q,-q^{11})}{f(-q^5,-q^7)}$ *is the continued fraction of order* 12 *defined by* (2.14).

The proof is similar to that of Lemma 4.1, and we leave this calculation as an exercise to the reader.

THEOREM 4.3. *Suppose that* $q = e^{-\pi/x}$. *Let* $G(q)$, $T(q)$ *and* $K(q)$ *be the continued fractions defined in* (2.10), (2.12) *and* (2.14), *respectively. Let* $J(q)$ *be the function defined in Lemma* 4.2. *Then*

$$H_{(3,1)}(q) = \frac{2\pi}{9x}\log\frac{8(1+G^3(q^2))}{1-8G^3(q^2)}, \tag{4.9}$$

$$H_{(4,1)}(q) = -\frac{\pi}{\sqrt{2}\,x}\log\frac{\sqrt{2}-1-T(q^4)}{1+(\sqrt{2}-1)T(q^4)}, \tag{4.10}$$

$$\begin{aligned} H_{(5,1)}(q) &= \frac{2\pi}{5x}\log 2 + \frac{\pi}{5x}\log\frac{\chi(-q^{10})}{\chi^5(-q^2)} - \frac{\pi}{\sqrt{5}\,x}\log\frac{\sqrt{5}\,\varphi(q^5)-\varphi(q)}{\sqrt{5}\,\varphi(q^5)+\varphi(q)} \\ &\quad - \frac{\pi}{5\sqrt{5}\,x}\log\frac{(1-\alpha^5R^5(q))(1-\beta^5R^5(q^2))}{(1-\beta^5R^5(q))(1-\alpha^5R^5(q^2))}, \end{aligned} \tag{4.11}$$

$$H_{(6,1)}(q) = \frac{\pi}{\sqrt{3}\,x}\log(2+\sqrt{3})\frac{1+(\sqrt{3}-1)J(q^6)+(2-\sqrt{3})K(q^6)}{1+(\sqrt{3}+1)J(q^6)+(2+\sqrt{3})K(q^6)}. \tag{4.12}$$

Proof. We begin by proving (4.9). If we set $(a,b) = (3,1)$, then (3.8) immediately reduces to

$$\begin{aligned} &H_{(3,1)}(q) \\ &= -\frac{\pi}{3x}\sum_{j=0}^{2}\cos\Big(\frac{(2j+1)\pi}{3}\Big)\log\prod_{m\in\mathbb{Z}}\Big(1-2\cos\Big(\frac{(2j+1)\pi}{3}\Big)q^{2|m|}+q^{4|m|}\Big) \\ &= -\frac{\pi}{3x}\log\prod_{m\in\mathbb{Z}}\frac{1-q^{2|m|}+q^{4|m|}}{1+2q^{2|m|}+q^{4|m|}} = -\frac{\pi}{3x}\log\frac{1}{4}\prod_{m\geq 1}\frac{(1-q^{2m}+q^{4m})^2}{(1+2q^{2m}+q^{4m})^2} \\ &= \frac{2\pi}{3x}\log 2 - \frac{2\pi}{3x}\log\prod_{m\geq 1}\frac{1+q^{6m}}{(1+q^{2m})^3} = \frac{2\pi}{3x}\log 2 - \frac{2\pi}{3x}\log\frac{\chi^3(-q^2)}{\chi(-q^6)}. \end{aligned} \tag{4.13}$$

Notice that we used $\chi(-q) = 1/(-q;q)_\infty$ in the last equality above. To finish the calculation, let us take $\alpha = 1-\phi^4(-q)/\phi^4(q)$ and $\beta = 1-\phi^4(-q^3)/\phi^4(q^3)$, so that β has degree 3 over α in the theory of modular equations. Then using [3, p. 124, Entry 12],

$$\chi(-q) = 2^{1/6}(1-x)^{1/12}(xq)^{-1/24},$$

we have

$$q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} = 2^{-1/3}\frac{(1-\alpha)^{1/12}\beta^{1/8}}{(1-\beta)^{1/4}\alpha^{1/24}}, \qquad \frac{\chi^3(-q)}{\chi(-q^3)} = 2^{1/3}\frac{(1-\alpha)^{1/4}\beta^{1/24}}{(1-\beta)^{1/12}\alpha^{1/8}}.$$

Since α and β admit birational parameterizations $\alpha = p(2+p)^3/(1+2p)^3$

and $\beta = p^3(2+p)/(1+2p)$ [3, p. 230, Entry 5(vi)], we deduce that

$$q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} = \left(\frac{p}{2(1+p)}\right)^{1/3}, \tag{4.14}$$

$$\frac{\chi^3(-q)}{\chi(-q^3)} = \left(\frac{2(1-p)^2}{(2+p)(1+2p)}\right)^{1/3}. \tag{4.15}$$

Now if $v = q^{1/3}\chi(-q)/\chi^3(-q^3) = G(q)$, as given in (2.11), then we solve (4.14) for p to obtain

$$p = \frac{1-4v^3+\sqrt{1-8v^3}}{4v^3}, \tag{4.16}$$

and it follows that

$$\frac{\chi^3(-q)}{\chi(-q^3)} = \left(\frac{1-8v^3}{1+v^3}\right)^{1/3} \tag{4.17}$$

by substituting (4.16) into (4.15). Replacing q by q^2 and substituting (4.17) into (4.13) completes the proof of (4.9).

Notice that if $(a,b) = (4,1)$, then (3.8) becomes

$$\begin{aligned} H_{(4,1)}(q) &= -\frac{\pi}{4x}\sum_{j=0}^{3}\cos\left(\frac{(2j+1)\pi}{4}\right) \\ &\qquad \times \log\prod_{m\in\mathbb{Z}}\left(1-2\cos\left(\frac{(2j+1)\pi}{4}\right)q^{2|m|}+q^{4|m|}\right) \\ &= -\frac{\sqrt{2}\,\pi}{4x}\log\prod_{m\in\mathbb{Z}}\frac{1-\sqrt{2}\,q^{2|m|}+q^{4|m|}}{1+\sqrt{2}\,q^{2|m|}+q^{4|m|}} \\ &= -\frac{\sqrt{2}\,\pi}{2x}\log\left((\sqrt{2}-1)\prod_{m\geq 1}\frac{1-\sqrt{2}\,q^{2m}+q^{4m}}{1+\sqrt{2}\,q^{2m}+q^{4m}}\right). \end{aligned}$$

Using (4.3), we are led to the closed form (4.10).

We set $\alpha = 2\cos\left(\frac{3}{5}\pi\right) = \frac{1-\sqrt{5}}{2}$, and $\beta = 2\cos\left(\frac{1}{5}\pi\right) = \frac{1+\sqrt{5}}{2}$. With $(a,b) = (5,1)$ in (3.8), we have

$$\begin{aligned} &H_{(5,1)}(q) \\ &= -\frac{\pi}{5x}\sum_{j=0}^{4}\cos\left(\frac{(2j+1)\pi}{5}\right)\log\prod_{m\in\mathbb{Z}}\left(1-2\cos\left(\frac{(2j+1)\pi}{5}\right)q^{2|m|}+q^{4|m|}\right) \\ &= -\frac{\pi}{5x}\log\prod_{m\in\mathbb{Z}}\frac{(1-\alpha q^{2|m|}+q^{4|m|})^{\alpha}(1-\beta q^{2|m|}+q^{4|m|})^{\beta}}{(1+q^{2|m|})^2} \\ &= -\frac{\pi}{5x}\log\prod_{m\in\mathbb{Z}}\frac{(1+q^{10|m|})^{1/2}}{(1+q^{2|m|})^{5/2}}\left(\frac{1-\beta q^{2|m|}+q^{4|m|}}{1-\alpha q^{2|m|}+q^{4|m|}}\right)^{\sqrt{5}/2} \end{aligned} \tag{4.18}$$

$$= -\frac{\pi}{5x}\log\frac{1}{4}\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{\sqrt{5}} - \frac{\pi}{5x}\log\frac{\chi^5(-q^2)}{\chi(-q^{10})}$$
$$-\frac{\pi}{\sqrt{5}\,x}\log\prod_{m\,\text{even}}\frac{1-\beta q^m+q^{2m}}{1-\alpha q^m+q^{2m}}.$$

Factorizations of certain theta-function identities of degree 5 are given by [1, p. 30, Entry 1.7.2(i),(ii)]:

$$\varphi(q)+\sqrt{5}\,\varphi(q^5) = \frac{(1+\sqrt{5})f(-q^2)}{\prod_{n\,\text{odd}}(1+\alpha q^n+q^{2n})\prod_{n\,\text{even}}(1-\beta q^n+q^{2n})},$$
$$\varphi(q)-\sqrt{5}\,\varphi(q^5) = \frac{(1-\sqrt{5})f(-q^2)}{\prod_{n\,\text{even}}(1-\alpha q^n+q^{2n})\prod_{n\,\text{odd}}(1+\beta q^n+q^{2n})},$$

from which we deduce that

(4.19)
$$\prod_{m\,\text{even}}\frac{1-\beta q^m+q^{2m}}{1-\alpha q^m+q^{2m}}\prod_{m\,\text{odd}}\frac{1+\alpha q^m+q^{2m}}{1+\beta q^m+q^{2m}} = \frac{(\sqrt{5}+1)(\sqrt{5}\,\varphi(q^5)-\varphi(q))}{(\sqrt{5}-1)(\sqrt{5}\,\varphi(q^5)+\varphi(q))}.$$

Now we use two of the most important formulas for the Rogers–Ramanujan continued fraction from Ramanujan's Lost Notebook [1, pp. 21–22, Entry 1.4.1],

$$\left(\frac{1}{\sqrt{t}}\right)^5 - (\alpha\sqrt{t})^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^n+q^{2n})^5},$$
$$\left(\frac{1}{\sqrt{t}}\right)^5 - (\beta\sqrt{t})^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\beta q^n+q^{2n})^5},$$

to obtain

$$\prod_{m\,\text{odd}}\frac{1+\beta q^m+q^{2m}}{1+\alpha q^m+q^{2m}} = \sqrt[5]{\frac{(1-\alpha^5R^5(q))(1-\beta^5R^5(q^2))}{(1-\beta^5R^5(q))(1-\alpha^5R^5(q^2))}}. \tag{4.20}$$

To complete the proof of (4.11), we substitute (4.19) and (4.20) into (4.18).

Now if $(a,b)=(6,1)$, we can easily prove (4.12) by applying Lemma 4.2. From (3.8), we have

$$H_{(6,1)}(q) = -\frac{\pi}{6x}\sum_{j=0}^{5}\cos\left(\frac{(2j+1)\pi}{6}\right) \tag{4.21}$$
$$\times\log\prod_{m\in\mathbb{Z}}\left(1-2\cos\left(\frac{(2j+1)\pi}{6}\right)q^{2|m|}+q^{4|m|}\right)$$

$$= -\frac{\sqrt{3}\,\pi}{6x} \log \prod_{m\in\mathbb{Z}} \frac{1-\sqrt{3}\,q^{2|m|}+q^{4|m|}}{1+\sqrt{3}\,q^{2|m|}+q^{4|m|}}$$

$$= \frac{\pi}{\sqrt{3}\,x} \log(2+\sqrt{3}) \prod_{m\geq 1} \frac{1+\sqrt{3}\,q^{2m}+q^{4m}}{1-\sqrt{3}\,q^{2m}+q^{4m}}.$$

Applying (4.8), we are led to the closed form (4.12). ■

Next we consider explicit examples of $H_{(a,b)}(q)$ from Theorem 4.3. We first derive examples for $H_{(3,1)}(q)$ from (4.9). It is clear that the formulas for $G(q)$ can also be used to evaluate $H_{(3,1)}(q)$. When $x = 1/\sqrt{2}$ we appeal to [1, p. 100, eq. (3.4.4)]. We have

$$G(e^{-\sqrt{2}\pi}) = \frac{-2+\sqrt{6}}{2}. \tag{4.22}$$

Thus we obtain

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{2m^2+(3n+1)^2} = \frac{\sqrt{2}\,\pi}{9} \log(4+2\sqrt{6}).$$

Similarly, set $x = 3\sqrt{2}$. We use [1, p. 100, eq. (3.4.5)] to find that

$$G^3(e^{-\sqrt{2}\pi/3}) = \frac{-2+\sqrt{6}}{4}. \tag{4.23}$$

Therefore

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{18m^2+(3n+1)^2} = \frac{\sqrt{2}\,\pi}{27} \log(44+18\sqrt{6}).$$

As another example, when $x = 1$ we appeal to [1, p. 100, eq. (3.4.3)] to obtain

$$G(e^{-2\pi}) = \frac{-(1+\sqrt{3})+\sqrt{6\sqrt{3}}}{4}, \tag{4.24}$$

and thus

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{m^2+(3n+1)^2} = \frac{2\pi}{9} \log\bigl(2+\sqrt{3}+\sqrt{9+6\sqrt{3}}\,\bigr). \tag{4.25}$$

Now we derive explicit examples from (4.10), which include the Ramanujan–Göllnitz–Gordon continued fraction on the right-hand side. When $x = 8$ we appeal to [9, p. 84, eq. (4.2)] to find that $T(e^{-\pi/2}) = \sqrt{\sqrt{2}+1} - \sqrt[4]{2}$, which yields

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(8m)^2+(4n+1)^2} = -\frac{\sqrt{2}\,\pi}{16} \log \frac{(\sqrt{2}-1)-\bigl(\sqrt{\sqrt{2}+1}-\sqrt[4]{2}\bigr)}{1+(\sqrt{2}-1)\bigl(\sqrt{\sqrt{2}+1}-\sqrt[4]{2}\bigr)}.$$

When $x = \frac{8}{3}\sqrt{3}$ we appeal to [9, p. 86]. We have

$$T(e^{-\pi\sqrt{3}/2}) = \sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}},$$

and hence

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(8m)^2+3(4n+1)^2} = -\frac{\sqrt{6}\,\pi}{48}\log\frac{(\sqrt{2}-1)-\big(\sqrt{\sqrt{6}+\sqrt{2}+1}-\sqrt{\sqrt{6}+\sqrt{2}}\big)}{1+(\sqrt{2}-1)\big(\sqrt{\sqrt{6}+\sqrt{2}+1}-\sqrt{\sqrt{6}+\sqrt{2}}\big)}.$$

We can find further evaluations for $H_{(4,1)}(q)$ by applying other formulas in [9, p. 84, eqs. (4.3), (4.4)] and [9, p. 86, Examples].

Now we examine the more difficult case of $H_{(6,1)}$. We consider $x = 6$, which yields $q^6 = e^{-\pi}$. We appeal to [13, Theorem 5.1] to find that

$$K(e^{-\pi}) = \frac{(6\sqrt{3}-9)^{1/4}-1}{(6\sqrt{3}-9)^{1/4}+1}. \tag{4.26}$$

We still need to examine $J(e^{-\pi})$. We apply [13, Lemma 3.1] first to obtain

$$J(q) = \frac{2q^{1/3}\chi(q)\psi(-q^3)}{\varphi(q)+\varphi(q^3)}.$$

So we can evaluate $J(e^{-\pi})$ from formulas for $\varphi(e^{-\pi})$, $\varphi(e^{-3\pi})$, $\psi(-e^{-3\pi})$ and $\chi(e^{-\pi})$. We appeal to [17, Lemma 5.1, Theorem 5.5], [18, Theorem 5.6] and [4, p. 326, Entry 2(viii)] respectively. For $a = \pi^{-1/4}/\Gamma(3/4)$ we have

$$\varphi(e^{-\pi}) = a, \qquad \psi(-e^{-3\pi}) = a2^{-3/4}3^{-1/2}e^{3\pi/8}(2\sqrt{3}-3)^{1/4},$$
$$\varphi(e^{-3\pi}) = a2^{-1}3^{-3/8}\sqrt{\sqrt{3}+1}, \qquad \chi(e^{-\pi}) = e^{-\pi/24}2^{1/4}.$$

Simplify the resulting quotient to obtain

$$J(e^{-\pi}) = \frac{\sqrt{2}\,(2-\sqrt{3})^{1/4}}{3^{3/8}+2^{-1/4}\sqrt{\sqrt{3}+1}}. \tag{4.27}$$

To finish the calculation, we just need to insert (4.26) and (4.27) into (4.12) and simplify. Hence

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^n}{(6m)^2+(6n+1)^2} = \frac{\pi}{6\sqrt{3}}\log\bigg(\frac{2+\sqrt{3}}{2}\big(5-\sqrt{3}+\sqrt{2}\cdot 3^{3/4}+\sqrt[4]{6\sqrt{3}-9}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})\big)\bigg).$$

5. Simplification of $F_{(8,1)}$ and explicit example

THEOREM 5.1. *Suppose that* $q = e^{-\pi/x}$. *Then*

(5.1) $$F_{(8,1)}(q)$$
$$= -\frac{\pi}{4x}\sqrt{2+\sqrt{2}}\log\frac{A(q)-\sqrt{1+\sqrt{2}/2}\,q\sqrt{C(q)}-\sqrt{1-\sqrt{2}/2}\,q\sqrt{D(q)}}{A(q)+\sqrt{1+\sqrt{2}/2}\,q\sqrt{C(q)}+\sqrt{1-\sqrt{2}/2}\,q\sqrt{D(q)}}$$
$$-\frac{\pi}{4x}\sqrt{2-\sqrt{2}}\log\frac{B(q)-\sqrt{1-\sqrt{2}/2}\,q\sqrt{C(q)}+\sqrt{1+\sqrt{2}/2}\,q\sqrt{D(q)}}{B(q)+\sqrt{1-\sqrt{2}/2}\,q\sqrt{C(q)}-\sqrt{1+\sqrt{2}/2}\,q\sqrt{D(q)}},$$

where
$$A(q) = \frac{\varphi(-q^{64})}{\psi(-q^{16})} + \sqrt{2}\,q^4\frac{\psi(-q^{32})}{\psi(-q^{16})}, \qquad C(q) = \frac{\varphi(q^{16})}{\psi(-q^{16})} + \frac{\varphi(q^{32})}{\psi(-q^{16})},$$
$$B(q) = \frac{\varphi(-q^{64})}{\psi(-q^{16})} - \sqrt{2}\,q^4\frac{\psi(-q^{32})}{\psi(-q^{16})}, \qquad D(q) = \frac{\varphi(q^{16})}{\psi(-q^{16})} - \frac{\varphi(q^{32})}{\psi(-q^{16})}.$$

Proof. If $(a, b) = (8, 1)$, then (1.2) reduces to

(5.2) $$F_{(8,1)}(q)$$
$$= -\frac{\pi}{4x}\sum_{j=0}^{7}\cos\left(\frac{(2j+1)\pi}{8}\right)\log\prod_{m=0}^{\infty}\left(1-2\cos\left(\frac{(2j+1)\pi}{8}\right)q^{2m+1}+q^{4m+2}\right)$$
$$= -\frac{\pi}{4x}\log\prod_{m=0}^{\infty}\left(\frac{1-2\cos(\frac{\pi}{8})q^{2m+1}+q^{4m+2}}{1+2\cos(\frac{\pi}{8})q^{2m+1}+q^{4m+2}}\right)^{2\cos(\frac{\pi}{8})}$$
$$\times\left(\frac{1-2\cos(\frac{3\pi}{8})q^{2m+1}+q^{4m+2}}{1+2\cos(\frac{3\pi}{8})q^{2m+1}+q^{4m+2}}\right)^{2\cos(\frac{3\pi}{8})}$$
$$= -\frac{\pi}{4x}\log\prod_{m=0}^{\infty}\left(\frac{(1-2\cos(\frac{\pi}{8})q^{2m+1}+q^{4m+2})(1-q^{2m+2})}{(1+2\cos(\frac{\pi}{8})q^{2m+1}+q^{4m+2})(1-q^{2m+2})}\right)^{2\cos(\frac{\pi}{8})}$$
$$\times\left(\frac{(1-2\cos(\frac{3\pi}{8})q^{2m+1}+q^{4m+2})(1-q^{2m+2})}{(1+2\cos(\frac{3\pi}{8})q^{2m+1}+q^{4m+2})(1-q^{2m+2})}\right)^{2\cos(\frac{3\pi}{8})}.$$

Letting $\xi = e^{\pi i/8}$ and using the Jacobi triple product identity (2.1), we find

$$F(q) := \prod_{m=0}^{\infty}\left(1+2\cos\left(\frac{\pi}{8}\right)q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2})$$
$$= (-\xi q; q^2)_\infty(-\bar{\xi}q; q^2)_\infty(q^2; q^2)_\infty = \sum_{n=-\infty}^{\infty}\xi^n q^{n^2}$$
$$= \sum_{n=-\infty}^{\infty}(-1)^n[q^{(8n)^2}+\xi q^{(8n+1)^2}+\xi^2 q^{(8n+2)^2}+\cdots+\xi^7 q^{(8n+7)^2}].$$

Note that $F(q)$ is real-valued, so the imaginary terms above sum to 0. Therefore we only need to consider the real parts. First, we have

$$\begin{aligned}\mathrm{Re}\sum_{n=-\infty}^{\infty}(-1)^n(\xi^2 q^{(8n+2)^2}+\xi^6 q^{(8n+6)^2})\\ &= 2\,\mathrm{Re}\sum_{n=-\infty}^{\infty}(-1)^n\xi^2 q^{(8n+2)^2}=\sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^n q^{4(4n+1)^2}\\ &= \sqrt{2}\Big(\sum_{n=0}^{\infty}(-1)^n q^{4(4n+1)^2}-\sum_{n=0}^{\infty}(-1)^n q^{4(4n+3)^2}\Big)\\ &= \sqrt{2}\sum_{n=0}^{\infty}(-1)^{n(n+1)/2}q^{4(2n+1)^2}=\sqrt{2}\,q^4\psi(-q^{32}).\end{aligned}$$

Now we consider

$$\begin{aligned}&\mathrm{Re}\sum_{n=-\infty}^{\infty}(-1)^n[\xi q^{(8n+1)^2}+\xi^7 q^{(8n+7)^2}]\\ &= \mathrm{Re}\Big(\xi\sum_{n=-\infty}^{\infty}(-1)^n[q^{(8n+1)^2}-q^{(8n+7)^2}]\Big)\\ &= 2\cos\Big(\frac{\pi}{8}\Big)\sum_{n=-\infty}^{\infty}(-1)^n q^{(8n+1)^2}=2\cos\Big(\frac{\pi}{8}\Big)q\sum_{n=-\infty}^{\infty}(-1)^n q^{16n(4n+1)}\\ &= 2\cos\Big(\frac{\pi}{8}\Big)qf(-q^{16\cdot 3},-q^{16\cdot 5})=\sqrt{2}\cos\Big(\frac{\pi}{8}\Big)\sqrt{\psi(-q^{16})[\varphi(q^{16})+\varphi(q^{32})]},\end{aligned}$$

where we apply (2.18) in the last identity. If we use (2.19), then we find

$$\begin{aligned}\mathrm{Re}\sum_{n=-\infty}^{\infty}(-1)^n[\xi^3 q^{(8n+3)^2}+\xi^5 q^{(8n+5)^2}]\\ &= \sqrt{2}\cos\Big(\frac{3\pi}{8}\Big)\sqrt{\psi(-q^{16})[\varphi(q^{16})-\varphi(q^{32})]}.\end{aligned}$$

Combining the results above, we are led to the closed form

$$\begin{aligned}F(q)&=\varphi(-q^{64})+\sqrt{2}q^4\psi(-q^{32})\\ &\quad+\sqrt{2}\cos\Big(\frac{\pi}{8}\Big)q\sqrt{\psi(-q^{16})[\varphi(q^{16})+\varphi(q^{32})]}\\ &\quad+\sqrt{2}\cos\Big(\frac{3\pi}{8}\Big)q\sqrt{\psi(-q^{16})[\varphi(q^{16})-\varphi(q^{32})]}.\end{aligned}\tag{5.3}$$

Similarly, we can derive a formula for the other factor in the denominator of (5.2). Thus,

$$G(q) := \prod_{m=0}^{\infty}\left(1+2\cos\left(\frac{3\pi}{8}\right)q^{2m+1}+q^{4m+2}\right)(1-q^{2m+2}) \tag{5.4}$$

$$= \varphi(-q^{64}) - \sqrt{2}\,q^4\psi(-q^{32}) + \sqrt{2}\cos\left(\frac{3\pi}{8}\right)q\sqrt{\psi(-q^{16})[\varphi(q^{16})+\varphi(q^{32})]}$$

$$-\sqrt{2}\cos\left(\frac{3\pi}{8}\right)q\sqrt{\psi(-q^{16})[\varphi(q^{16})-\varphi(q^{32})]}.$$

To complete the proof, we just need to apply (5.3), (5.4) and the facts that

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2} \quad \text{and} \quad \cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}. \ \blacksquare$$

The formula for $a = 12$ can be deduced in a similar fashion. However, the formula is more complicated and thus we do not give it here. We will need to apply Lemma 2.3 to obtain formulas similar to (2.18) and (2.19), namely,

$$f(q^5,q^7) = \sqrt{\frac{f(-q^6)\varphi(q^3)\chi(-q^6)+f(-q^2)\varphi(-q^3)\chi(-q^2)}{2\chi(-q^2)}},$$

$$f(q,q^{11}) = \sqrt{\frac{f(-q^6)\varphi(q^3)\chi(-q^6)-f(-q^2)\varphi(-q^3)\chi(-q^2)}{2q^2\chi(-q^2)}}.$$

We conclude this section by proving a formula for $F_{(8,1)}$ from (5.1). In principle, these calculations are straightforward exercises if the values of $\varphi(q)$, $\varphi(q^2)$, $\varphi(-q^2)$, $\varphi(-q^4)$, $\psi(-q)$ and $\psi(-q^2)$ are known. However, (5.1) is a long equation, so in practice, we only identify one instance where q^{16} is reasonably simple, that is, when $q^{16} = e^{-\pi}$. We appeal to [17, Theorems 5.5, 5.7] and [18, Theorems 5.6, 5.7]. For $a = \pi^{-1/4}/\Gamma(3/4)$ we have

$$\varphi(e^{-\pi}) = a, \qquad \varphi(-e^{-4\pi}) = a2^{-7/16}(\sqrt{2}+1)^{1/2},$$

$$\varphi(e^{-2\pi}) = a2^{-1}(\sqrt{2}+2)^{1/2}, \qquad \psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8},$$

$$\varphi(-e^{-2\pi}) = a2^{-1/8}, \qquad \psi(-e^{-2\pi}) = a2^{-15/16}e^{\pi/4}(\sqrt{2}-1)^{1/4}.$$

After simplification, we obtain

$$A(e^{-\pi}) = 2^{15/16}e^{-\pi/8}\{(\sqrt{2}+1)^{1/4}+(\sqrt{2}-1)^{1/4}\},$$

$$B(e^{-\pi}) = 2^{15/16}e^{-\pi/8}\{(\sqrt{2}+1)^{1/4}-(\sqrt{2}-1)^{1/4}\},$$

$$C(e^{-\pi}) = e^{-\pi/8}(2^{3/4}+(\sqrt{2}+1)^{1/2}),$$

$$D(e^{-\pi}) = e^{-\pi/8}(2^{3/4}-(\sqrt{2}+1)^{1/2}).$$

With all the calculations above, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+n}}{(16m)^2+(8n+1)^2}$$
$$=-\frac{\pi}{64}\sqrt{2+\sqrt{2}}\log\frac{\sqrt{2}a-\sqrt{2+\sqrt{2}}\,c-\sqrt{2-\sqrt{2}}\,d}{\sqrt{2}a+\sqrt{2+\sqrt{2}}\,c+\sqrt{2-\sqrt{2}}\,d}$$
$$-\frac{\pi}{64}\sqrt{2-\sqrt{2}}\log\frac{\sqrt{2}b-\sqrt{2-\sqrt{2}}\,c+\sqrt{2+\sqrt{2}}\,d}{\sqrt{2}b+\sqrt{2-\sqrt{2}}\,c-\sqrt{2+\sqrt{2}}\,d},$$

where

$$a=2^{15/16}\{(\sqrt{2}+1)^{1/4}+(\sqrt{2}-1)^{1/4}\},\qquad c=2^{3/4}+(\sqrt{2}+1)^{1/2},$$
$$b=2^{15/16}\{(\sqrt{2}+1)^{1/4}-(\sqrt{2}-1)^{1/4}\},\qquad d=2^{3/4}-(\sqrt{2}+1)^{1/2}.$$

Curiously, $\sqrt{2+\sqrt{2}}$ is the connective constant of the honeycomb lattice (see [11]).

6. Examination of $G_{(a,b)}$ and explicit examples. The authors of [5] examined $F_{(a,b)}$ for $a\in\{3,4,5,6\}$, and we have just considered the case $a=8$ in the previous section. Applying Theorem 3.3, we can easily express $G_{(a,b)}$ for $a\in\{3,5,6,10,12\}$ in terms of $F_{(a,b)}$. Moreover, it can be easily derived from (3.7) that

$$G_{(2,1)}(q)=-\frac{\pi}{x}\log\prod_{m=-\infty}^{\infty}\frac{(1-q^{2m+1})^2}{(1+q^{2m+1})^2} \tag{6.1}$$
$$=-\frac{2\pi}{x}\log\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}=-\frac{2\pi}{x}\log\frac{\chi(-q)}{\chi(q)},$$

where $\chi(q)$ is defined in (2.4). Combining (3.9), (6.1) and the fact that $F_{(2,1)}=0$, we can examine the case when $a=4$. Indeed, we have

$$G_{(4,1)}(q)=-\frac{\pi}{x}\log\frac{\chi(-q)}{\chi(q)}. \tag{6.2}$$

By iterating, we can now examine the cases where $a\in\{8,16\}$.

Now we produce explicit examples for $G_{(a,b)}(q)$. We first consider the simple case when $(a,b)=(3,1)$. From (3.10) and (4.17), we can easily derive that

$$G_{(3,1)}(q)=\frac{2\pi}{9x}\log\frac{1+G^3(q)}{1-8G^3(q)}, \tag{6.3}$$

where $G(q)$ is Ramanujan's cubic continued fraction (2.10). We can use formulas for $G(q)$ to evaluate $G_{(3,1)}(q)$. When $x=1/\sqrt{2}$ we appeal to (4.22).

After simplification, it follows from (6.3) that

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{m^2+2(3n+1)^2}=\frac{\sqrt{2}\,\pi}{9}\log\frac{2+\sqrt{6}}{4}.$$

Similarly, when $x=\sqrt{3}$, we appeal to [1, p. 105] to find that $G(e^{-\pi/\sqrt{3}})=\frac{\sqrt{3}-1}{4^{1/3}}$. After simplification, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{3m^2+(3n+1)^2}=\frac{2\pi}{9\sqrt{3}}\log\frac{5+3\sqrt{3}}{2}.$$

Now we examine $G_{(6,1)}(q)$ from (3.9). It follows from [5, eq. (3.3)] and (6.3) that

$$G_{(6,1)}(q)=\frac{\pi}{9x}\log\frac{(1+v^3)(1-8u^3)}{(1-8v^3)(1+u^3)},$$

where $u=G(-q)$ and $v=G(q)$. When $x=1$ we appeal to [8, p. 350, eqs. (4.1) and (4.2)] to find that

$$G(-e^{-\pi})=\frac{1-\sqrt{3}}{2},\qquad G(e^{-\pi})=\frac{(1+\sqrt{3})\big(-(1+\sqrt{3})+\sqrt{6\sqrt{3}}\,\big)}{4},$$

which yield

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{m^2+(6n+1)^2}=\frac{\pi}{9}\log\big(1+3^{3/4}\sqrt{2-\sqrt{3}}\,\big). \tag{6.4}$$

Next we consider $G_{(4,1)}(q)$ and then $G_{(8,1)}(q)$. Recall from (6.2) that

$$G_{(4,1)}(q)=-\frac{\pi}{x}\log\frac{\chi(-q)}{\chi(q)}=-\frac{\pi}{x}\log\frac{\psi(-q)}{\psi(q)}=-\frac{\pi}{2x}\log\frac{\varphi(-q)}{\varphi(q)}. \tag{6.5}$$

The formulas for $\varphi(q)$ and $\psi(q)$ can be used to evaluate $G_{(4,1)}$. For example, many explicit evaluations can be found in [4, p. 325], [17] and [18]. Set $x=4$. By [4, p. 325, Entry 1], for $a=\pi^{1/4}/\Gamma(3/4)$ we have

$$\varphi(e^{-\pi/4})=a(1+2^{-1/4}),\qquad \varphi(-e^{-\pi/4})=a(1-2^{-1/4}).$$

It follows from (6.5) that

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{16m^2+(4n+1)^2}=\frac{\pi}{8}\log\frac{\sqrt[4]{2}+1}{\sqrt[4]{2}-1}. \tag{6.6}$$

Using (3.9), we can obtain the formula for $G_{(8,1)}(e^{-\pi/4})$ from $F_{(4,1)}(e^{-\pi/4})$ and $G_{(4,1)}(e^{-\pi/4})$. We first evaluate $F_{(4,1)}(q)$ at $x=4$ from [5, eq. (3.4)]. Similarly to the evaluation of [5, eq. (3.18)], we appeal to [4, Examples 9.4] to find that $\alpha_4=(\sqrt{2}-1)^4$, and thus

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m+n}}{16m^2+(4n+1)^2}=\frac{\pi}{4\sqrt{2}}\log\frac{1+\sqrt{\sqrt{2}-1}}{1-\sqrt{\sqrt{2}-1}}. \tag{6.7}$$

Substituting (6.6) and (6.7) into (3.9) leads to

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{16m^2+(8n+1)^2}=\frac{\pi}{16}\log\frac{\sqrt[4]{2}+1}{\sqrt[4]{2}-1}+\frac{\pi}{8\sqrt{2}}\log\frac{1+\sqrt{\sqrt{2}-1}}{1-\sqrt{\sqrt{2}-1}}.$$

7. Examination of $J_{(a,b,s,t)}(q)$ and explicit examples. Simplification of $J_{(a,b,s,t)}(q)$ is difficult when $(s,t)\neq(1,0)$, $(0,1)$ or $(1,1)$. However, we can get several nice evaluations for $(a,b)=(2,1)$.

THEOREM 7.1. *Assume that s and t are any real numbers with at least one not being an even integer, and* $\operatorname{Re} x>0$. *With $q=e^{-\pi/x}$, we have*

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(2n+1)^2}=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}} \tag{7.1}$$

$$\times\log\prod_{m\geq 0}\frac{\left(1-2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+s}\right)\left(1-2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s}\right)}{\left(1+2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+s}\right)\left(1+2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s}\right)}.$$

Proof. The proof is straightforward. We apply (7.1) with $a=2$ and $b=1$ to obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(2n+1)^2}$$

$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}\log\prod_{m\in\mathbb{Z}}\frac{(1-e^{\frac{\pi it}{2}}q^{|2m+s|})(1-e^{-\frac{\pi it}{2}}q^{|2m+s|})}{(1+e^{\frac{\pi it}{2}}q^{|2m+s|})(1+e^{-\frac{\pi it}{2}}q^{|2m+s|})}$$

$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}$$

$$\times\log\prod_{m\geq 0}\frac{(1-e^{\frac{\pi it}{2}}q^{2m+s})(1-e^{-\frac{\pi it}{2}}q^{-(2m+s)})(1-e^{\frac{\pi it}{2}}q^{2m+2-s})(1-e^{-\frac{\pi it}{2}}q^{-(2m+2-s)})}{(1+e^{\frac{\pi it}{2}}q^{2m+s})(1+e^{-\frac{\pi it}{2}}q^{-(2m+s)})(1+e^{\frac{\pi it}{2}}q^{2m+2-s})(1+e^{-\frac{\pi it}{2}}q^{-(2m+2-s)})}$$

$$=-\frac{\pi}{2x}e^{-\frac{\pi it}{2}}\log\prod_{m\geq 0}\frac{(1-2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+2s})(1-2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s})}{(1+2\cos(\frac{\pi t}{2})q^{2m+s}+q^{4m+2s})(1+2\cos(\frac{\pi t}{2})q^{2m+2-s}+q^{4m+4-2s})}.\ \blacksquare$$

THEOREM 7.2. *With* $\operatorname{Re} x>0$ *and $q=e^{-\pi/x}$, we have*

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{\psi(q)\psi(-q^3)}{\psi(-q)\psi(q^3)}, \tag{7.2}$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{1}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2} \tag{7.3}$$

$$=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\left(-\frac{G(q^{-1/3})\psi(q)\psi(q^3)}{G(-q^{1/3})\psi(-q)\psi(-q^3)}\right),$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{\varphi^2(-q^2)}{\varphi(-q^{\frac{2}{3}})\varphi(-q^6)}, \tag{7.4}$$

where $G(q)$ is Ramanujan's cubic continued fraction (2.10).

Proof. If we set $t=\frac{2}{3}$, then (7.1) immediately reduces to

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\pi ims}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2} \tag{7.5}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\prod_{m\geq 0}\frac{(1-q^{2m+s}+q^{4m+2s})(1-q^{2m+2-s}+q^{4m+4-2s})}{(1+q^{2m+s}+q^{4m+2s})(1+q^{2m+2-s}+q^{4m+4-2s})}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{(q^s;q^2)_\infty(q^{2-s};q^2)_\infty(-q^{3s};q^6)_\infty(-q^{6-3s};q^6)_\infty}{(-q^s;q^2)_\infty(-q^{2-s};q^2)_\infty(q^{3s};q^6)_\infty(q^{6-3s};q^6)_\infty}.$$

We begin by proving (7.2). If $s=1$, then (7.5) becomes

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=-\frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{(q;q^2)_\infty(-q^3;q^6)_\infty}{(-q;q^2)_\infty(q^3;q^6)_\infty}$$
$$=\frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{\chi(q)\chi(-q^3)}{\chi(-q)\chi(q^3)}=\frac{\pi}{x}e^{-\frac{\pi i}{3}}\log\frac{\psi(q)\psi(-q^3)}{\psi(-q)\psi(q^3)},$$

where we applied (2.6) in the last identity.

Next, we prove (7.3). Notice that if we set $s=\frac{1}{3}$, then (7.5) becomes

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{1}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2} \tag{7.6}$$
$$=-\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{(q^{1/3};q^2)_\infty(q^{5/3};q^2)_\infty(-q;q^6)_\infty(-q^5;q^6)_\infty}{(-q^{1/3};q^2)_\infty(-q^{5/3};q^2)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}.$$

To manipulate the q-products on the right side of (7.6), we first replace q by q^3 to obtain

$$\frac{(q;q^6)_\infty(q^5;q^6)_\infty(-q^3;q^{18})_\infty(-q^{15};q^{18})_\infty}{(-q;q^6)_\infty(-q^5;q^6)_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty}=\frac{(q;q^2)_\infty(-q^3;q^6)_\infty^2(q^9;q^{18})_\infty}{(-q;q^2)_\infty(q^3;q^6)_\infty^2(-q^9;q^{18})_\infty}$$
$$=\frac{\chi(-q)\chi^2(q^3)\chi(-q^9)}{\chi(q)\chi^2(-q^3)\chi(q^9)}.$$

We recall from (2.11) that

$$G(-q)=-q^{1/3}\chi(q)/\chi^3(q^3),\qquad G(q)=q^{1/3}\chi(-q)/\chi^3(-q^3). \tag{7.7}$$

After replacing q by $q^{1/3}$ in the above identity and simplifying, we have

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{1}{3}\pi im}e^{\frac{2}{3}\pi in}}{(xm)^2+(2n+1)^2}=\frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\left(-\frac{G(-q^{1/3})\psi(q)\psi(q^3)}{G(q^{1/3})\psi(-q)\psi(-q^3)}\right).$$

Now it remains to prove (7.4). Similarly to the proof of (7.3), we set $t=\frac{2}{3}$

in (7.5), manipulate the resulting q-products and simplify to obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi i m}e^{\frac{2}{3}\pi i n}}{(xm)^2+(2n+1)^2} = \frac{\pi}{2x}e^{-\frac{\pi i}{3}}\log\frac{f^2(-q^2)\chi^2(-q^2)}{f(-q^{2/3})f(-q^6)\chi(-q^{2/3})\chi(-q^6)}.$$

Finally, we use (2.7) to complete the proof. ∎

Now we derive some explicit examples from Theorem 7.2. All of our identities follow from well-known q-series evaluations. We first examine an example from (7.2). This case is relatively easy to evaluate. When $x=1$ we appeal to [18, Theorems 5.6, 5.7]. For $a=\pi^{-1/4}/\Gamma(3/4)$ we have

$$\psi(-e^{-\pi}) = a2^{-3/4}e^{\pi/8}, \tag{7.8}$$

$$\psi(e^{-\pi}) = a2^{-5/8}e^{\pi/8}, \tag{7.9}$$

$$\psi(-e^{-3\pi}) = a2^{-3/4}3^{-1/2}e^{3\pi/8}(2\sqrt{3}-3)^{1/4}, \tag{7.10}$$

$$\psi(e^{-3\pi}) = \frac{ae^{3\pi/8}}{2^{1/8}3^{3/8}\sqrt{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}}. \tag{7.11}$$

With all the evaluations above, from (7.2) we get

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi i n}}{m^2+(2n+1)^2} = \pi e^{-\pi i/3}\log\frac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})}. \tag{7.12}$$

If we equate the real and imaginary parts of (7.12), then lattice sums involving sine and cosine functions can be evaluated. We obtain, respectively,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m\cos\left(\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2} = \frac{\pi}{2}\log\frac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})},$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m\sin\left(\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2} = -\frac{\sqrt{3}\,\pi}{2}\log\frac{1+\sqrt{2}\sqrt[4]{3}+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})}.$$

Similarly, when $x=1/\sqrt{3}$, we have [18, Theorems 4.7(iii), 4.10(x)]

$$\frac{\psi(-e^{-\sqrt{3}\pi})}{\psi(-e^{-3\sqrt{3}\pi})} = 3^{1/4}e^{-\sqrt{3}\pi/4}\frac{\sqrt{3}}{\sqrt[3]{4}-1},$$

$$\frac{\psi(e^{-\sqrt{3}\pi})}{\psi(e^{-3\sqrt{3}\pi})} = 3^{1/4}e^{-\sqrt{3}\pi/4}\frac{3^{1/6}(1-\sqrt{3}+\sqrt{3}\sqrt[3]{4})^{1/3}}{2^{1/12}(\sqrt[3]{2}-1)^{2/3}(1+\sqrt{3})^{1/6}}.$$

Thus we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^m e^{\frac{2}{3}\pi i n}}{m^2+3(2n+1)^2} = \frac{\sqrt{3}}{9}\pi e^{-\pi i/3}\log\frac{1-\sqrt{3}+\sqrt{3}\sqrt[3]{4}}{\sqrt[4]{2}(\sqrt{3}+1)^{1/2}}. \tag{7.13}$$

To calculate further examples from (7.2), we rewrite the ψ-quotient on the right-hand side of (7.2) in terms of Ramanujan's cubic continued fraction. We appeal to (2.6), (7.7) and [4, p. 330, eq. (4.6)] to deduce that

$$\frac{\varphi(q)}{\varphi(q^3)} = \left(1 + 8q\frac{\chi^3(q)}{\chi^9(q^3)}\right)^{1/4} = (1 - 8qG^3(-q))^{1/4},$$

and therefore we can rewrite (7.2) as

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{2/3\pi in}}{(xm)^2 + (2n+1)^2} = \frac{\pi}{8x} e^{-\pi i/3} \log \frac{1 - 8G^3(-q)}{1 - 8G^3(q)}. \tag{7.14}$$

Besides (7.12) and (7.13), we can derive more examples from (7.2) by applying formulas for $G(q)$ and $G(-q)$. For instance, when $x = 3$, we have [1, p. 105], [6, Corollary 4.6]

$$G(-e^{-\pi/3}) = -\left(\frac{1+\sqrt{3}}{4}\right)^{1/3}, \tag{7.15}$$

$$G(e^{-\pi/3}) = \frac{1}{2}\left(1 - \left(\frac{3-b}{1+b}\right)^2\right)^{1/3} \quad \text{with} \quad b = \sqrt{2\sqrt{3}+3}, \tag{7.16}$$

and thus we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{(-1)^m e^{2/3\pi in}}{9m^2 + (2n+1)^2} = \frac{\pi}{12} e^{-\pi i/3} \log \frac{\sqrt{3+2\sqrt{3}}(1 + \sqrt{3+2\sqrt{3}})}{3 - \sqrt{3+2\sqrt{3}}}. \tag{7.17}$$

Now we examine (7.3). When $x = 1$, we have similar evaluations for $\psi(-e^{-\pi})$, $\psi(e^{-\pi})$, $\psi(-e^{-3\pi})$, $\psi(e^{-3\pi})$, $G(-e^{-\pi/3})$ and $G(e^{-\pi/3})$ to the previous example, namely, (7.8), (7.9), (7.10), (7.11), (7.15) and (7.16). By a direct computation, we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} &\frac{e^{\frac{1}{3}\pi im} e^{\frac{2}{3}\pi in}}{m^2 + (2n+1)^2} \\ &= \frac{\pi}{2} e^{-\pi i/3} \log 2^{1/4}(2+\sqrt{3})^{1/4}\big(\sqrt{2\sqrt{3}+3}+1\big)\big(1 + \sqrt{2}\,\sqrt[4]{3} + \sqrt{3}\big)^{-1/2}.\end{aligned}$$

After further simplification, we obtain the very neat and nice formula

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{e^{\frac{1}{3}\pi im} e^{\frac{2}{3}\pi in}}{m^2 + (2n+1)^2} = \frac{\pi}{4} e^{-\pi i/3} \log(2+\sqrt{3}). \tag{7.18}$$

Equate the real and imaginary parts to obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{\cos\big(\frac{1}{3}\pi m + \frac{2}{3}\pi n\big)}{m^2 + (2n+1)^2} = \frac{\pi}{8} \log(2+\sqrt{3}),$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin\left(\frac{1}{3}\pi m+\frac{2}{3}\pi n\right)}{m^2+(2n+1)^2} = -\frac{\sqrt{3}\,\pi}{8}\log(2+\sqrt{3}).$$

Next, we examine (7.4). Theoretically, the calculation is also straightforward if the values of the φ-functions on the right side are known. In practice, it is actually difficult to simultaneously obtain the values of $\varphi(-q^{2/3})$, $\varphi(-q^2)$ and $\varphi(-q^6)$. However, we can rewrite the right side in terms of cubic continued fractions. From Ramanujan's Lost Notebook [1, p. 96, eq. (3.3.10)], we have

$$\frac{\varphi(-q^{1/3})}{\varphi(-q^3)} = 1-2G(q).$$

In [17, Theorem 4.3], J. Yi proved that for $P = \varphi(q)/\varphi(q^3)$ and $Q = \varphi(q^3)/\varphi(q^9)$,

$$\left(\frac{Q}{P}\right)^2 = PQ+\frac{3}{PQ}-3. \tag{7.19}$$

Now we apply (7.19) with $P = \varphi(-q^{2/3})/\varphi(-q^2)$ and $Q = \varphi(-q^2)/\varphi(-q^6)$ to find that

$$\frac{Q}{P} = \frac{\varphi^2(-q^2)}{\varphi(-q^{2/3})\varphi(-q^6)}, \qquad PQ = \frac{\varphi(-q^{2/3})}{\varphi(-q^6)} = 1-2G(q^2).$$

To evaluate Q/P, we only need to know the value of the relative cubic continued fraction. We give a couple of examples here.

If we set $x = \sqrt{2}$, then $G(q^2) = G(e^{-\sqrt{2}\pi}) = (-1+\sqrt{6})/2$, as given in (4.22). It follows that $PQ = 3-\sqrt{6}$. Applying (7.19), we have

$$\frac{\varphi^2(-e^{-\sqrt{2}\pi})}{\varphi(-e^{-\sqrt{2}\pi/3})\varphi(-e^{-3\sqrt{2}\pi})} = \sqrt{3}, \tag{7.20}$$

which implies

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi i m}e^{\frac{2}{3}\pi i n}}{2m^2+(2n+1)^2} = \frac{\pi}{4\sqrt{2}}e^{-\pi i/3}\log 3. \tag{7.21}$$

Again, if we equate the real parts and imaginary parts of (7.21), then we obtain, respectively,

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\cos\left(\frac{2}{3}\pi m+\frac{2}{3}\pi n\right)}{2m^2+(2n+1)^2} = \frac{\pi}{8\sqrt{2}}\log 3,$$

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin\left(\frac{2}{3}\pi m+\frac{2}{3}\pi n\right)}{2m^2+(2n+1)^2} = -\frac{\sqrt{3}\,\pi}{8\sqrt{2}}\log 3.$$

When $x=1$, the value of $G(e^{-2\pi})$ (see (4.24)) yields

$$\frac{\varphi(-e^{-2\pi/3})}{\varphi(-e^{-2\pi})}\,\frac{\varphi(-e^{-2\pi})}{\varphi(-e^{-6\pi})}=1-2G(e^{-2\pi})=\frac{3+\sqrt{3}+\sqrt{6\sqrt{3}}}{2}.$$

Substituting this last result into (7.19) and simplifying, we deduce that

$$\frac{\varphi^2(-e^{-2\pi})}{\varphi(-e^{-2\pi/3})\varphi(-e^{-6\pi})}=3^{1/4}. \tag{7.22}$$

Thus we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{\frac{2}{3}\pi im}e^{\frac{2}{3}\pi in}}{m^2+(2n+1)^2}=\frac{\pi}{8}e^{-\pi i/3}\log 3. \tag{7.23}$$

8. Examination of $J(a,0,s,t)$ and explicit examples. Now we examine $J(a,0,s,t)$ from Theorem 3.4 in two cases: $a=1$ and $a=2$. Calculation easily shows

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+n^2}=\sum_{m\neq0}\frac{e^{\pi ims}}{(xm)^2}-\frac{\pi}{x}\log\prod_{m=-\infty}^{\infty}(1-2\cos(\pi t)q^{|s+2m|}+q^{2|s+2m|}) \tag{8.1}$$

and

$$\begin{aligned}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{\pi ims}e^{\pi int}}{(xm)^2+(2n)^2}&=\sum_{m\neq0}\frac{e^{\pi ims}}{(xm)^2}-\frac{\pi}{2x}\log\prod_{m=-\infty}^{\infty}\left(1-2\cos\left(\frac{\pi t}{2}\right)q^{|s+2m|}+q^{2|s+2m|}\right)\\&\quad\times\left(1+2\cos\left(\frac{\pi t}{2}\right)q^{|s+2m|}+q^{2|s+2m|}\right).\end{aligned} \tag{8.2}$$

Before we derive explicit examples from (8.1) and (8.2), let us recall the definition of the Weber–Ramanujan class invariants G_n and g_n from (2.8). The table at the end of Weber's book [16, pp. 721–726] contains the values of 105 class invariants. Without the knowledge of class field theory, Ramanujan calculated class invariants independently for different reasons. His table of 46 class invariants does not contain any values that are in Weber's book. As G. N. Watson [15] remarked, "For reasons which had commended themselves to Weber and Ramanujan independently, it is customary to determine G_n for odd values of n, and g_n for even values of n." With the help of the properties of χ, i.e., (2.6), (2.7) and

(2.9), we can calculate many values of χ-functions using the values of class invariants in the tables [16, pp. 721–726], [4, pp. 189–204]. For instance,

$$\chi(e^{-\pi}) = 2^{1/4}e^{-\pi/24}G_1 = e^{-\pi/24}2^{1/4},$$
$$\chi(-e^{-2\pi}) = 2^{1/4}e^{-\pi/12}g_4 = e^{-\pi/12}2^{3/8},$$
$$\chi(e^{-3\pi}) = 2^{1/4}e^{-\pi/8}G_9 = e^{-\pi/8}2^{1/12}(1+\sqrt{3})^{1/3},$$
$$\chi(-e^{-6\pi}) = 2^{1/4}e^{-\pi/4}g_{36} = e^{-\pi/4}2^{1/8}\big(2+\sqrt{3}+\sqrt{9+6\sqrt{3}}\big)^{1/3},$$
$$\chi(e^{-\pi/3}) = e^{\pi/9}\chi(e^{-3\pi}) = e^{-\pi/72}2^{1/12}(1+\sqrt{3})^{1/3},$$
$$\chi(e^{-\pi/2}) = e^{3\pi/48}\chi(e^{-2\pi}) = e^{3\pi/48}\frac{\psi(e^{-2\pi})\chi(-e^{-2\pi})}{\psi(-e^{-2\pi})} = e^{-\pi/48}2^{1/16}(\sqrt{2}+1)^{1/4}.$$

EXAMPLE 8.1.

$$\sum_{(m,n)\neq(0,0)} \frac{(-1)^{m+n}}{m^2+n^2} = -\pi\log 2, \tag{8.3}$$

$$\sum_{(m,n)\neq(0,0)} \frac{(-1)^{m+n}}{(2m)^2+n^2} = -\frac{\pi}{4}\log(4+3\sqrt{2}), \tag{8.4}$$

$$\sum_{(m,n)\neq(0,0)} \frac{(-1)^{n}}{m^2+(2n)^2} = \frac{\pi}{4}\log\frac{4+3\sqrt{2}}{2}, \tag{8.5}$$

$$\sum_{(m,n)\neq(0,0)} \frac{(-1)^{m+n}}{3m^2+7n^2} = -\frac{\pi}{\sqrt{21}}\log(2^{-1/3}(\sqrt{7}-\sqrt{3})(3+\sqrt{7})^{2/3}), \tag{8.6}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}\right)\cos\left(\frac{\pi n}{3}\right)}{m^2+n^2} = \frac{\pi}{3}\log(2+\sqrt{3}), \tag{8.7}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{m^2+n^2} = \frac{\pi}{6}\log\frac{2-\sqrt{3}}{3\sqrt{3}}, \tag{8.8}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{2m^2+n^2} = -\frac{\pi}{2\sqrt{2}}\log 3, \tag{8.9}$$

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{m^2+(2n)^2} = \frac{\pi}{6}\log\frac{2-\sqrt{3}}{3^{3/4}}. \tag{8.10}$$

Note that (8.3) is the classical lattice evaluation [7, eq. (9.2.4)]. By interchanging the order of m and n and using the special values of the cosine function, we very easily see that

$$\sum_{(m,n)\neq(0,0)} \frac{(-1)^n \cos\left(\frac{\pi m}{2}\right)}{m^2+(2n)^2} = \sum_{(m,n)\neq(0,0)} \frac{(-1)^m \cos\left(\frac{\pi n}{2}\right)}{(2m)^2+n^2}$$

$$= \sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{2}\right)\cos\left(\frac{\pi n}{2}\right)}{m^2+n^2} = \frac{1}{4}\sum_{(m,n)\neq(0,0)} \frac{(-1)^{m+n}}{m^2+n^2} = -\frac{\pi}{4}\log 2.$$

Identities (8.3), (8.4), (8.5), (8.7) and (8.8) can be found in [2, Ex. 18, Appendix C]. However, those authors can only rigorously establish (8.3), (8.4), (8.5) and (8.7). The authors of [2] obtain (8.8) experimentally, and moreover, they have a misprint in their evaluation: $\frac{\pi}{6}\log\left(\frac{2-\sqrt{3}}{\sqrt{3}}\right)$ instead of $\frac{\pi}{6}\log\left(\frac{2-\sqrt{3}}{3\sqrt{3}}\right)$ on the right-hand side of (8.8). Using (8.1) and (8.2), we can derive all these identities from well-known q-series evaluations.

We derive some explicit formulas from (8.1) first. If we set $s = t = 1$, then (8.1) immediately reduces to

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{(-1)^{m+n}}{(xm)^2+n^2} = -\frac{4\pi}{x}\log\chi(q) + \sum_{m\neq 0}\frac{(-1)^m}{(xm)^2}.$$

Note that $\sum_{m\neq 0}(-1)^m/m^2 = -\pi^2/6$. When $x = 1$, then $\chi(q) = \chi(e^{-\pi}) = e^{-\pi/24}2^{1/4}$, and therefore we have (8.3). Similarly, when $x = 2$, we have $\chi(q) = \chi(e^{-\pi/2}) = e^{-\pi/48}2^{1/16}(\sqrt{2}+1)^{1/4}$. Thus we obtain (8.4). We can obtain many additional formulas using the explicit values of the class invariants G_n and g_n. For instance, when $x = \sqrt{3}/\sqrt{7}$, we have $G_{7/3} = 2^{-1/3}(\sqrt{7}-\sqrt{3})^{1/4}(3+\sqrt{7})^{1/6}$ [4, p. 341]. This completes the evaluation of (8.6).

If we set $s = t = 1/3$ and then $s = 1/3$, $t = -1/3$, we obtain

$$\sum_{(m,n)\neq(0,0)} \frac{e^{\pi i m/3}e^{\pi i n/3}}{m^2+n^2} = \sum_{(m,n)\neq(0,0)} \frac{e^{\pi i m/3}e^{-\pi i n/3}}{m^2+n^2}$$

$$= -\frac{\pi}{x}\log\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} + \sum_{m\neq 0}\frac{e^{\pi i m/3}}{(xm)^2}.$$

Equate the real parts of each side to find that

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}+\frac{\pi n}{3}\right)}{m^2+n^2} = \sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}-\frac{\pi n}{3}\right)}{m^2+n^2}$$

$$= -\frac{\pi}{x}\log\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} + \sum_{m\neq 0}\frac{\cos\left(\frac{\pi m}{3}\right)}{(xm)^2}.$$

Therefore we also have

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{\pi m}{3}\right)\cos\left(\frac{\pi n}{3}\right)}{m^2+n^2} = -\frac{\pi}{x}\log\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} + \sum_{m\neq 0}\frac{\cos\left(\frac{\pi m}{3}\right)}{(xm)^2}.$$

When $x=1$, we have

$$\frac{\chi^2(q)}{\chi(q^{1/3})\chi(q^3)} = \frac{\chi^2(e^{-\pi})}{\chi(e^{-\pi/3})\chi(e^{-3\pi})} = e^{\pi/18}2^{-1/3}(\sqrt{3}-1)^{2/3}.$$

Note that $\sum_{m\neq 0}\frac{\cos(\pi m/3)}{m^2} = \pi^2/18$. The last two identities lead to (8.7).

Now we examine a more complicated case when $s=t=2/3$. Similarly to the previous case, we find that

(8.11)
$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{(xm)^2+n^2} = -\frac{\pi}{x}\log\frac{f^2(-q^2)}{f(-q^{2/3})f(-q^6)} + \sum_{m\neq 0}\frac{\cos\left(\frac{2\pi m}{3}\right)}{(xm)^2}.$$

To calculate the theta function quotient on the right side above, we first apply (2.6) to obtain

$$\frac{f^2(-q^2)}{f(-q^{2/3})f(-q^6)} = \frac{\varphi^2(-q^2)}{\varphi(-q^{2/3})\varphi(-q^6)}\,\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)}.$$

Consider the case $x=1$. Recall that we have (7.22) for the φ-quotient. So it remains to calculate the χ-quotient

$$\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)} = \frac{\chi(-q^{2/3})\chi(-q^2)\chi(-q^6)}{\chi^3(-q^2)} = q^{-2/9}G(q^{2/3})\chi(-q^2)\chi(-q^6).$$

We appeal to [1, p. 39, eq. (3.3.9)], (7.15) and (7.16) to find that

$$G(e^{-2\pi/3}) = -G(e^{-\pi/3})G(-e^{-\pi/3}) = 2^{-5/3}(\sqrt{3}-1)^{1/3}\left(\sqrt{2\sqrt{3}+3}-1\right),$$

which yields

$$\frac{\chi(-e^{-2\pi/3})\chi(-e^{-6\pi})}{\chi^2(-e^{-2\pi})} = e^{-\pi/9}2^{-1/6}(\sqrt{3}-1)^{1/3}(2+\sqrt{3})^{1/3}. \tag{8.12}$$

Notice that $\sum_{m\neq 0}\frac{\cos(2\pi m/3)}{m^2} = -\pi^2/9$. Substituting all these results into (8.11) and simplifying, we complete the proof of (8.8). Similarly, when $x=\sqrt{2}$, we first appeal to [4, p. 200] and (4.23) to find that $g_2=1$, $g_{18}=(\sqrt{2}+\sqrt{3})^{1/3}$ and $G(e^{-\sqrt{2}\pi/3}) = \frac{1}{\sqrt{2}}(-\sqrt{2}+\sqrt{3})^{1/3}$, and therefore

$$\begin{aligned}\frac{\chi(-e^{-\sqrt{2}\pi/3})\chi(-e^{-3\sqrt{2}\pi})}{\chi^2(-e^{-\sqrt{2}\pi})} &= e^{\sqrt{2}\pi/9}G(e^{-\sqrt{2}\pi/3})\chi(-e^{-\sqrt{2}\pi})\chi(-e^{-3\sqrt{2}\pi})\\ &= e^{-\sqrt{2}\pi/18}.\end{aligned}$$

Substituting (7.20) and the result above into (8.11), and simplifying, we complete the proof of (8.9).

We conclude this section by deriving (8.10) from (8.2). If we set $s = t = 2/3$, then (8.2) reduces to

$$\sum_{(m,n)\neq(0,0)} \frac{\cos\left(\frac{2\pi m}{3}\right)\cos\left(\frac{2\pi n}{3}\right)}{(xm)^2+(2n)^2}$$
$$= -\frac{\pi}{2x}\log\left(\frac{f^2(-q^2)}{f(-q^{2/3})f(-q^6)}\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)}\right) + \sum_{m\neq 0}\frac{\cos\left(\frac{2\pi m}{3}\right)}{(xm)^2}$$
$$= -\frac{\pi}{2x}\log\left(\frac{\varphi^2(-q^2)}{\varphi(-q^{2/3})\varphi(-q^6)}\left(\frac{\chi(-q^{2/3})\chi(-q^6)}{\chi^2(-q^2)}\right)^2\right) + \sum_{m\neq 0}\frac{\cos\left(\frac{2\pi m}{3}\right)}{(xm)^2}.$$

Set $x = 1$. Substituting (7.22) and (8.12) into the identity above, and simplifying, we obtain (8.10).

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