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A generalization of the classical circle problem

by

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1. Introduction and results. Let $r(n)$ denote the number of representations of n as a sum of squares of two integers. The classical circle problem is to study

$$P(x)=\sum_{0\le n\le x} r(n)-\pi x.$$

The best known upper bound is due to Huxley [6]:

$$P(x)\ll x^{131/416}(\log x)^{2.26}. \tag{1.1}$$

It is conjectured that for any $\varepsilon>0$,

$$P(x)\ll_\varepsilon x^{1/4+\varepsilon}.$$

It is well known that

$$\int_0^X P^2(x)\,dx = CX^{3/2}+Q(X), \tag{1.2}$$

where $C\approx 1.68396$. The best known bound for $Q(X)$ is due to Lau and Tsang [7] (see also Nowak [9] for a similar result):

$$Q(X)\ll X(\log X)(\log\log X).$$

This implies that

$$|P(N)|\ge 1.5N^{1/4}$$

for infinitely many positive integers N. Hence

$$\sum_{0\le n\le x} r(n)=\pi x+o(x^{1/4}) \text{ cannot hold.}$$

Let $k\ge 2$ be a fixed integer and let $A=\{a_1\le a_2\le\cdots\}$ be an infinite sequence of nonnegative integers. For $x\ge 0$ let $r_k(A,x)$ denote the number

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of solutions of

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \le x.$$

For a positive constant c, let

$$P_k(A, c, x) = r_k(A, x) - cx.$$

In particular, if $A = \{0, (-1)^2, 1^2, (-2)^2, 2^2, (-3)^2, 3^2, \ldots\}$, then

$$r_2(A, x) = \sum_{0 \le n \le x} r(n).$$

In 1956, Erdős and Fuchs [2] proved the following unusual result:

THEOREM A. *If A is an infinite sequence of nonnegative integers, then*

$$r_2(A, n) = cn + o(n^{1/4}(\log n)^{-1/2}) \text{ cannot hold}$$

for any constant $c > 0$.

Jurkat (unpublished), and later Montgomery and Vaughan [8] improved the Erdős–Fuchs theorem by eliminating the log power on the right-hand side:

THEOREM B. *If A is an infinite sequence of nonnegative integers, then*

$$r_2(A, n) = cn + o(n^{1/4}) \text{ cannot hold}$$

for any constant $c > 0$.

Up to now, the Erdős–Fuchs theorem has been extended in various directions. See [1], [3], [4], [5], [10] and [11].

Recently, the authors [1] proved that

$$|P_k(A, c, n)| = |r_k(A, n) - cn| \ge 0.04([k/2]!)^{3/2}(cn)^{1/4}$$

for infinitely many positive integers n.

Motivated by the Erdős–Fuchs theorem and (1.2), we consider the asymptotic properties of

$$\int_0^X P_2^2(A, c, x)\, dx.$$

Since A is a general sequence, the method for the classical circle problem cannot be applied here. Note that not even the assumption $r_2(A, x) = cx + o(x)$ guarantees

$$\int_0^X P_2^2(A, c, x)\, dx = O(X^{3/2}).$$

For example, let $A = \{0, 1^2, 2^2, 3^2, \ldots\}$; by (1.1) we find that

$$r_2(A,x) = 1 + \frac{1}{4}\sum_{1\le n\le x} r(n) + [\sqrt{x}] = \frac{1}{4}\pi x + \sqrt{x} + \frac{1}{4}P(x) + O(1)$$
$$= \frac{1}{4}\pi x + \sqrt{x} + O(x^{1/3}).$$

In this case, we have $c = \frac{1}{4}\pi$ and

$$P_2(A,c,x) = r_2(A,x) - cx = \sqrt{x} + O(x^{1/3}). \tag{1.3}$$

By (1.3) we get

$$\int_0^X P_2^2(A,c,x)\,dx = \int_0^X (\sqrt{x} + O(x^{1/3}))^2\,dx = \frac{1}{2}X^2 + O(X^{11/6}).$$

Now we consider the following problem:

Problem 1.1. *Is it true that for any infinite sequence A of nonnegative integers, any $c > 0$ and $\varepsilon > 0$ we have*

$$\int_0^X P_2^2(A,c,x)\,dx \gg_{A,c,\varepsilon} X^{3/2-\varepsilon}?$$

In this paper, we prove that under the natural assumption of $r_2(A,x) = cx + o(x)$ the answer to Problem 1.1 is affirmative.

Theorem 1.2. *Let A be an infinite sequence of nonnegative integers, $k \ge 2$ be a fixed integer and c be a positive constant. Then for any $\varepsilon > 0$ the estimate*

$$\int_0^M P_k^2(A,c,x)\,dx \ge (H(k,c)(\Gamma(5/2))^{-1} - \varepsilon)M^{3/2}$$

holds for infinitely many positive integers M, where $\tau_k = k - 2[k/2]$ and

$$H(k,c) = 2^{-9/2}e^2c^{1/2}\left(\frac{k-\tau_k}{k+2\tau_k}\right)^2\left(\frac{k+2\tau_k}{3(k+\tau_k)}\left[\frac{k}{2}\right]!\right)^{3k/(k-\tau_k)}.$$

Corollary 1.3. *Let A be an infinite sequence of nonnegative integers, $k \ge 2$ be a fixed integer, and c, β be positive constants with $\beta < 1$. Assume that $r_k(A,x) = cx + O(x^\beta)$. Then for any $\varepsilon > 0$,*

$$|\{0 \le n \le M : |P_k(A,c,n)| \ge ((\Gamma(5/2))^{-1/2}\sqrt{H(k,c)} - \varepsilon)M^{1/4}\}| \gg M^{3/2-2\beta}$$

for infinitely many positive integers M.

Theorem 1.4. *Let A be an infinite sequence of nonnegative integers, $k \ge 2$ be a fixed integer and c be a positive constant. Assume that $r_k(A,x) =$*

$cx + o(x)$. *Then for any* $\varepsilon > 0$,

$$\int_0^M P_k^2(A,c,x)\,dx \ge (5^{-3/2}H(k,c)-\varepsilon)\Big(\frac{M}{\log M}\Big)^{3/2}$$

for all sufficiently large numbers M, *where* $H(k,c)$ *is as in Theorem* 1.2.

We pose the following conjecture:

Conjecture 1.5. *For any infinite sequence* A *of nonnegative integers, any integer* $k \ge 2$ *and any positive constant* c,

$$\int_0^M P_k^2(A,c,x)\,dx \gg_{A,c,k} M^{3/2}$$

for all sufficiently large numbers M.

Remark. By (1.2), the case of the circle problem shows that if Conjecture 1.5 is true then it is sharp. In the above results and the conjecture we have the corresponding conclusions for

$$\sum_{n=0}^{M} P_k^2(A,c,n).$$

These can be derived from Lemma 2.2 of Section 2. In fact, the corresponding results are contained in the proofs.

2. Proofs. Throughout this paper, let $z = re(\alpha)$, where $e(\alpha) = e^{2\pi i\alpha}$, $r = 1 - 1/N$ and α is a real number. We write $F(z) = \sum_{a\in A} z^a$, $A(n) = \sum_{a\in A,\, a\le n} 1$ (counting repetitions).

Lemma 2.1. *Let* $\beta > 0$ *and* $r = 1 - 1/N$, *where* N *is a large positive integer. Then*

$$\sum_{n=0}^{\infty} n^\beta r^n = \Gamma(\beta+1)N^{\beta+1}(1 + o_N(1)).$$

The proof is similar to that of [1, Lemma 2.3].

Lemma 2.2. *For positive integers* M, *we have*

$$\int_0^{M+1} P_k^2(A,c,x)\,dx = (1+o(1))\sum_{n=0}^{M} P_k^2(A,c,n) + O(M\log M).$$

Proof. Let n be a nonnegative integer. Then for $n \le x < n+1$ we have

$$P_k(A,c,x) = r_k(A,x) - cx = r_k(A,n) - cn + O(1) = P_k(A,c,n) + O(1).$$

Thus

$$\int_n^{n+1} P_k^2(A,c,x)\,dx = P_k^2(A,c,n) + O(P_k(A,c,n)) + O(1).$$

Hence

$$\int_0^{M+1} P_k^2(A,c,x)\,dx$$

$$= \sum_{n=0}^{M} P_k^2(A,c,n) + O\Big(\sum_{n=0}^{M} |P_k(A,c,n)|\Big) + O(M)$$

$$= \sum_{n=0}^{M} P_k^2(A,c,n) + O\Big(\sqrt{M}\Big(\sum_{n=0}^{M} P_k^2(A,c,n)\Big)^{1/2}\Big) + O(M)$$

$$= \Big(\sum_{n=0}^{M} P_k^2(A,c,n)\Big)^{1/2}\Big(\Big(\sum_{n=0}^{M} P_k^2(A,c,n)\Big)^{1/2} + O(\sqrt{M})\Big) + O(M)$$

$$= (1+o(1))\sum_{n=0}^{M} P_k^2(A,c,n) + O(M\log M). \ \blacksquare$$

To prove Theorems 1.2 and 1.4, we first prove the following result.

THEOREM 2.3. *Let A be an infinite sequence of nonnegative integers, $k \ge 2$ be a fixed integer and c be a positive constant. Assume that $r_k(A,x) = cx + o(x)$ if k is odd. Then for any $\varepsilon > 0$,*

$$\sum_{n=0}^{\infty} P_k^2(A,c,n) r^n \ge (H(k,c)-\varepsilon) N^{3/2}$$

for all sufficiently large numbers N, where $H(k,c)$ is as in Theorem 1.2.

Proof. Since $1-1/N \ge 1-1/[N]$ and $[N]^{3/2} = N^{3/2}(1+o_N(1))$, it is enough to prove Theorem 2.3 for all sufficiently large integers N.

Suppose that there is an $\varepsilon_0 > 0$ such that

$$\sum_{n=0}^{\infty} P_k^2(A,c,n) r^n < (H(k,c)-\varepsilon_0) N^{3/2} \tag{2.1}$$

for infinitely many positive integers N. Then $\sum_{n=0}^{\infty} P_k^2(A,c,n) z^n$ is absolutely convergent for $|z| < 1$. Since so also is $\sum_{n=0}^{\infty} z^n$, the same is true of

$$\sum_{n=0}^{\infty} (1+P_k^2(A,c,n)) z^n.$$

As

$$|P_k(A,c,n)| \le 1 + P_k^2(A,c,n),$$

the series

$$\sum_{n=0}^{\infty} P_k(A,c,n) z^n$$

is absolutely convergent for $|z| < 1$. Since $0 \le r_k(A, n) \le cn + |P_k(A, c, n)|$, also the series $\sum_{n=0}^{\infty} r_k(A, n)z^n$ is absolutely convergent for $|z| < 1$. By

$$\frac{1}{1-r}\Big(\sum_{i=1}^{T} r^{a_i}\Big)^k \le \sum_{n=0}^{\infty} r_k(A, n)r^n,$$

we know that $\sum_{a\in A} z^a$ converges absolutely for $|z| < 1$. For $|z| < 1$, we have

$$\frac{1}{1-z}F^k(z) = \sum_{n=0}^{\infty} r_k(A, n)z^n = \frac{cz}{(1-z)^2} + \sum_{n=0}^{\infty} P_k(A, c, n)z^n.$$

That is,

$$F^k(z) = \frac{cz}{1-z} + (1-z)\sum_{n=0}^{\infty} P_k(A, c, n)z^n. \tag{2.2}$$

Using the idea of Jurkat, by differentiation of (2.2), we have

$$\begin{aligned} kF^{k-1}(z)F'(z) = {} & \frac{c}{(1-z)^2} - \sum_{n=0}^{\infty} P_k(A, c, n)z^n \\ & + (1-z)\sum_{n=1}^{\infty} nP_k(A, c, n)z^{n-1}. \end{aligned} \tag{2.3}$$

Let δ be a positive constant to be determined later, $m = [\delta c^{-1/2}N^{1/2}]$, and let

$$J = \int_0^1 |kF^{k-1}(z)F'(z)| \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha,$$

$$J_1 = c\int_0^1 \frac{1}{|1-z|^2} \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha,$$

$$J_2 = \int_0^1 \left|\sum_{n=0}^{\infty} P_k(A, c, n)z^n\right| \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha,$$

$$J_3 = \int_0^1 \left|(1-z)\sum_{n=1}^{\infty} nP_k(A, c, n)z^{n-1}\right| \cdot \left|\frac{1-z^m}{1-z}\right|^2 d\alpha.$$

By (2.3), we have

$$J \le J_1 + J_2 + J_3. \tag{2.4}$$

By Cauchy's inequality, (2.1) and (2.2) we have

$$F^k(r^2) = \frac{cr^2}{1-r^2} + (1-r^2)\sum_{n=0}^{\infty} P_k(A,c,n)r^{2n} \tag{2.5}$$
$$= \frac{c}{2}N(1+o_N(1))$$
$$+ O\Big(\frac{1}{N}\Big(\sum_{n=0}^{\infty} r^{2n}\Big)^{1/2}\Big(\sum_{n=0}^{\infty} P_k^2(A,c,n)r^{2n}\Big)^{1/2}\Big)$$
$$= \frac{c}{2}N(1+o_N(1)) + O(N^{1/4}) = \frac{c}{2}N(1+o_N(1)).$$

By (2.1), (2.3), Cauchy's inequality and Lemma 2.1 (noting that $r^{2n} \le r^n$), we have

$$kF^{k-1}(r^2)F'(r^2) \tag{2.6}$$
$$= \frac{c}{(1-r^2)^2} - \sum_{n=0}^{\infty} P_k(A,c,n)r^{2n} + (1-r^2)\sum_{n=1}^{\infty} nP_k(A,c,n)r^{2n-2}$$
$$= \frac{c}{4}N^2(1+o_N(1)) + O\Big(\Big(\sum_{n=0}^{\infty} r^{2n}\Big)^{1/2}\Big(\sum_{n=0}^{\infty} P_k^2(A,c,n)r^{2n}\Big)^{1/2}\Big)$$
$$+ O\Big(\frac{1}{N}\Big(\sum_{n=0}^{\infty} n^2 r^{2n}\Big)^{1/2}\Big(\sum_{n=0}^{\infty} P_k^2(A,c,n)r^{2n}\Big)^{1/2}\Big)$$
$$= \frac{c}{4}N^2(1+o_N(1)) + O(N^{5/4}) + O\Big(\frac{1}{N}N^{9/4}\Big) = \frac{c}{4}N^2(1+o_N(1)).$$

By (2.5) and (2.6) we have

$$F'(r^2) = \frac{1}{k}2^{-1-1/k}c^{1/k}N^{1+1/k}(1+o_N(1)). \tag{2.7}$$

If $k = 2l$ is even, similarly to the proof of [1, Theorem 1.1], by (2.5), (2.7), $0 < F(r^4) < F(r^2)$ and $0 < F'(r^4) < F'(r^2)$, we find that

$$J \ge [k/2]!2^{-3/2}\delta N^2(1+o_N(1)). \tag{2.8}$$

If $k = 2l+1$ is odd, then by $r_k(A,x) = cx + o(x)$ and

$$A^k(M) \ge \sum_{a_{i_1}+\cdots+a_{i_k}\le M} 1 = r_k(A,M),$$

we have

$$A(M) \ge \sqrt[k]{cM}(1+o_M(1)).$$

Thus, similarly to the proof of [1, Theorem 1.1], by (2.5), (2.7), $0 < F(r^4) < F(r^2)$ and $0 < F'(r^4) < F'(r^2)$, we see that

$$J \ge [k/2]!2^{-3/2+1/(2k)}\frac{k}{k+1}\delta^{1+1/k}N^2(1+o_N(1)). \tag{2.9}$$

By (2.8) and (2.9) we get

$$J \geq [k/2]! 2^{-3/2+\tau_k/(2k)} \frac{k}{k+\tau_k} \delta^{1+\tau_k/k} N^2 (1+o_N(1)), \tag{2.10}$$

where $\tau_k = k - 2[k/2]$. Similarly to the proof of [1, Theorem 1.1], we deduce

$$J_1 \leq \frac{1}{2} c m^2 N (1+o_N(1)) = \frac{1}{2} \delta^2 N^2 (1+o_N(1)). \tag{2.11}$$

By Cauchy's inequality, Parseval's formula and (2.1) we have

$$\begin{aligned} J_2 &\leq m^2 \int_0^1 \Big| \sum_{n=0}^{\infty} P_k(A,c,n) z^n \Big| \, d\alpha \\ &\leq m^2 \Big(\int_0^1 \Big| \sum_{n=0}^{\infty} P_k(A,c,n) z^n \Big|^2 \, d\alpha \Big)^{1/2} \\ &= m^2 \Big(\sum_{n=0}^{\infty} |P_k(A,c,n)|^2 r^{2n} \Big)^{1/2} \\ &= O(m^2 N^{3/4}) = O(N^{7/4}). \end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned} J_3 &= \int_0^1 \Big| \sum_{n=1}^{\infty} n P_k(A,c,n) z^{n-1} \Big| \cdot \Big| \frac{1-z^m}{1-z} (1-z^m) \Big| \, d\alpha \\ &\leq \Big(\int_0^1 \Big| \sum_{n=1}^{\infty} n P_k(A,c,n) z^{n-1} \Big|^2 \, d\alpha \Big)^{1/2} \cdot \Big(\int_0^1 \Big| \frac{1-z^m}{1-z} (1-z^m) \Big|^2 \, d\alpha \Big)^{1/2} \\ &= \Big(\sum_{n=1}^{\infty} n^2 P_k^2(A,c,n) r^{2n-2} \Big)^{1/2} \cdot \Big((1+r^{2m}) \sum_{j=0}^{m-1} r^{2j} \Big)^{1/2} \\ &\leq (2m)^{1/2} \Big(\sum_{n=1}^{\infty} n^2 P_k^2(A,c,n) r^{2n-2} \Big)^{1/2}. \end{aligned}$$

Let $f(x) = x^2 r^x$. Then

$$f(x) \leq f\Big(-\frac{2}{\log r} \Big) = \frac{4e^{-2}}{\log^2 r} < 4e^{-2} N^2.$$

Thus, by (2.1) we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^2 P_k^2(A,c,n) r^{2n-2} &\leq 4e^{-2} N^2 r^{-2} \sum_{n=1}^{\infty} P_k^2(A,c,n) r^n \\ &\leq 4e^{-2} N^2 r^{-2} (H(k,c) - \varepsilon_0) N^{3/2} \\ &= 4e^{-2} (H(k,c) - \varepsilon_0) N^{7/2} (1+o_N(1)). \end{aligned}$$

Hence

$$(2.13)\qquad \begin{aligned} J_3 &\le 2\sqrt{2}e^{-1}(H(k,c)-\varepsilon_0)^{1/2}m^{1/2}N^{7/4}(1+o_N(1))\\ &\le 2\sqrt{2}e^{-1}(H(k,c)-\varepsilon_0)^{1/2}c^{-1/4}\delta^{1/2}N^{2}(1+o_N(1)). \end{aligned}$$

By (2.4) and (2.10)–(2.13) we have

$$\begin{aligned} &[k/2]!2^{-3/2+\tau_k/(2k)}\frac{k}{k+\tau_k}\delta^{1+\tau_k/k}N^2\\ &\quad\le \frac{1}{2}\delta^2N^2+O(N^{7/4})+2\sqrt{2}e^{-1}(H(k,c)-\varepsilon_0)^{1/2}c^{-1/4}\delta^{1/2}N^2+o(N^2). \end{aligned}$$

Dividing by N^2 and letting $N\to\infty$, we find that

$$[k/2]!2^{-3/2+\tau_k/(2k)}\frac{k}{k+\tau_k}\delta^{1+\tau_k/k}\le \frac{1}{2}\delta^2+2\sqrt{2}\,e^{-1}(H(k,c)-\varepsilon_0)^{1/2}c^{-1/4}\delta^{1/2}.$$

So

$$(H(k,c)-\varepsilon_0)^{1/2}\ge \frac{\sqrt{2}}{4}ec^{1/4}\bigg([k/2]!2^{-3/2+\tau_k/(2k)}\frac{k}{k+\tau_k}\delta^{1/2+\tau_k/k}-\frac{1}{2}\delta^{3/2}\bigg).$$

Taking

$$\delta=3^{-k/(k-\tau_k)}\frac{1}{\sqrt{2}}\bigg(1+\frac{\tau_k}{k+\tau_k}\bigg)^{k/(k-\tau_k)}([k/2]!)^{k/(k-\tau_k)},$$

we get

$$\begin{aligned} (H(k,c)-\varepsilon_0)^{1/2}&\ge \frac{\sqrt{2}}{4}ec^{1/4}\bigg([k/2]!2^{-3/2+\tau_k/(2k)}\frac{k}{k+\tau_k}\delta^{1/2+\tau_k/k}-\frac{1}{2}\delta^{3/2}\bigg)\\ &=2^{-9/4}ec^{1/4}\frac{k-\tau_k}{k+2\tau_k}\bigg(\frac{k+2\tau_k}{3(k+\tau_k)}\bigg[\frac{k}{2}\bigg]!\bigg)^{3k/(2k-2\tau_k)}\\ &=(H(k,c))^{1/2}, \end{aligned}$$

a contradiction. This completes the proof of Theorem 2.3. ∎

Proof of Theorem 1.2. Suppose that there exists an $\varepsilon_0>0$ such that

$$(2.14)\qquad S_k(A,c,M)=\sum_{n=0}^{M}P_k^2(A,c,n)\le (H(k,c)-\varepsilon_0)(\Gamma(5/2))^{-1}M^{3/2}$$

for all sufficiently large integers M. Then $P_k^2(A,c,M)\ll M^{3/2}$. This means that $r_k(A,x)=cx+O(x^{3/4})$. Since

$$\begin{aligned} \sum_{n=0}^{\infty}P_k^2(A,c,n)r^n&=\sum_{n=0}^{\infty}(S_k(A,c,n)-S_k(A,c,n-1))r^n\\ &=\sum_{n=0}^{\infty}S_k(A,c,n)r^n-\sum_{n=0}^{\infty}S_k(A,c,n-1)r^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} S_k(A,c,n) r^n - \sum_{n=0}^{\infty} S_k(A,c,n) r^{n+1} = \frac{1}{N} \sum_{n=0}^{\infty} S_k(A,c,n) r^n,$$

by Theorem 2.3 we have

$$\sum_{n=0}^{\infty} S_k(A,c,n) r^n \geq \left(H(k,c) - \frac{1}{2}\varepsilon_0\right) N^{5/2}.$$

On the other hand, by (2.14) and Lemma 2.1 we get

$$\sum_{n=0}^{\infty} S_k(A,c,n) r^n \leq O(1) + (H(k,c) - \varepsilon_0)(\Gamma(5/2))^{-1} \sum_{n=0}^{\infty} n^{3/2} r^n$$
$$= (H(k,c) - \varepsilon_0) N^{5/2}(1 + o(1)),$$

a contradiction. Therefore

$$\sum_{n=0}^{M} P_k^2(A,c,n) \geq (H(k,c)(\Gamma(5/2))^{-1} - \varepsilon) M^{3/2}$$

for infinitely many positive integers M. The proof of Theorem 1.2 is completed by an appeal to Lemma 2.2. ∎

Proof of Corollary 1.3. Let $\delta = (\Gamma(5/2))^{-1/2}\sqrt{H(k,c)} - \varepsilon > \varepsilon$. Since $r_k(A,x) = cx + O(x^\beta)$, there exists a constant $C > 0$ such that

$$|P_k(A,c,n)| = |r_k(A,n) - cn| < Cn^\beta$$

for all $n \geq 1$. By Theorem 1.2 and Lemma 2.2 we have

$$\sum_{n=0}^{M} P_k^2(A,c,n) \geq \left(H(k,c)(\Gamma(5/2))^{-1} - \frac{1}{2}\varepsilon^2\right) M^{3/2}$$

for infinitely many positive integers M. Since

$$\sum_{n=0}^{M} P_k^2(A,c,n)$$
$$= \sum_{\substack{0 \leq n \leq M \\ |P_k(A,c,n)| < \delta M^{1/4}}} P_k^2(A,c,n) + \sum_{\substack{0 \leq n \leq M \\ |P_k(A,c,n)| \geq \delta M^{1/4}}} P_k^2(A,c,n)$$
$$\leq \delta^2 M^{1/2} \sum_{\substack{0 \leq n \leq M \\ |P_k(A,c,n)| < \delta M^{1/4}}} 1 + C^2 M^{2\beta} \sum_{\substack{0 \leq n \leq M \\ |P_k(A,c,n)| \geq \delta M^{1/4}}} 1$$
$$\leq \delta^2 M^{3/2} + \delta^2 M^{1/2} + C^2 M^{2\beta} |\{0 \leq n \leq M : |P_k(A,c,n)| \geq \delta M^{1/4}\}|,$$

we have

$$\begin{aligned}&|\{0 \le n \le M : |P_k(A,c,n)| \ge \delta M^{1/4}\}| \\ &\qquad \ge (H(k,c)(\Gamma(5/2))^{-1} - \frac{1}{2}\varepsilon^2 - \delta^2 - \delta^2 M^{-1})C^{-2}M^{3/2-2\beta} \\ &\qquad \ge (\Gamma(5/2))^{-1/2}\sqrt{H(k,c)}\,C^{-2}\varepsilon M^{3/2-2\beta}\end{aligned}$$

for infinitely many positive integers M. ∎

Proof of Theorem 1.4. By Theorem 2.3,

$$\sum_{n=0}^{\infty} P_k^2(A,c,n)r^n \ge \left(H(k,c) - \frac{1}{2}\varepsilon\right)N^{3/2}$$

for all sufficiently large N. Since $r_k(A,x) = cx + o(x)$, we have

$$P_k^2(A,c,n) = o(n^2).$$

Thus

$$\sum_{n>5N\log N} P_k^2(A,c,n)r^n = o\Big(\sum_{n>5N\log N} n^2 r^n\Big).$$

Let $f(x) = x^2 r^{x/2}$. Then

$$\begin{aligned}f'(x) &= 2xr^{x/2} + \frac{1}{2}x^2 r^{x/2}\log r = \frac{1}{2}xr^{x/2}(4 + x\log r) \\ &< \frac{1}{2}xr^{x/2}\left(4 - \frac{x}{N}\right) < 0\end{aligned}$$

for $x \ge 5N\log N$. Hence

$$\log f(n) \le \log f(5N\log N) = 2\log(5N\log N) + \frac{5}{2}N\log N\log r < 0$$

for $n \ge 5N\log N$ and sufficiently large N. Thus $f(n) < 1$ for $n \ge 5N\log N$ and sufficiently large N. Hence

$$\begin{aligned}\sum_{n>5N\log N} P_k^2(A,c,n)r^n &= o\Big(\sum_{n>5N\log N} n^2 r^n\Big) \\ &= o\Big(\sum_{n>5N\log N} r^{n/2}\Big) = o\left(\frac{r^{2.5N\log N}}{1-\sqrt{r}}\right) \\ &= o(e^{2.5N\log N\log r}N(1+\sqrt{r})) = o(1).\end{aligned}$$

Thus

$$\sum_{n\le 5N\log N} P_k^2(A,c,n) \ge \sum_{n\le 5N\log N} P_k^2(A,c,n)r^n \ge (H(k,c) - \varepsilon)N^{3/2}$$

for all sufficiently large N. Let M be any sufficiently large number. Let N be a positive number with $M = 5N\log N$. Then

$$N = \frac{M}{5\log M}(1 + o(1))$$

and

$$\sum_{n\le M} P_k^2(A,c,n) \ge (5^{-3/2}H(k,c)-\varepsilon)\left(\frac{M}{\log M}\right)^{3/2}.$$

By Lemma 2.2 the proof of Theorem 1.4 is complete. ∎

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