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Quantitative analysis of the Satake parameters of GL_2 representations with prescribed local representations

by

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1. Introduction. Let f be a non-dihedral primitive cusp form of weight k and level N with Fourier expansion $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$, normalized so that $a_1 = 1$. The *Sato–Tate conjecture for modular forms* asserts that the normalized Hecke eigenvalues $p^{-(k-1)/2}a_p$ for prime $p \nmid N$ are equidistributed relative to the measure

$$d\mu_\infty(x) = \begin{cases} \pi^{-1}\sqrt{1-x^2/4}\,dx & \text{for } x \in [-2,2], \\ 0 & \text{otherwise.} \end{cases}$$

See also [10, Section 21.2] and [15, Chapter 4, Section 7]. The Sato–Tate conjecture is a consequence of the analytic continuation of symmetric power L-functions to $\operatorname{Re} s \geq 1$ (Serre [20], Murty [17]), which in turns follows from Langlands' functoriality conjecture. Although Langlands' conjecture is still unsettled, the Sato–Tate conjecture is now a theorem by the breakthrough of Barnet-Lamb, Geraghty, Harris and Taylor (see [1]).

Another viewpoint is the vertical version of the problem: fix a prime p and determine the distribution of the eigenvalues of the Hecke operator T_p on a parametric family of cusp forms as the parameter goes to infinity. Different cases were investigated by several authors: for Maass forms by Bruggeman [4] and Sarnak [19]; for holomorphic forms by Serre [21] and Conrey–Duke–Farmer [5]; for Hilbert modular forms by Li [14].

A quantitative version of the distribution of the eigenvalues of T_p on a family of modular forms was given by Murty and Sinha [16]. Lau and Wang [13] gave a quantitative version for Maass forms with level 1. In this paper, we extend their results to Hilbert modular forms and further to some GL_2 automorphic representations whose local components at a finite set of finite places are specified. The latter perspective is new.

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All the previous results except [14] are based on the classical trace formulas for modular forms or Maass forms. Our proof is based on Arthur's trace formula on $\mathrm{GL}_2(F)$ ([8], [9]), where F is a totally real algebraic number field with degree $r \geq 2$ over $\mathbb{Q}$. For a special class of test functions, we derive a simple trace formula (3.3) on $\mathrm{GL}_2(F)$. The formula gives us an estimation which suffices for our purpose. Although the formula can be made more explicit, we do not pursue this direction in this paper. Furthermore, the representation theory of GL_2 allows us to refine the trace formula. Namely we can have a trace formula on Hilbert modular forms not only with given weight and level but also with prescribed local representations at some finite places. This is otherwise difficult to obtain by classical methods. Our formula (3.3) is also a generalization of [14, Theorem 3.21]. Other salient points include Lemma 4.3 and a variant of the Erdős–Turán inequality (Proposition 7.1) which are our keys to the quantitative result in both level and weight aspects.

Let $\sigma_1, \ldots, \sigma_r$ be the embeddings of F into $\mathbb{R}$ and $\infty_1, \ldots, \infty_r$ be the corresponding valuations. Let $\mathcal{O}$ be the ring of integers of F. For an integral ideal $\mathfrak{a}$, denote by $\mathrm{N}(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$ the ideal norm of $\mathfrak{a}$. For $\alpha \in \mathcal{O}$, denote by $\mathrm{N}(\alpha) = \mathrm{N}((\alpha))$ the absolute norm of α. For an even integer $\mathrm{k} \geq 2$, we denote by π_{k} the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight k with trivial central character.

Let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}$ and $\underline{\mathrm{k}} = (\mathrm{k}_1, \ldots, \mathrm{k}_r)$ be an r-tuple of even integers with $\mathrm{k}_i \geq 4$. Let $\Pi_{\underline{\mathrm{k}}}(\mathfrak{N})$ be the set of cuspidal automorphic representations π in L_0^2 for which

1. $\pi_{\mathrm{fin}} = \hat{\bigotimes}_{v<\infty} \pi_v$ contains a non-zero $K_0(\mathfrak{N})$-fixed vector,
2. $\pi_{\infty_i} = \pi_{\mathrm{k}_i}$ for $i = 1, \ldots, r$.

(See Section 2 for the notation.) The set $\Pi_{\underline{\mathrm{k}}}(\mathfrak{N})$ is finite [3]. At each finite unramified place v of π, $\pi_v = \mathrm{Ind}_{B(F_v)}^{G(F_v)} \chi$. Here B is the set of upper triangular matrices of G,

$$\chi\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\right) = \left|\frac{a}{d}\right|_v^{1/2} \chi_1(a)\chi_2(d)$$

and χ_1, χ_2 are unramified characters of F_v. Let ϖ_v be a uniformizer of F_v and write $\alpha_{iv} = \chi_i(\varpi)$ for $i = 1, 2$. The values α_{1v}, α_{2v} are called the *Satake parameters* of π_v. We define $\lambda_v(\pi) = \alpha_{1v} + \alpha_{2v}$. For $\pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N})$, the Ramanujan conjecture was settled (see [2], [18]) and hence $|\lambda_v(\pi)| \leq 2$. In this paper we give a quantitative analysis of $\lambda_v(\pi)$ where some local representations of π are prescribed.

Let $\mathcal{S} = \{w_1, \ldots, w_\iota\}$ be a set consisting of non-archimedean valuations and for all i let $\mathfrak{q}_i$ be the prime ideal corresponding to w_i. The set $\mathcal{S}$ can be taken to be empty. Let ρ_{w_i} be a supercuspidal representation of

$Z(F_{w_i})\backslash \mathrm{GL}_2(F_{w_i})$, where Z is the center of GL_2. Let $\mathfrak{q}_i^{c_i}$ be the conductor of ρ_{w_i}. Write $\underline{\rho} = (\rho_{w_1}, \ldots, \rho_{w_\iota})$ and

$$\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho}) = \{\pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N}) : \pi_{w_i} \cong \rho_{w_i} \text{ for } i = 1, \ldots, \iota\}.$$

Note that $\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho})$ is non-empty only if $\mathfrak{M} \mid \mathfrak{N}$ where $\mathfrak{M} = \prod_{i=1}^{\iota} \mathfrak{q}_i^{c_i}$. We associate to the family $\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho})$ two quantities:

$$C_{\underline{\mathrm{k}}} = \prod_{i=1}^{r} \frac{\mathrm{k}_i - 1}{4\pi} \quad \text{and} \quad d_{\underline{\rho}} = \prod_{i=1}^{\iota} d_{\rho_{w_i}}, \tag{1.1}$$

where $d_{\rho_{w_i}}$ is the formal degree of ρ_{w_i} (see Section 6).

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_T$ be prime ideals $\notin \mathcal{S}$ whose corresponding valuations are $v_1, \ldots, v_T$ respectively. We will show that the set

$$\{(\lambda_{v_1}(\pi), \ldots, \lambda_{v_T}(\pi)) : \pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})\} \quad \text{with} \quad \Big(\mathfrak{N}, \prod_{t=1}^{T} \mathfrak{p}_t \prod_{i=1}^{\iota} \mathfrak{q}_i\Big) = (1)$$

is equidistributed with respect to the measure $\prod_{t=1}^{T} d\mu_{v_t}$ as $\mathrm{N}(\mathfrak{N}) + C_{\underline{\mathrm{k}}} \to \infty$. Here, for a non-archimedean valuation v with corresponding prime ideal $\mathfrak{p}$, the measure is defined as

$$d\mu_v(x) = \frac{\mathrm{N}(\mathfrak{p}) + 1}{(\mathrm{N}(\mathfrak{p})^{1/2} + \mathrm{N}(\mathfrak{p})^{-1/2})^2 - x^2}\, d\mu_\infty(x). \tag{1.2}$$

Furthermore the rate of convergence will be estimated. Define the counting function

$$N_I(\mathfrak{p}_1, \ldots, \mathfrak{p}_T; \mathfrak{N}, \underline{\rho}) = \#\{\pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho}) : (\lambda_{v_1}(\pi), \ldots, \lambda_{v_T}(\pi)) \in I\}$$

for any $I = \prod_{t=1}^{T} [\alpha_t, \beta_t] \subseteq [-2, 2]^T$.

Our main result is the following.

THEOREM 1.1. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_T$ be distinct prime ideals with $(\mathfrak{p}_1 \cdots \mathfrak{p}_T, \mathfrak{M}) = (1)$. Let $\mathfrak{N}$ be an ideal of $\mathcal{O}$ with $(\mathfrak{N}, \mathfrak{p}_1 \cdots \mathfrak{p}_T \mathfrak{M}) = (1)$. Then*

$$\frac{N_I(\mathfrak{p}_1, \ldots, \mathfrak{p}_T; \mathfrak{N}\mathfrak{M}, \underline{\rho})}{\#\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})} = \int_I \prod_{t=1}^{T} d\mu_{v_t} + O\bigg(\frac{T \log \mathrm{N}(\mathfrak{p}_1 \cdots \mathfrak{p}_T)}{\log(C_{\underline{\mathrm{k}}}\, \mathrm{N}(\mathfrak{N}))}\bigg) \tag{1.3}$$

where $C_{\underline{\mathrm{k}}}$ and $d\mu_v$, for $v = v_1, \ldots, v_T$, are defined in (1.1) and (1.2) respectively. The implied O-constant depends only on F and ρ_{w_i}, $i = 1, \ldots, \iota$.

REMARK 1.2. (i) We see (from (8.3) below) that as $\mathrm{N}(\mathfrak{N}) + C_{\underline{\mathrm{k}}} \to \infty$,

$$\#\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M}, \underline{\rho}) = (1 + o(1))\, \mathrm{meas}(\overline{G}(F)\backslash \overline{G}(\mathbb{A})) C_{\underline{\mathrm{k}}} d_{\underline{\rho}}\, \mathrm{N}(\mathfrak{N}) \prod_{\mathfrak{p}_v^2 \mid \mathfrak{N}} (1 - q_v^{-2}).$$

(ii) Since the left side of (1.3) is at most 1, it suffices to consider the case

$$T \log \mathrm{N}(\mathfrak{p}_1 \cdots \mathfrak{p}_T) \le \delta \log(C_{\underline{\mathrm{k}}}\, \mathrm{N}(\mathfrak{N}))$$

for some small absolute constant $\delta > 0$.

We immediately deduce some generalizations of [14, Theorem 1.1].

Corollary 1.3. *For $j = 1, 2, \dots$, let $\underline{\mathrm{k}}^{(j)}$ be an r-tuple whose entries are even numbers ≥ 4 and let $\mathfrak{N}^{(j)}$ be an integral ideal relatively prime to $\mathfrak{p}_1 \cdots \mathfrak{p}_T \mathfrak{q}_1 \cdots \mathfrak{q}_\iota$. Suppose $\mathrm{N}(\mathfrak{N}^{(j)}) + C_{\underline{\mathrm{k}}^{(j)}} \to \infty$ as $j \to \infty$. Then*

$$\{(\lambda_{v_1}(\pi), \dots, \lambda_{v_T}(\pi)) : \pi \in \Pi_{\underline{\mathrm{k}}^{(j)}}(\mathfrak{N}^{(j)}\mathfrak{M}, \underline{\rho})\}$$

is equidistributed with respect to the measure $\prod_{t=1}^{T} \mu_{v_t}$.

Taking $\mathcal{S} = \emptyset$, we have

Corollary 1.4. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_T$ be distinct prime ideals. Suppose $\mathfrak{N}$ is an integral ideal with $(\mathfrak{N}, \mathfrak{p}_1 \cdots \mathfrak{p}_T) = (1)$. Then*

$$\frac{\#\{\pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N}) : (\lambda_{v_1}(\pi), \dots, \lambda_{v_T}(\pi)) \in I\}}{\#\Pi_{\underline{\mathrm{k}}}(\mathfrak{N})} = \int_I \prod_{t=1}^{T} d\mu_{v_t} + O\biggl(\frac{T \log \mathrm{N}(\mathfrak{p}_1 \cdots \mathfrak{p}_T)}{\log(C_{\underline{\mathrm{k}}}\,\mathrm{N}(\mathfrak{N}))}\biggr),$$

where the implied O-constant depends only on F.

2. Notation. Prime ideals are usually denoted by $\mathfrak{p}$ or $\mathfrak{q}$. The prime ideal corresponding to a non-archimedean valuation v is denoted by $\mathfrak{p}_v$. For every non-archimedean valuation v, write $q_v = \mathrm{N}(\mathfrak{p}_v)$. If $\mathfrak{a}$ is a fractional ideal of F, we use $[\mathfrak{a}]$ to represent the corresponding ideal class in the ideal class group of F. Write $h(F)$ for the class number of F. Let $\mathbb{A} = \mathbb{A}_F$ be the set of adeles, and $\hat{\mathcal{O}} = \prod_{v<\infty} \mathcal{O}_v$. Let $\mathbb{A}_{\mathrm{fin}}$ be the set of finite adeles. For $\alpha = (\alpha_v)_{v<\infty} \in \mathbb{A}_{\mathrm{fin}}$, the norm $\mathrm{N}(\alpha)$ is defined by $\prod_{v<\infty} \mathrm{N}(\mathfrak{p}_v)^{\mathrm{ord}_v \alpha_v}$.

Let $G = \mathrm{GL}_2$ and Z be its center. The identity element of G is denoted by e. We write $\overline{G} = Z\backslash G$ so that $Z(\mathbb{A})\backslash G(\mathbb{A})$ is abbreviated as $\overline{G}(\mathbb{A})$, etc. For any $S \subseteq G$, we write $\overline{S}$ for the image of S in $\overline{G}$. Let $K_{\infty_i} = \mathrm{SO}_2(\mathbb{R})$, which is a compact subgroup in $G(F_{\infty_i})$, and $K_\infty = \prod_{i=1}^{r} K_{\infty_i}$. For $v < \infty$, we take $K_v = \mathrm{GL}_2(\mathcal{O}_v)$, the standard maximal compact subgroup of $G(F_v)$.

For any valuation v, let dg_v be a Haar measure on $G(F_v)$. For a non-archimedean valuation v, the Haar measure on $G(F_v)$ is normalized by $\operatorname{meas} K_v = 1$.

Next, $g \in G(\mathbb{R})$ can be expressed as

$$\begin{pmatrix} 1 & \\ & \pm 1 \end{pmatrix} z \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad y, z > 0,\ \theta \in [0, 2\pi),\ x \in \mathbb{R}.$$

The measure on $G(\mathbb{R})$ is given by

$$dg = \frac{dz}{z}\,\frac{dxdy}{y^2}\,\frac{d\theta}{2\pi}.$$

For an archimedean valuation v, the Haar measure on $G(F_v)$ is defined by identifying $G(F_v)$ with $G(\mathbb{R})$.

Let Y' be a positive number and C' be a compact subset of $\mathbb{R}$. Put

$$\mathcal{D} = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} : x \in C',\, y > Y' \right\}$$

and $\mathfrak{S}'_\infty = \prod_{i=1}^r \mathcal{D} K_{\infty_i}$. As in [14, Section 3], we have (a variant of) the Siegel domain

$$\mathfrak{S}' = \mathfrak{S}'_\infty \times K'_{\mathrm{fin}} = \prod_{i=1}^{r} \mathcal{D} K_{\infty_i} \times \prod_{v<\infty} K'_v, \tag{2.1}$$

where K'_v is an open compact set and is equal to K_v for almost all v. When C' and K'_{fin} are sufficiently large and Y' is sufficiently small, we have

$$\overline{G}(\mathbb{A}) = \overline{G}(F)\mathfrak{S}'. \tag{2.2}$$

There exists a positive integer P such that K'_{fin} and $K'^{-1}_{\mathrm{fin}} \subseteq P^{-1} M_2(\widehat{\mathcal{O}})$. Let $Q = P^2$. The choices of Y', C', Q and K'_{fin} only depend on F. Throughout the paper, $Y', C', Q, K'_{\mathrm{fin}}$ are fixed as above and all the implied constants in $\ll$- or O-notation may depend on Y', C', Q, K'_{fin}, F unless otherwise stated.

Let $L^2(\overline{G}(F)\backslash\overline{G}(\mathbb{A}))$ be the space of square integrable functions on $\overline{G}(F)\backslash\overline{G}(\mathbb{A})$, and let L^2_0 be the subspace of cuspidal functions. The restriction of the right regular representation R to L^2_0 decomposes into a discrete sum of irreducible cuspidal representations, each of which can be factorized as a restricted tensor product $\hat{\bigotimes}_v \pi_v$.

For a ring R, $M_2(R)$ denotes the set of two by two matrices with entries in R. Let $\mathfrak{n}, \mathfrak{N}$ be two integral ideals of $\mathcal{O}$. Define the set

$$M(\mathfrak{n}_v, \mathfrak{N}_v) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_v) : c \in \mathfrak{N}_v,\ (\det g)\mathcal{O}_v = \mathfrak{n}_v \right\}.$$

Write $M(\mathfrak{n}, \mathfrak{N}) = \prod_{v<\infty} M(\mathfrak{n}_v, \mathfrak{N}_v)$. Denote by $\chi^{\mathfrak{n}_v}_{\mathfrak{N}_v}$ the characteristic function of $\overline{M(\mathfrak{n}_v, \mathfrak{N}_v)}$.

Let

$$K_0(\mathfrak{N}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_v : c \in \mathfrak{N}_v \right\}, \qquad K_0(\mathfrak{N}) = \prod_{v<\infty} K_0(\mathfrak{N}_v),$$

$$K(\mathfrak{N}_v) = \left\{ g \in K_v : g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{N}_v} \right\}, \qquad K(\mathfrak{N}) = \prod_{v<\infty} K(\mathfrak{N}_v).$$

Let further

$$\begin{aligned}(2.3)\qquad \psi(\mathfrak{N}_v) &= (\operatorname{meas} \overline{K_0(\mathfrak{N}_v)})^{-1} = [K_v : K_0(\mathfrak{N}_v)] \\ &= \begin{cases} q_v^{\operatorname{ord}_v \mathfrak{N}-1}(q_v+1) & \text{if } \operatorname{ord}_v \mathfrak{N} > 0, \\ 1 & \text{if } \operatorname{ord}_v \mathfrak{N} = 0. \end{cases}\end{aligned}$$

Globally define $\psi(\mathfrak{N}) = \prod_{v<\infty} \psi(\mathfrak{N}_v) = (\operatorname{meas} K_0(\mathfrak{N}))^{-1}$. Plainly one has

$$(2.4)\qquad \mathrm{N}(\mathfrak{N}) \le \psi(\mathfrak{N}) \le d(\mathfrak{N})\,\mathrm{N}(\mathfrak{N}) \ll_\varepsilon \mathrm{N}(\mathfrak{N})^{1+\varepsilon}.$$

Here for an integral ideal $\mathfrak{a}$ of $\mathcal{O}$, the divisor function $d(\mathfrak{a})$ is defined as

$$d(\mathfrak{a}) = \prod_{\mathfrak{p}_v|\mathfrak{a}} (\operatorname{ord}_v \mathfrak{a} + 1).$$

Here and below, ε denotes an arbitrarily small positive constant whose value may differ at each occurrence. We extend the divisor function to the set of fractional ideals by setting $d(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not an integral ideal.

Write $\mathcal{V}(\mathfrak{N})$ for the set of valuations dividing $\mathfrak{N}$. For $\emptyset \subseteq S \subseteq \mathcal{V}(\mathfrak{N})$ define

$$(2.5)\qquad \mathfrak{N}_S = \mathfrak{N}/\prod_{v\in S} \mathfrak{p}_v,$$

$$(2.6)\qquad \widetilde{\psi}(\mathfrak{N}) = \sum_{S\subseteq\mathcal{V}(\mathfrak{N})} (-1)^{|S|}\psi(\mathfrak{N}_S),$$

where $S = \emptyset$ is included in the summation. Then $\mathrm{N}(\mathfrak{N}) \ll \widetilde{\psi}(\mathfrak{N}) \le \mathrm{N}(\mathfrak{N})$ as

$$(2.7)\qquad \widetilde{\psi}(\mathfrak{N}) = \mathrm{N}(\mathfrak{N}) \prod_{\mathfrak{p}_v^2|\mathfrak{N}} (1 - q_v^{-2}).$$

3. The trace formula for a class of test functions. Let k be a positive integer. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{R})$. Define

$$(3.1)\qquad f_k(g) = \begin{cases} \dfrac{k-1}{4\pi}\,\dfrac{(\det g)^{k/2}(2i)^k}{(-b+c+(a+d)i)^k} & \text{if } \det g > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\underline{\mathrm{k}} = (\mathrm{k}_1, \dots, \mathrm{k}_r)$, define

$$f_{\underline{\mathrm{k}}} = \prod_{i=1}^r f_{\mathrm{k}_i}.$$

Let f be a function on $\overline{G}(\mathbb{A})$ which factorizes into $f = \prod_v f_v$, where f_v is a function of $\overline{G}(F_v)$. Let $f_\infty = \prod_{i=1}^r f_{\infty_i} = f_{\underline{\mathrm{k}}}$. Throughout the paper $f_{\mathrm{fin}} = \prod_{v<\infty} f_v$ is compactly supported modulo the center, locally constant and for almost all v, f_v is the characteristic function of $\overline{K}_v$. The right regular action of f on L^2 is given by

$$R(f)\phi(x) = \int\limits_{\overline{G}(\mathbb{A})} f(g)\phi(xg)\,dg.$$

Because f_{fin} is compactly supported, there exist $g_1, \dots, g_L \in M_2(\widehat{\mathcal{O}})$ such that

$$\operatorname{Supp} f_{\mathrm{fin}} \subseteq \bigcup_{i=1}^{L} Z(\mathbb{A}_{\mathrm{fin}}) g_i K_{\mathrm{fin}}.$$

Taking $\mathfrak{n}_i = (\det g_i)$ and $\mathfrak{N} = \mathcal{O}$, we have

$$\operatorname{Supp} f_{\mathrm{fin}} \subseteq \bigcup_{\ell=1}^{L} Z(\mathbb{A}_{\mathrm{fin}}) M(\mathfrak{n}_\ell, \mathfrak{N}). \tag{3.2}$$

Note that $Z(\mathbb{A}_{\mathrm{fin}})M(\mathfrak{n}_i, \mathfrak{N}) = Z(\mathbb{A}_{\mathrm{fin}})M(\mathfrak{n}_j, \mathfrak{N})$ if $\mathfrak{n}_i/\mathfrak{n}_j$ is the square of an ideal, and they are disjoint otherwise. We can therefore assume that $\mathfrak{n}_i/\mathfrak{n}_j$ is not the square of an ideal for $i \neq j$ and (3.2) is thus a disjoint union.

Unless otherwise stated, throughout the rest of the paper we assume f_{fin} satisfies (3.2) with integral ideals $\mathfrak{N}$, $\mathfrak{n}_1, \dots, \mathfrak{n}_L$ such that $\mathfrak{n}_i/\mathfrak{n}_j$ is not the square of an ideal for $i \neq j$.

THEOREM 3.1. *Let $f = f_{\underline{k}} f_{\mathrm{fin}}$ be given as above. Then*

$$\operatorname{tr} R(f) = \operatorname{meas}(\overline{G}(F)\backslash\overline{G}(\mathbb{A}))f(e) + \int\limits_{\overline{G}(F)\backslash\overline{G}(\mathbb{A})} \sum_{\gamma \text{ elliptic in } \overline{G}(F)} f(g^{-1}\gamma g)\,dg \tag{3.3}$$

and the right hand side is absolutely convergent. Here γ is said to be elliptic *if it is not conjugate to an upper triangular matrix over F.*

Proof. The proof of the trace formula for $L = 1$ is given in [14, Section 3]. The proof there can be easily generalized to $L \geq 2$ by the linearity of the trace formula. ∎

See also [6] and [7] for discussions of trace formulas for non-compactly supported functions.

4. Preliminary treatment of the elliptic term. Let $\mathfrak{B}$ be a set of representatives of the class group of F. In particular, the representatives can be chosen such that the norms are $\leq (r!/r^r)d_F^{1/2}$, where d_F is the discriminant of F (see [12, Theorem V.4.4]). For $\ell = 1, \dots, L$, let $\{\mathfrak{b}_{\ell 1}, \mathfrak{b}_{\ell 2}, \dots, \mathfrak{b}_{\ell t_\ell}\} \subseteq \mathfrak{B}$ be the solutions of $[\mathfrak{b}]^2[\mathfrak{n}_\ell] = [(1)]$. For all ℓ and t, there exists $\eta_{\ell t} \in \mathcal{O}$ such that

$$\mathfrak{b}_{\ell t}^2 \mathfrak{n}_\ell = (\eta_{\ell t}). \tag{4.1}$$

We can assume $\eta_{\ell t}$ satisfies $\sigma_1(\eta_{\ell t}) \gg \mathrm{N}(\eta_{\ell t})$ and $|\sigma_i(\eta_{\ell t})| \geq 1$ for $i = 2, \ldots, r$ by the following proposition.

PROPOSITION 4.1. *For any $\eta \in \mathcal{O}$, there exists a unit $u \in \mathcal{O}^*$ such that $\sigma_1(\eta u) \gg \mathrm{N}(\eta)$ and $\sigma_i(\eta u) \geq 1$ for $i \geq 2$.*

Proof. For $x \in F^*$, write $\lambda(x) = (\log|\sigma_1(x)|, \ldots, \log|\sigma_r(x)|)$. It is known that $\Lambda = \{(\lambda(x) : x \in \mathcal{O}^*\}$ is a lattice in $L_0 = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : x_1 + \cdots + x_r = 0\}$ with maximum rank. Let $\mathcal{P}$ be a fundamental parallelogram of L/Λ with one of the vertices at the origin and, for $i \geq 2$, the x_i-coordinate of all the vertices non-negative and bounded by a constant A. For $\eta \in \mathcal{O}$, $\lambda(\eta) - (\log \mathrm{N}(\eta), 0, \ldots, 0) \in L$. There exists $u \in \mathcal{O}^*$ such that $\lambda(\eta) - (\log \mathrm{N}(\eta), 0, \ldots, 0) + \lambda(u) \in \mathcal{P}$. Then for $i \geq 2$,

$$\log|\sigma_i(\eta u)| \geq 0$$

and

$$\log|\sigma_1(\eta u)| = \log \mathrm{N}(\eta) - \sum_{i=1}^{2} \log|\sigma_i(\eta u)| \geq \log \mathrm{N}(\eta) - (n-1)A. \quad \blacksquare$$

We fix a set of representatives of $\mathcal{O}^*/\{u^2 : u \in \mathcal{O}^*\}$, namely

$$\{u_1, \ldots, u_{2^r}\}. \tag{4.2}$$

For $\gamma \in G(F)$, denote by $[\gamma]$ the conjugacy class containing γ. Let G_γ be the centralizer of $\gamma \in G(F)$. Suppose $\mathfrak{o}$ is a conjugacy class in $G(F)$. Then $\det\gamma$ has the same value for any $\gamma \in \mathfrak{o}$. This value is denoted by $\det \mathfrak{o}$.

Write $\mathcal{E}(f)$ for the elliptic part of the trace formula, i.e., the integral on the right side of (3.3). By [14, Proposition 2.4] and [11, Section 26], the elliptic part can be written as

$$\begin{aligned}
\mathcal{E}(f) &= \sum_{\ell,t,j,\eta=\eta_{\ell t}u_j} \frac{1}{2} \int\limits_{\overline{G}(F)\backslash\overline{G}(\mathbb{A})} \sum_{\substack{\gamma \text{ elliptic in } G(F)\\ \det\gamma=\eta}} f(g^{-1}\gamma g)\, dg \\
&= \sum_{\ell,t,j,\eta=\eta_{\ell t}u_j} \frac{1}{2} \sum_{\substack{\text{elliptic conjugacy classes } \mathfrak{o}\\ \text{of } G(F) \text{ with } \det\mathfrak{o}=\eta}} \int\limits_{\overline{G}(F)\backslash\overline{G}(\mathbb{A})} \sum_{\gamma\in\mathfrak{o}} f(g^{-1}\gamma g)\, dg \\
&= \sum_{\ell,t,j,\eta=\eta_{\ell t}u_j} \frac{1}{2} \sum_{\substack{\text{elliptic conjugacy classes } [\gamma]\\ \text{of } G(F) \text{ with } \det\gamma=\eta}} \\
&\qquad\qquad \times \int\limits_{\overline{G}(F)\backslash\overline{G}(\mathbb{A})} \sum_{\delta\in G_\gamma(F)\backslash G(F)} f(g^{-1}\delta^{-1}\gamma\delta g)\, dg \\
&= \sum_{\ell,t,j,\eta=\eta_{\ell t}u_j} \frac{1}{2} \sum_{\substack{\text{elliptic conjugacy classes } [\gamma]\\ \text{of } G(F) \text{ with } \det\gamma=\eta}} \int\limits_{Z(\mathbb{A})G_\gamma(F)\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\, dg.
\end{aligned}$$

Here we use the fact that $G_\gamma(F)\backslash G(F) = Z(\mathbb{A})G_\gamma(F)\backslash Z(\mathbb{A})G(F)$. The above then leads to

$$\mathcal{E}(f) = \sum_{\ell,t,j,\eta=\eta_{\ell t}u_j} \frac{1}{2} \sum_{\substack{\text{elliptic conjugacy classes } [\gamma] \\ \text{of } G(F) \text{ with } \det\gamma=\eta}} \mu_\gamma \int_{Z(\mathbb{A})G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\,dg,$$

where

$$\mu_\gamma = \operatorname{meas}(Z(\mathbb{A})G_\gamma(F)\backslash G_\gamma(\mathbb{A})).$$

Note that

$$\int_{Z(\mathbb{A})G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\,dg = \prod_v \int_{Z(F_v)G_\gamma(F_v)\backslash G(F_v)} f_v(g^{-1}\gamma g)\,dg. \tag{4.3}$$

If γ is not elliptic for some archimedean place ∞_i, then by [11, Proposition 26.3],

$$\int_{Z(F_{\infty_i})G_\gamma(F_{\infty_i})\backslash G(F_{\infty_i})} f_{\infty_i}(g^{-1}\gamma g)\,dg = 0.$$

Hence the left hand side of (4.3) is non-zero only if γ is elliptic in any archimedean embedding in $G(F_{\infty_i})$.

Given elliptic $\gamma, \gamma' \in G(F)$, we say that γ and γ' are *conjugate* if they have the same characteristic polynomial. Suppose the characteristic polynomial of γ is $x^2 - \tau x + \eta$. Then γ is elliptic in $G(F_{\infty_i})$ if and only if $\sigma_i(\tau)^2 < 4\sigma_i(\eta)$ for $i = 1, \ldots, r$. Furthermore, by [14, Proposition 2.4], if $\det\gamma = \eta_{\ell t}u_j$ and $f(g^{-1}\gamma g) \neq 0$ for some g, then

$$\beta_{\ell t}^{-1} g_{\mathrm{fin}}^{-1} \gamma g_{\mathrm{fin}} \in \prod_{v<\infty} M_2(\mathcal{O}_v), \tag{4.4}$$

where $\beta_{\ell t} \in \widehat{\mathcal{O}}$ satisfies $(\beta_{\ell t v}) = \mathfrak{b}_{\ell t v}$ for all non-archimedean valuations v. Therefore

$$\operatorname{tr}\gamma \in \beta_{\ell t v}\mathcal{O}_v \quad \text{for all } v < \infty.$$

We rewrite the above as follows.

Proposition 4.2. *The elliptic term in Theorem* 3.1 *is given by*

$$\mathcal{E}(f_{\underline{\mathrm{k}}} f_{\mathrm{fin}}) = \frac{1}{2}\sum_\eta \sum_\tau \mu_{\gamma_{\eta\tau}} \prod_v \int_{Z(F_v)G_{\gamma_{\eta\tau}}(F_v)\backslash G(F_v)} f_v(g^{-1}\gamma_{\eta\tau} g)\,dg.$$

Here η runs through

$$\eta = \eta_{\ell t}u_j, \quad \ell = 1, \ldots, L,\ t = 1, \ldots, t_\ell,\ j = 1, \ldots, 2^r, \tag{4.5}$$

τ runs through

$$\tau \in \beta_{\ell t}\widehat{\mathcal{O}} \cap \mathcal{O}, \quad \sigma_i(\tau)^2 < 4\sigma_i(\eta) \quad \textit{for } i = 1, \ldots, r, \tag{4.6}$$

and $\gamma_{\eta\tau} = \left(\begin{smallmatrix} 0 & -\eta \\ 1 & \tau \end{smallmatrix}\right)$ is an elliptic element with characteristic function $x^2 - \tau x + \eta$.

The next lemma is our key to bounding the elliptic term uniformly with respect to weight $\underline{\mathrm{k}}$.

LEMMA 4.3. *Let $k \geq 4$ be an integer. For elliptic $\gamma \in G(\mathbb{R})$,*

$$\Big| \int_{Z(\mathbb{R})G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f_k(g^{-1}\gamma g)\, dg \Big| \leq \int_{Z(\mathbb{R})G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} |f_4(g^{-1}\gamma g)|\, dg.$$

Proof. Write

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let $ae^{i\theta}$ and $ae^{-i\theta}$ with $a, \theta \in \mathbb{R}$ be the eigenvalues of γ. By [11, (26.6)],

$$\int_{Z(\mathbb{R})G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f_k(g^{-1}\gamma g)\, dg = \int_{\mathrm{SL}_2(\mathbb{R})} f_k(g^{-1}k_\theta g)\, dg + \int_{\mathrm{SL}_2(\mathbb{R})} f_k(g^{-1}k_{-\theta} g)\, dg. \tag{4.7}$$

By the calculation in [11, p. 301],

$$\int_{\mathrm{SL}_2(\mathbb{R})} f_k(g^{-1}k_\theta g)\, dg = \frac{ie^{-i(k-1)\theta}}{2\sin\theta}.$$

Denote the above expression by $\Phi_k(\theta)$. The absolute value of the left side of (4.7) is

$$|\Phi_k(\theta) + \Phi_k(-\theta)| \leq |\Phi_k(\theta)| + |\Phi_k(-\theta)| = |\Phi_4(\theta)| + |\Phi_4(-\theta)|.$$

Our assertion follows by $|\Phi_4(\theta)| \leq \int_{\mathrm{SL}_2(\mathbb{R})} |f_4(g^{-1}k_\theta g)|\, dg$ and (4.7) with $|f_4|$ in place of f_k. Here $|f_4|(x) = |f_4(x)|$. ∎

PROPOSITION 4.4. *Write $\underline{4} = (4, \ldots, 4)$. Then*

$$|\mathcal{E}(f_{\underline{\mathrm{k}}} f_{\mathrm{fin}})| \leq \frac{1}{2} \sum_\eta \sum_\tau \sum_\gamma \int_{\mathfrak{S}'} |(f_{\underline{4}} f_{\mathrm{fin}})(g^{-1}\gamma g)|\, dg.$$

Here η runs through (4.5), *τ runs through* (4.6) *and γ runs through*

$$\gamma \in G(F) \cap Q^{-1}\beta_{\ell t} M_2(\widehat{\mathcal{O}}), \qquad \operatorname{tr}\gamma = \tau, \qquad \det\gamma = \eta. \tag{4.8}$$

Proof. By Proposition 4.2, (4.3) and the previous lemma,

$$\begin{aligned} |\mathcal{E}(f_{\mathrm{k}} f_{\mathrm{fin}})| &\leq \frac{1}{2} \sum_\eta \sum_\tau \mu_{\gamma_{\eta\tau}} \int_{Z(\mathbb{A})G_{\gamma_{\eta\tau}}(\mathbb{A})\backslash G(\mathbb{A})} |(f_{\underline{4}} f_{\mathrm{fin}})(g^{-1}\gamma_{\eta\tau} g)|\, dg \\ &= \frac{1}{2} \sum_\eta \sum_\tau \int_{Z(\mathbb{A})G_{\gamma_{\eta\tau}}(F)\backslash G(\mathbb{A})} |(f_{\underline{4}} f_{\mathrm{fin}})(g^{-1}\gamma_{\eta\tau} g)|\, dg \end{aligned}$$

$$= \frac{1}{2}\sum_{\eta}\sum_{\tau}\sum_{\gamma\in[\gamma_{\eta\tau}]}\int\limits_{\overline{G}(F)\backslash\overline{G}(\mathbb{A})} |(f_{\underline{4}}f_{\mathrm{fin}})(g^{-1}\gamma g)|\,dg$$

$$\le \frac{1}{2}\sum_{\eta}\sum_{\tau}\sum_{\gamma\in[\gamma_{\eta\tau}]}\int\limits_{\mathfrak{S}'} |(f_{\underline{4}}f_{\mathrm{fin}})(g^{-1}\gamma g)|\,dg.$$

If $f(g^{-1}\gamma g)\neq 0$ for some $g\in\mathfrak{S}'$, then by (4.4), $\beta_{\ell t}^{-1}\gamma\in K'_{\mathrm{fin}}M_2(\widehat{\mathcal{O}})K'^{-1}_{\mathrm{fin}} \subseteq Q^{-1}M_2(\widehat{\mathcal{O}})$. Thus we can replace $\sum_{\gamma\in[\gamma_{\eta\tau}]}$ by a summation of γ satisfying (4.8). This gives the result. ∎

5. Estimation of the elliptic terms. Because in (2.1), K'_v is compact, there exist $\alpha_1,\ldots,\alpha_{s_v}\in K'_v$ such that

$$K'_v\subseteq\bigcup_{i=1}^{s_v}\alpha_i K_v.$$

If $K'_v=K_v$, then we can take $s_v=1$. So $s_v=1$ for almost all v.

PROPOSITION 5.1. *Let v be a non-archimedean valuation. Let s_v be given as above. Let $\mathfrak{N}_v=\mathfrak{p}_v^m$ where m is a positive integer. Let $\mathfrak{n}_v$ be an integral ideal of $\mathcal{O}_v$. Let $f_v=\chi_{\mathfrak{N}_v}^{\mathfrak{n}_v}$. Suppose that $\gamma\in G(F_v)$ is such that $\det\gamma\in\mathfrak{n}_v$ and $\Delta=\Delta(\gamma)=(\mathrm{tr}\,\gamma)^2-4\det\gamma\neq 0$. If $\mathrm{ord}_v\,\Delta<m$, then*

$$\int\limits_{K'_v}|f_v(k^{-1}\gamma k)|\,dk\le s_v(m+1)(1-q_v^{-2})^{-1}(1-q_v^{-1})^{-1}\frac{q_v^{\mathrm{ord}_v\,\Delta/2}}{q_v^m}.$$

Proof. First we notice that

$$\int\limits_{K'_v}|f_v(k^{-1}\gamma k)|\,dk\le\sum_{i=1}^{s_v}\int\limits_{\alpha_iK_v}|f_v(k^{-1}\gamma k)|\,dk=\sum_{i=1}^{s_v}\int\limits_{K_v}|f_v(k^{-1}\alpha_i^{-1}\gamma\alpha_i k)|\,dk.$$

Suppose $\int_{K_v}|f_v(k^{-1}\alpha_i^{-1}\gamma\alpha_ik)|\,dk\neq 0$. Then by [14, Lemma 2.3], there exists $k_0\in K_v$ such that

$$\delta=k_0^{-1}\alpha_i^{-1}\gamma\alpha_ik_0\equiv\begin{pmatrix}a&b\\&d\end{pmatrix}\pmod{\mathfrak{N}_v}.$$

Because dk is a Haar measure on K_v and f_v is bi-$K(\mathfrak{N}_v)$-invariant,

$$\int\limits_{K_v}|f_v(k^{-1}\alpha_i^{-1}\gamma\alpha_ik)|\,dk=\int\limits_{K_v}|f_v(k^{-1}\delta k)|\,dk \tag{5.1}$$
$$=\sum_{k\in K_v/K(\mathfrak{N}_v)}\mathrm{meas}(K(\mathfrak{N}_v))|f_v(k^{-1}\delta k)|.$$

Write $k=\left(\begin{smallmatrix}x&y\\z&w\end{smallmatrix}\right)\in K_v$. Then $k^{-1}\delta k\in K_0(\mathfrak{N}_v)$ if and only if

$$z((a-d)x+bz)\equiv 0\pmod{\mathfrak{N}_v}. \tag{5.2}$$

Let $i = \operatorname{ord}_v(a-d)$. Because $(a-d)^2 \equiv \Delta \pmod{\mathfrak{N}_v}$, $\operatorname{ord}_v \Delta = 2i$ if $\operatorname{ord}_v \Delta < m$.

Let $j = \operatorname{ord}_v z$. For fixed a, b, d, z, the congruence (5.2) is solvable if and only if $\min(i+j, m) \le 2j + \operatorname{ord}_v b$ and the number of solutions of x mod $\mathfrak{N}_v$ is $q_v^{\min(i+j,m)}$.

For fixed $j < m$, the number of z mod $\mathfrak{N}_v$ with $\operatorname{ord}_v z = j$ is $\le q_v^{m-j}$. The number of solutions of x mod $\mathfrak{N}_v$ of the congruence (5.2) is $q_v^{\min(i+j,m)}$. The numbers of w, y mod $\mathfrak{N}_v$ are both q_v^m. Hence for such j, the number of $k \in K_v/K(\mathfrak{N})$ such that $k^{-1}\delta k \in K_0(\mathfrak{N}_v)$ is $\le q_v^{m-j} q_v^{\min(i+j,m)} q_v^{2m} \le q_v^{m-j} q_v^{i+j} q_v^{2m} = q^{3m+i}$.

For fixed $j \ge m$, the number of $z \in \mathcal{O}_v/\mathfrak{N}_v$ such that $\operatorname{ord}_v z \ge m$ is 1. Hence for such j, the number of $k \in K_v/K(\mathfrak{N})$ satisfying $k^{-1}\delta k \in K_0(\mathfrak{N}_v)$ is $\le q_v^m q_v^{2m} \le q_v^{3m+i}$.

Because $m \ge 1$,

$$\begin{aligned}(\operatorname{meas} K(\mathfrak{N}_v))^{-1} &= [K_v : K(\mathfrak{N}_v)] = [K_v : K(\mathfrak{p}_v)][K(\mathfrak{p}_v) : K(\mathfrak{p}_v^m)] \\ &= (q_v^2 - 1)(q_v^2 - q_v) q_v^{4(m-1)} = q_v^{4m}(1 - q_v^{-2})(1 - q_v^{-1}).\end{aligned}$$

Summing up over $j = 0, 1, \ldots, m$, the integral (5.1) is

$$\le (m+1)(1 - q_v^{-2})^{-1}(1 - q_v^{-1})^{-1} \frac{q_v^{3m+i}}{q_v^{4m}}.$$

The proposition follows easily. ∎

PROPOSITION 5.2. *Let $\eta = \eta_{\ell t} u_j$ be as in* (4.5). *Then*

$$\sum_{\tau} \sum_{\gamma} \int_{\mathfrak{S}'_\infty} |f_{\underline{4}}(g^{-1}\gamma g)| \ll \mathrm{N}(\eta)^{3/2}. \tag{5.3}$$

Here τ runs through (4.6) *and γ runs through* (4.8).

Proof. By (4.8), $\gamma = \left(\begin{smallmatrix} a/Q & b/Q \\ c/Q & d/Q \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}$ and $ad - bc = Q^2\eta$.

Let M be a positive number such that $[-M, M]^r$ contains a fundamental domain of $\mathbb{R}^r/\{(\sigma_1(x), \ldots, \sigma_r(x)) : x \in \mathcal{O}\}$. Let

$$g_i = \begin{pmatrix} 1 & x_i \\ & 1 \end{pmatrix} \begin{pmatrix} y_i^{1/2} & \\ & y_i^{-1/2} \end{pmatrix} k_{\theta_i} \in \mathfrak{S}'_{\infty_i}.$$

Then by [14, p. 359], for $|\varepsilon_{i1}|, |\varepsilon_{i2}|, |\varepsilon_{i3}| \le M$,

$$\begin{aligned}&|f_4(g_i^{-1}\sigma_i(\gamma) g_i)| \\ &\quad\le \frac{12}{\pi} \frac{Q^4 \sigma_i(\eta)^2}{((\sigma_i(a) - \sigma_i(c)x_i)^2 + (\sigma_i(d) + \sigma_i(c)x_i)^2 + (Y'\sigma_i(c))^2 + 2Q^2\sigma_i(\eta))^2} \\ &\quad\le \frac{C_i Q^4 \sigma_i(\eta)^2}{((\sigma_i(a) - \sigma_i(c)x_i + \varepsilon_{i1})^2 + (\sigma_i(d) + \sigma_i(c)x_i + \varepsilon_{i2})^2 + (Y'(\sigma_i(c) + \varepsilon_{i3}))^2 + 2Q^2\sigma_i(\eta))^2},\end{aligned}$$

where

$$C_i = \frac{12}{\pi}\left(1+\frac{M}{Q\sqrt{2\sigma_i(\eta)}}\right)^8\left(1+\frac{Y'M}{Q\sqrt{2\sigma_i(\eta)}}\right)^4 \ll 1.$$

Here we have used Proposition 4.1: $\sigma_i(\eta) > 1$ for $i = 2,\ldots,n$ and $\sigma_1(\eta) \gg \mathrm{N}(\eta) \geq 1$.

By [14, p. 359, last line],

$$\begin{aligned}(5.3) \ll \prod_{i=1}^{r}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sigma_i(\eta)^2\,du_i\,dv_i\,dw_i}{(u_i^2+v_i^2+(Y'w_i)^2+2Q^2\sigma_i(\eta))^2} \\ = \prod_{i=1}^{r}\frac{\sigma_i(\eta)^{3/2}}{\sqrt{2}QY'}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{du_i\,dv_i\,dw_i}{(u_i^2+v_i^2+w_i^2+1)^2}.\end{aligned}$$

The proposition follows easily. ∎

Proposition 5.3. *Let $\mathfrak{M}$ be an integral ideal and $\alpha > 0$. Let η be given as in the previous proposition. Then*

$$\sum_{\tau}\sum_{\gamma,\,\Delta(\gamma)\in\mathfrak{M}}\int_{\mathfrak{S}'_\infty} |f_{\underline{4}}(g^{-1}\gamma g)|\,dg \ll \mathrm{N}(\eta)^{3/2+\alpha}\,\mathrm{N}(\mathfrak{M})^{-\alpha}, \tag{5.4}$$

where τ runs through (4.6) *and γ runs through* (4.8) *with $\Delta(\gamma)\in\mathfrak{M}$.*

Proof. Since $|\sigma_i(\Delta(\gamma))| = 4\sigma_i(\eta) - \sigma_i(\tau)^2 \leq 4\sigma_i(\eta)$, $\mathrm{N}(\mathfrak{M}) \leq \mathrm{N}(\Delta(\gamma)) \leq 4^r\,\mathrm{N}(\eta)$. By the previous proposition, (5.4) $\ll \mathrm{N}(\eta)^{3/2} = \mathrm{N}(\eta)^{3/2+\alpha}\,\mathrm{N}(\eta)^{-\alpha} \ll \mathrm{N}(\eta)^{3/2+\alpha}\,\mathrm{N}(\mathfrak{M})^{-\alpha}$. ∎

Theorem 5.4. *Let $f = f_{\underline{\mathrm{k}}} f_{\mathrm{fin}}$ be given as in Section 3. Suppose Φ is an upper bound for $|f_{\mathrm{fin}}|$. Then for any $\varepsilon > 0$,*

$$\mathcal{E}(f) \ll_\varepsilon \Phi\,\mathrm{N}(\mathfrak{N})^{-1/2+\varepsilon}\sum_{\ell=1}^{L}\mathrm{N}(\mathfrak{n}_\ell)^2.$$

Proof. By Proposition 4.4,

$$|\mathcal{E}(f)| \leq \frac{1}{2}\sum_{\eta}\sum_{\tau}\sum_{\gamma}\int_{\mathfrak{S}'} |(f_{\underline{4}} f_{\mathrm{fin}})(g^{-1}\gamma g)|\,dg,$$

where η runs through (4.5), τ runs through (4.6) and γ runs through (4.8). We partition γ according to the values $m_v(\gamma) = \min(\mathrm{ord}_v\,\Delta(\gamma), \mathrm{ord}_v\,\mathfrak{N})$ for all non-archimedean valuations v. Then

$$|\mathcal{E}(f)| \leq \frac{1}{2}\sum_{\eta}\sum_{\tau}\sum_{\mathfrak{M}|\mathfrak{N}}\sum_{\substack{\gamma,\,m_v(\gamma)=\mathrm{ord}_v\,\mathfrak{M}\\ \text{for all } v<\infty}}\int_{\mathfrak{S}'} |(f_{\underline{4}} f_{\mathrm{fin}})(g^{-1}\gamma g)|\,dg.$$

Fix $\eta = \eta_{\ell t}u_j$, $\mathfrak{M}$ and γ with $\det\gamma = \eta$. Suppose v is a non-archimedean valuation such that $\mathrm{ord}_v\,\mathfrak{M} < \mathrm{ord}_v\,\mathfrak{N}$. By (4.1) and (4.8), $\det(\beta_{\ell t v}^{-1}\gamma) \in \mathfrak{n}_{\ell v}$.

Hence by Proposition 5.1, we have

$$\int_{K'_v} \chi_{\mathfrak{N}_v}^{\mathfrak{n}_{\ell v}}(g_v^{-1}\gamma g_v)\,dg_v = \int_{K'_v} \chi_{\mathfrak{N}_v}^{\mathfrak{n}_{\ell v}}(g_v^{-1}\beta_{\ell t v}^{-1}\gamma g_v)\,dg_v$$
$$\leq s_v(\mathrm{ord}_v\,\mathfrak{N}+1)(1-q_v^{-2})^{-1}(1-q_v^{-1})^{-1}\frac{q_v^{\mathrm{ord}_v\,\Delta(\gamma)/2}}{q_v^{\mathrm{ord}_v\,\mathfrak{N}}}.$$

For other v, we use the trivial estimation

$$\int_{K'_v} \chi_{\mathfrak{N}_v}^{\mathfrak{n}_{\ell v}}(g_v^{-1}\gamma g_v)\,dg_v \leq \mathrm{meas}\,K'_v.$$

Note that $s_v = 1$, $\mathrm{meas}\,K'_v = 1$ for all but finitely many v, and their values only depend on our choice in (2.1). So

$$\int_{\mathfrak{S}'} |(f_{\underline{4}}f_{\mathrm{fin}})(g^{-1}\gamma g)|$$
$$\ll_\varepsilon \Phi d(\mathfrak{N})(\mathrm{meas}\,K')\Big(\prod_{v<\infty} s_v\Big)\Big(\prod_{\mathrm{ord}_v\,\mathfrak{M}<\mathrm{ord}_v\,\mathfrak{N}} \frac{q_v^{\mathrm{ord}_v\,\mathfrak{M}/2}}{q_v^{\mathrm{ord}_v\,\mathfrak{N}}}\Big)$$
$$\times\Big(\prod_{\mathrm{ord}_v\,\mathfrak{N}>0}(1-q_v^{-2})^{-1}(1-q_v^{-1})^{-1}\Big)\int_{\mathfrak{S}'_\infty}|f_{\underline{4}}(g_\infty^{-1}\gamma g_\infty)|\,dg_\infty$$
$$\ll_\varepsilon \Phi d(\mathfrak{N})\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2}\Big(\prod_{\mathrm{ord}_v\,\mathfrak{M}<\mathrm{ord}_v\,\mathfrak{N}} \frac{q_v^{\mathrm{ord}_v\,\mathfrak{M}/2}}{q_v^{\mathrm{ord}_v\,\mathfrak{N}}}\Big)\int_{\mathfrak{S}'_\infty}|f_{\underline{4}}(g_\infty^{-1}\gamma g_\infty)|\,dg_\infty.$$

Therefore

$$\mathcal{E}(f) \ll_\varepsilon \Phi d(\mathfrak{N})\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2}\sum_\eta\sum_\tau\sum_{\mathfrak{M}|\mathfrak{N}}\sum_{\substack{\gamma,\,m_v(\gamma)=\mathrm{ord}_v\,\mathfrak{M}\\ \text{for all } v<\infty}}\Big(\prod_{\mathrm{ord}_v\,\mathfrak{M}<\mathrm{ord}_v\,\mathfrak{N}} \frac{q_v^{\mathrm{ord}_v\,\mathfrak{M}/2}}{q_v^{\mathrm{ord}_v\,\mathfrak{N}}}\Big)$$
$$\times\int_{\mathfrak{S}'_\infty}|f_{\underline{4}}(g_\infty^{-1}\gamma g_\infty)|\,dg_\infty$$
$$\leq \Phi d(\mathfrak{N})\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2}\sum_\eta\sum_{\mathfrak{M}|\mathfrak{N}}\Big(\prod_{\mathrm{ord}_v\,\mathfrak{M}<\mathrm{ord}_v\,\mathfrak{N}} \frac{q_v^{\mathrm{ord}_v\,\mathfrak{M}/2}}{q_v^{\mathrm{ord}_v\,\mathfrak{N}}}\Big)$$
$$\times\sum_\tau\sum_{\gamma,\,\Delta(\gamma)\in\mathfrak{M}}\int_{\mathfrak{S}'_\infty}|f_{\underline{4}}(g_\infty^{-1}\gamma g_\infty)|\,dg_\infty$$
$$\ll \Phi d(\mathfrak{N})\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2}\sum_\eta\sum_{\mathfrak{M}|\mathfrak{N}}\Big(\prod_{\mathrm{ord}_v\,\mathfrak{M}<\mathrm{ord}_v\,\mathfrak{N}} \frac{q_v^{\mathrm{ord}_v\,\mathfrak{N}/2}}{q_v^{\mathrm{ord}_v\,\mathfrak{N}}}\Big)\frac{\mathrm{N}(\eta)^{3/2+\alpha}}{\mathrm{N}(\mathfrak{M})^\alpha}.$$

Here we use Proposition 5.3 in the last step. Let $\alpha = 1/2$. Then

$$\mathcal{E}(f) \ll_\varepsilon \Phi d(\mathfrak{N})\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2} \times \sum_{\eta}\sum_{\mathfrak{M}|\mathfrak{N}} \frac{\mathrm{N}(\eta)^2}{\prod_{\mathrm{ord}_v \mathfrak{M} < \mathrm{ord}_v \mathfrak{N}} q_v^{\mathrm{ord}_v(\mathfrak{N}/2)} \prod_{\mathrm{ord}_v \mathfrak{M} = \mathrm{ord}_v \mathfrak{N}} q_v^{\mathrm{ord}_v \mathfrak{N}/2}}$$

$$\le \Phi\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2}\,\mathrm{N}(\mathfrak{N})^{\varepsilon/2} \sum_{\eta}\sum_{\mathfrak{M}|\mathfrak{N}} \frac{\mathrm{N}(\eta)^2}{\mathrm{N}(\mathfrak{N})^{1/2}} \ll_\varepsilon \Phi\,\mathrm{N}(\mathfrak{N})^{-1/2+\varepsilon} \sum_{\ell=1}^{L} \mathrm{N}(\mathfrak{n}_\ell)^2. \blacksquare$$

6. Prescribed local representations. Recall $\mathcal{S} = \{w_1, \ldots, w_\iota\}$ and ρ_{w_i} is a supercuspidal representation of $Z(F_{w_i})\backslash G(F_{w_i})$ with conductor $\mathfrak{q}_i^{c_i}$, $i = 1, \ldots, \iota$. Also, $\mathfrak{M} = \prod_{i=1}^{\iota} \mathfrak{q}_i^{c_i}$.

Let u_{w_i} be a $K_0(\mathfrak{q}_i^{c_i})$-invariant unit vector of ρ_{w_i}. It is unique up to a scalar multiple of a complex number with norm one. Define

$$f_{w_i}(g) = d_{\rho_{w_i}} \overline{\langle \rho_{w_i}(g) u_{w_i}, u_{w_i}\rangle},$$

where $d_{\rho_{w_i}}$ is the formal degree of ρ_{w_i} defined by

$$d_{\rho_{w_i}}^{-1} = \int_{\overline{G}(F_{w_i})} |\langle \rho_{w_i}(g) u_{w_i}, u_{w_i}\rangle|^2\, dg.$$

Let $\mathfrak{n} = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_T^{m_T}$. Let $\mathfrak{N}$ be an ideal relatively prime to $\mathfrak{p}_1 \cdots \mathfrak{p}_T \mathfrak{q}_1 \cdots \mathfrak{q}_\iota$. For $v \notin \mathcal{S}$, define $f_v = \psi(\mathfrak{N}_v)\chi_{\mathfrak{N}_v}^{\mathfrak{n}_v}$. Set

$$f_{\mathfrak{N}}^{\mathfrak{n}} = f_{\underline{\mathrm{k}}} \cdot \prod_{w\in\mathcal{S}} f_w \cdot \prod_{v<\infty,\, v\notin\mathcal{S}} f_v. \tag{6.1}$$

Let

$$A_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho}) = \bigoplus_{\pi\in\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M})} \mathbb{C} u_{\infty_1} \otimes \cdots \otimes u_{\infty_r} \otimes u_{w_1} \cdots \otimes u_{w_\iota} \otimes \prod_{\text{other } v} \pi_v^{K_0(\mathfrak{N}_v)}.$$

Here u_{∞_i} is a lowest weight unit vector and $\pi_v^{K_0(\mathfrak{N}_v)}$ is the set of $K_0(\mathfrak{N}_v)$-invariant vectors of π_v.

Proposition 6.1. *Let* $\mathfrak{n}$, $\mathfrak{N}$ *and* f *be given as above. Then* $R(f)$ *maps* $L^2(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}))$ *to* $A_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho})$ *and annihilates* $A_{\underline{\mathrm{k}}}(\mathfrak{N}, \underline{\rho})^\perp$. *Moreover*

$$\operatorname{tr} R(f) = \mathrm{N}(\mathfrak{n})^{1/2} \sum_{\pi\in\Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})} d\left(\frac{\mathfrak{N}\mathfrak{M}}{\mathfrak{c}(\pi)}\right) \prod_{t=1}^{T} X_{m_t}(\lambda_{v_t}(\pi)).$$

Here $d(\mathfrak{a})$ *denotes the divisor function* (*see Section* 2), $\mathfrak{c}(\pi)$ *is the conductor of* π *and* X_m *is the polynomial defined by*

$$X_m(2\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}.$$

Proof. For a square integrable representation π_{w_i} of $\overline{G}(F_{w_i})$, $\pi_{w_i}(f_{w_i})$ is the orthogonal projection to $\mathbb{C}u_{w_i}$ if $\pi_{w_i} \cong \rho_{w_i}$ and is the zero map otherwise. The proposition can then be proved in a similar fashion to [14, Propositions 2.2 and 3.5]. We also use the fact that for $\pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})$, $\pi_{w_i} \cong \rho_{w_i}$ and thus $\mathfrak{c}(\pi_{w_i}) = \mathfrak{M}_{w_i}$. ■

Define

$$\tilde{f}_{\mathfrak{N}}^{\mathfrak{n}} = \sum_{S \subseteq \mathcal{V}(\mathfrak{N})} (-1)^{|S|} f_{\mathfrak{N}_S}^{\mathfrak{n}}, \tag{6.2}$$

(see (2.5)). The summation includes $S = \emptyset$.

THEOREM 6.2.

$$\operatorname{tr} R(\tilde{f}_{\mathfrak{N}}^{\mathfrak{n}}) = \mathrm{N}(\mathfrak{n})^{1/2} \sum_{\pi \in \Pi_{\underline{\mathrm{k}}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})} \prod_{t=1}^{T} X_{m_t}(\lambda_{v_t}(\pi)).$$

Proof. By the previous proposition, it suffices to show that for fixed $\pi \in \Pi_{\mathrm{k}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})$,

$$\sum_{S \subseteq \mathcal{V}(\mathfrak{N})} (-1)^{|S|} d\left(\frac{\mathfrak{N}_S \mathfrak{M}}{\mathfrak{c}(\pi)}\right) = 1.$$

The left hand side can be factorized as

$$\prod_{v \in \mathcal{V}(\mathfrak{N})} \left(d\left(\frac{\mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{N}}}{\mathfrak{c}(\pi_v)}\right) - d\left(\frac{\mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{N}}}{\mathfrak{p}_v \mathfrak{c}(\pi_v)}\right)\right) \prod_{v \in \mathcal{S}} d\left(\frac{\mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{M}}}{\mathfrak{c}(\pi_v)}\right) \prod_{\text{other } v} d\left(\frac{\mathcal{O}}{\mathfrak{c}(\pi_v)}\right).$$

For $v \in \mathcal{V}(\mathfrak{N})$, $d(\mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{N}}/\mathfrak{c}(\pi_v)) - d(\mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{N}}/\mathfrak{p}_v\mathfrak{c}(\pi_v))$ is obviously equal to 1. For $v \in \mathcal{S}$, because $\pi_v \cong \rho_v$, the conductor of ρ_v is $\mathfrak{M}_v$ and so $d(\mathfrak{p}_v^{\operatorname{ord}_v \mathfrak{M}}/\mathfrak{c}(\pi_v)) = 1$. For all other v, π_v is an unramified induced representation and thus the conductor is $\mathcal{O}_v$. Therefore $d(\mathcal{O}_v/\mathfrak{c}(\pi_v)) = 1$. The theorem follows easily. ■

Since $\prod_{v \in \mathcal{S}} f_v$ is a compactly supported function on $\prod_{v \in \mathcal{S}} \overline{G}(F_v)$, as in the discussion before (3.2) we can show that there exist integral ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_L$ divisible by $\mathfrak{q}_1, \dots, \mathfrak{q}_\iota$ only such that $\mathfrak{m}_i/\mathfrak{m}_j$ is not a square of an ideal for all $i \neq j$ and

$$\operatorname{Supp} \prod_{v \in \mathcal{S}} f_v \subseteq \bigcup_{i=1}^{L} \prod_{v \in \mathcal{S}} Z(F_v) M(\mathfrak{m}_{iv}, \mathcal{O}_v). \tag{6.3}$$

So f satisfies (3.2) with $\mathfrak{n}_i = \mathfrak{n}\mathfrak{m}_i$.

THEOREM 6.3. *Suppose $\mathfrak{n} = \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_T^{m_T}$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_L$ are given as above. Then*

$$\sum_{\pi\in\Pi_{\underline{k}}(\mathfrak{N}\mathfrak{M},\underline{\rho})}\prod_{t=1}^{T}X_{m_t}(\lambda_{v_t}(\pi)) = \mathrm{meas}(\overline{G}(F)\backslash\overline{G}(\mathbb{A}))C_{\underline{k}}d_{\underline{\rho}}\delta_{2|\underline{m}}\,\mathrm{N}(\mathfrak{n})^{-1/2}\widetilde{\psi}(\mathfrak{N})$$
$$+ O_\varepsilon\Big(d_{\underline{\rho}}\,\mathrm{N}(\mathfrak{N})^{1/2+\varepsilon}\,\mathrm{N}(\mathfrak{n})^{3/2}\sum_{i=1}^{L}\mathrm{N}(\mathfrak{m}_i)^2\Big).$$

Here $C_{\underline{k}}$, $d_{\underline{\rho}}$ *and* $\widetilde{\psi}(\mathfrak{N})$ *are respectively defined in* (1.1) *and* (2.6). *Moreover, we set* $\delta_{2|\underline{m}} = 1$ *if all the entries of* $\underline{m} = (m_1,\dots,m_T)$ *are even and zero otherwise.*

Proof. Note that $\mathrm{N}(\mathfrak{n}_i) = \mathrm{N}(\mathfrak{n})\,\mathrm{N}(\mathfrak{m}_i)$ and

$$f^{\mathfrak{n}}_{\mathfrak{N}_S}(e) = f_\infty(e)\prod_{i=1}^{\iota}f_{w_i}(e)\prod_{\text{other }v}f_v(e) = C_{\underline{k}}d_{\underline{\rho}}\delta_{2|\underline{m}}\psi(\mathfrak{N}_S).$$

For $v\notin\mathcal{S}$ we have $|f_v(g)|\le\psi(\mathfrak{N}_v)$. For $v\in\mathcal{S}$,

$$|f_v(g)| = |d_{\rho_v}\langle\rho_v(g)u_v,u_v\rangle| \le d_{\rho_v}\|\rho_v(g)u_v\|\|u_v\| = d_{\rho_v}.$$

Applying Theorem 5.4 with $\Phi = \psi(\mathfrak{N}_S)d_{\underline{\rho}}$ to the elliptic part in (3.3), we have

$$\operatorname{tr}R(f^{\mathfrak{n}}_{\mathfrak{N}_S}) = \mathrm{meas}(\overline{G}(F)\backslash\overline{G}(\mathbb{A}))C_{\underline{k}}d_{\underline{\rho}}\delta_{2|\underline{m}}\psi(\mathfrak{N}_S)$$
$$+ O_\varepsilon\Big(d_{\underline{\rho}}\psi(\mathfrak{N}_S)\,\mathrm{N}(\mathfrak{N}_S)^{-1/2+\varepsilon/2}\,\mathrm{N}(\mathfrak{n})^2\sum_{i=1}^{L}\mathrm{N}(\mathfrak{m}_i)^2\Big).$$

The theorem follows from (6.2) and the fact that

$$\sum_{S\subseteq\mathcal{V}(\mathfrak{N})}\psi(\mathfrak{N}_S)\,\mathrm{N}(\mathfrak{N}_S)^{-1/2+\varepsilon/2} \le 2^{|\mathcal{V}(\mathfrak{N})|}\psi(\mathfrak{N})\,\mathrm{N}(\mathfrak{N})^{-1/2+\varepsilon/2} \ll_\varepsilon \mathrm{N}(\mathfrak{N})^{1/2+\varepsilon}. \ \blacksquare$$

7. A variant of the Erdős–Turán inequality. In this section, we generalize a variant of the Erdős–Turán inequality in Murty and Sinha [16] to higher-dimensional situations.

Let $\{\underline{x}_n\}_{n=1}^\infty$ be a sequence in the box $[0,1/2]^T$. Write $\underline{x}_n = (x_{1n},\dots,x_{Tn})$ and consider the sequence $\{x_{tn}\}_{n=1}^\infty$. For all $t = 1,\dots,T$ and $m\in\mathbb{Z}$, assume that the limit

$$c_t(m) = \lim_{V\to\infty}\frac{1}{V}\sum_{n\le V}\cos(2\pi m x_{tn})$$

exists and suppose

$$\sum_{m=1}^{\infty}|c_t(m)| < \infty. \tag{7.1}$$

We define a measure μ_t on $[0,1]$ by

$$d\mu_t = 2F_t(x)dx,$$

where

$$F_t(x) = \sum_{m=-\infty}^{\infty} c_t(m)e(mx) = 1 + 2\sum_{m=1}^{\infty} c_t(m)\cos(2\pi mx).$$

If F_t is a non-negative function, then μ_t is a probability measure on the interval $[0,1/2]$. (Note that $\mu_t[0,1/2]=1$.) Define the product measure

$$\boldsymbol{\mu} = \prod_{t=1}^{T} \mu_t,$$

whose total volume on $[0,1/2]^T$ is 1. Here we are concerned with the discrepancy between $\{\underline{x}_n\}$ and $\boldsymbol{\mu}$. For any (Lebesgue) measurable set $S \subset [0,1/2]^T$, we write

$$N_S(V) = \#\{n \le V : \underline{x}_n \in S\} \quad \text{and} \quad D_S(V) = |N_S(V) - V\boldsymbol{\mu}(S)|.$$

Let $\underline{a} = (a_1,\dots,a_T)$ and $\underline{b} = (b_1,\dots,b_T)$. Whenever $a_t \le b_t$ for all t, we write $[\underline{a},\underline{b}] = \prod_{t=1}^{T}[a_t,b_t] \subset \mathbb{R}^T$. Our aim is to estimate $D_I(V)$ for $I = [\underline{a},\underline{b}]$ in terms of

$$\Delta_t(m,V) = \Big|\sum_{n\le V}\cos(2\pi m x_{tn}) - Vc_t(m)\Big| \qquad (\text{for } m\in\mathbb{Z}),$$

$$\boldsymbol{\Delta}(\underline{m},V) = \Big|\sum_{n\le V}\mathbf{cos}(2\pi \underline{m}\diamond \underline{x}_n) - V\mathbf{c}(\underline{m})\Big| \qquad (\text{for } \underline{m}\in\mathbb{Z}^T),$$

where for $\underline{m} = (m_1,\dots,m_T)$,

$$\mathbf{cos}(2\pi\underline{m}\diamond\underline{x}_n) = \prod_{t=1}^{T}\cos(2\pi m_t x_{tn}) \quad \text{and} \quad \mathbf{c}(\underline{m}) = \prod_{t=1}^{T} c_t(m_t).$$

Then we have the following result; its proof is postponed to Section 9.

PROPOSITION 7.1. *Given a sequence $\{\underline{x}_n\}_{n=1}^{\infty}$ in $[0,1/2]^T$ and $I=[\underline{a},\underline{b}] \subset [0,1/2]^T$, we have*

$$D_I(V) \le \sum_{\underline{m}\in([-M,M]\cap\mathbb{Z})^T} w(\underline{m})\boldsymbol{\Delta}(\underline{m},V) + 10\frac{T}{M+1}\sum_{1\le m\le M}\max_{1\le t\le T}\Delta_t(m,V) + 12C_F\frac{VT}{M+1}$$

for any integers $V, M\ge 1$, where

$$w(\underline{m}) = (2\pi)^T\prod_{t=1}^{T}\bigg(\min\Big(\frac{1}{\pi|m_t|}, b_t - a_t\Big) + \frac{2}{M+1}\bigg),$$

$$C_F = \max_{1\le t\le T}\|F_t\|_\infty + \prod_{t=1}^{T}\|F_t\|_\infty$$

with $\|F_t\|_\infty = \max_{x\in[0,1]}|F_t(x)|$.

REMARK 7.2. $\boldsymbol{\Delta}(\underline{0}, V) = 0$ if V is an integer, and $\Delta_t(m, V) = \boldsymbol{\Delta}(m\underline{e}_t, V)$ where $\underline{e}_j$ is the standard basis vector whose jth entry is 1 and others are 0. For later use, we note that for $M \gg 1$,

$$\sum_{\underline{m} \in ([-M,M] \cap \mathbb{Z})^T} w(\underline{m}) \ll (5 \log M)^T.$$

8. Proof of Theorem 1.1. Let $\theta_v(\pi)$ be $\theta \in [0, 1/2]$ such that $2\cos(2\pi\theta) = \lambda_v(\pi)$ ([1]). Given $I \subset [-2, 2]$, we choose $I' \subset [0, 1/2]$ so that $\theta_v(\pi) \in I'$ if and only if $\lambda_v(\pi) \in I$. In view of Proposition 7.1, we are led to evaluate $\Delta_t(m, V)$ and $\boldsymbol{\Delta}(\underline{m}, V)$.

Let $v = v_t$ where $1 \le t \le T$ and consider

$$\sum_{\pi \in \Pi_{\underline{k}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})} \cos(2\pi m \theta_v(\pi)). \tag{8.1}$$

Let X_m be defined as in Proposition 6.1. Note $X_{-1} \equiv 0$ and $X_{-2} \equiv -1$. For $m \ge 0$, we have

$$2\cos m\theta = X_m(2\cos\theta) - X_{m-2}(2\cos\theta). \tag{8.2}$$

Take $\mathfrak{n} = \mathfrak{p}_t^m$ in Theorem 6.3 and write

$$V = |\Pi_{\underline{k}}(\mathfrak{N}\mathfrak{M}, \underline{\rho})|.$$

For $m = 0$ we obtain

$$V = \mathrm{meas}(\overline{G}(F) \backslash \overline{G}(\mathbb{A})) C_{\underline{k}} d_{\underline{\rho}} \widetilde{\psi}(\mathfrak{N}) + O_\varepsilon\Big(d_{\underline{\rho}} \mathrm{N}(\mathfrak{N})^{1/2+\varepsilon} \sum_{i=1}^{L} \mathrm{N}(\mathfrak{m}_i)^2\Big). \tag{8.3}$$

(With (2.7) Remark 1.2(i) follows.) Moreover, for $m \ge 1$, the sum in (8.1) equals

$$\frac{V}{2}\delta_{2|m}(\mathrm{N}(\mathfrak{p}_t^m)^{-1/2} - \mathrm{N}(\mathfrak{p}_t^{m-2})^{-1/2}) \\ + O_\varepsilon\Big(d_{\underline{\rho}} \mathrm{N}(\mathfrak{N})^{1/2+\varepsilon} \mathrm{N}(\mathfrak{p}_t^m)^{3/2} \sum_{i=1}^{L} \mathrm{N}(\mathfrak{m}_i)^2\Big).$$

Hence we set $c_t(0) = 1$, and for $m \neq 0$,

$$c_t(m) = \begin{cases} \frac{1}{2}(\mathrm{N}(\mathfrak{p}_t)^{-|m|/2} - \mathrm{N}(\mathfrak{p}_t)^{-|m|/2+1}) & \text{if } 2 \mid m, \\ 0 & \text{otherwise.} \end{cases}$$

([1]) The same symbol π is used for the mathematical constant and the automorphic representation but it is clear from the context.

The condition in (7.1) is clearly fulfilled and

$$F_t(x) = 1 + (1 - \mathrm{N}(\mathfrak{p}_t)) \sum_{\substack{m\ge 1\\ 2|m}} \mathrm{N}(\mathfrak{p}_t)^{-m/2}\cos(2\pi m x)$$
$$= \frac{2(\mathrm{N}(\mathfrak{p}_t)+1)\sin^2(2\pi x)}{(\mathrm{N}(\mathfrak{p}_t)^{1/2}+\mathrm{N}(\mathfrak{p}_t)^{-1/2})^2 - 4\cos^2(2\pi x)}.$$

With the change of variable $y = -2\cos(2\pi x)$, the measure $d\mu_t$ on $[0,1/2]$ is pushed forward to $d\mu_{v_t}$ on $[-2,2]$.

Finally we prove the following.

Proposition 8.1. *For any $\underline{m} \in \mathbb{Z}^T$ and $\underline{\theta}(\pi) = (\theta_{v_1}(\pi),\dots,\theta_{v_T}(\pi)) \in [0,1/2]^T$,*

$$\Big| \sum_{\pi\in \Pi_{\underline{k}}(\mathfrak{N}\mathfrak{M},\underline{\rho})} \mathbf{cos}(2\pi \underline{m}\diamond \underline{\theta}(\pi)) - V\mathbf{c}(\underline{m})\Big| \ll d_{\underline{\rho}}\,\mathrm{N}(\mathfrak{N})^{1/2+\varepsilon}\,\mathrm{N}(\mathfrak{n})^{3/2}\sum_{i=1}^{L}\mathrm{N}(\mathfrak{m}_i)^2,$$

where $\mathfrak{n} = \mathfrak{p}_1^{m_1}\cdots\mathfrak{p}_T^{m_T}$, and the implied constant depends on F only.

Proof. Let $\underline{m} = (m_1,\dots,m_T)$. Evidently we may assume all m_t are nonnegative. Partition the index set $\{t : t = 1,\dots,T\}$ into three sets S_0, S_1, S' according as $m_t = 0$, $m_t = 1$ or $m_t \ge 2$ respectively. By (8.2) we write

$$\mathbf{cos}(2\pi\underline{m}\diamond\underline{\theta}(\pi))$$
$$= 2^{|S_0|-T}\sum_{S\subseteq S'}(-1)^{|S|}\prod_{t\in S_1\cup(S'\setminus S)} X_{m_t}(\lambda_{v_t}(\pi))\prod_{t\in S} X_{m_t-2}(\lambda_{v_t}(\pi)).$$

Summing over $\pi \in \Pi_{\underline{k}}(\mathfrak{N}\mathfrak{M},\underline{\rho})$, from Theorem 6.3 we get

$$V\delta_{2|\underline{m}}\,\mathrm{N}(\mathfrak{n})^{-1/2}\prod_{\mathfrak{p}|\mathfrak{n}}\frac{1}{2}(1-\mathrm{N}(\mathfrak{p})) + O_\varepsilon\Big(d_{\underline{\rho}}\,\mathrm{N}(\mathfrak{N})^{1/2+\varepsilon}\,\mathrm{N}(\mathfrak{n})^{3/2}\sum_{i=1}^{L}\mathrm{N}(\mathfrak{m}_i)^2\Big).$$

The main term is obviously $V\prod_{t=1}^T c_t(m_t) = V\mathbf{c}(\underline{m})$. ∎

Let $I\subset[-2,2]^T$. Then by the discussion at the beginning of this section

$$\bigg|\frac{N_I(\mathfrak{p}_1,\dots,\mathfrak{p}_T;\mathfrak{N}\mathfrak{M},\underline{\rho})}{\#\Pi_{\underline{k}}(\mathfrak{N}\mathfrak{M},\underline{\rho})} - \int_I\prod_{t=1}^T d\mu_{v_t}\bigg|$$

equals $V^{-1}D_{I'}(V)$ for some $I'\subset[0,1/2]^T$. From Propositions 7.1 and 8.1, this is

$$(8.4)\quad \ll \frac{T}{M+1} + V^{-1}(5\log M)^T d_{\underline{\rho}}\,\mathrm{N}(\mathfrak{N})^{1/2+\varepsilon}\,\mathrm{N}(\mathfrak{p}_1\cdots\mathfrak{p}_T)^{3M/2}\sum_{i=1}^{L}\mathrm{N}(\mathfrak{m}_i)^2,$$

where the implied constant depends only on F. Let

$$M = \left[\frac{\log(C_{\underline{\mathrm{k}}}\,\mathrm{N}(\mathfrak{N}))}{8\log \mathrm{N}(\mathfrak{p}_1\cdots\mathfrak{p}_T)}\right].$$

The first term in (8.4) dominates by (8.3) and (2.4). (Recall Remark 1.2(ii).) Theorem 1.1 follows readily.

9. Proof of Proposition 7.1. Finally we complete the postponed proof, based on some auxiliary results in [13]. We follow the notation therein.

Suppose firstly $I = [\underline{a}, \underline{b}]$ is a subset of the open box $(0, 1/2)^T$. As in [13, Section 2], we write briefly $\widetilde{\varphi}_t$ for the periodic function $\widetilde{\varphi}_{u_t,v_t}$ of period 1, which coincides with the characteristic function of the union $[u_t, v_t] \cup [-v_t, -u_t]$ except for the end-points of the two intervals at which the values of $\widetilde{\varphi}_t$ are 1/2. Adopting the definition in [13, (2.9)], we write

$$\Phi_{\underline{u},\underline{v}}(\underline{x}) = \prod_{t=1}^{T} \widetilde{\varphi}_t(x_t).$$

We pick $\underline{u}_1, \underline{v}_1, \underline{u}_2, \underline{v}_2 \in ((M+1)^{-1}\mathbb{Z} \cap [0, 1/2])^T$ so that if $I_i = [\underline{u}_i, \underline{v}_i]$ $(i = 1, 2)$, then $I_1 \subset I \subset I_2$ with $|\boldsymbol{\mu}(I) - \boldsymbol{\mu}(I_i)| \le 2T \max_{1\le t\le T} \|F_t\|_\infty/(M+1)$ for $i = 1, 2$, and

$$\sum_{n\le V} \Phi_{\underline{u}_1,\underline{v}_1}(\underline{x}_n) \le N_I(V) \le \sum_{n\le V} \Phi_{\underline{u}_2,\underline{v}_2}(\underline{x}_n). \tag{9.1}$$

This reduces to evaluating

$$\Big|\sum_{n\le V} \Phi_{\underline{u},\underline{v}}(\underline{x}_n) - V\boldsymbol{\mu}(I)\Big| \tag{9.2}$$

for $I = [\underline{u}, \underline{v}]$ with $\underline{u}, \underline{v} \in ((M+1)^{-1}\mathbb{Z} \cap [0, 1/2])^T$, and $D_I(V)$ differs from (9.2) for $(\underline{u}, \underline{v}) = (\underline{u}_1, \underline{v}_1)$ or $(\underline{u}_2, \underline{v}_2)$ by no more than $2C_F VT/(M+1)$. The function $\widetilde{\varphi}_t$ is well approximated by a trigonometric sum $\widetilde{\alpha}_t$,

$$\widetilde{\alpha}_t(x) = \widehat{\alpha}_t(0) + \sum_{1\le|m|\le M} \widehat{\alpha}_t(m)\cos(2\pi m x),$$

where

$$|\widehat{\alpha}_t(m)| \le 2\pi \min\left(\frac{1}{\pi|m|}, v_t - u_t\right). \tag{9.3}$$

(This follows from [13, (2.7)] with the facts $|t(1-t)\cot \pi t| \le 1/2$ for $0 < t < 1$ and $|e(\theta) - 1| \le 2\min(1, \pi|\theta|)$.)

Set

$$\widetilde{\boldsymbol{\alpha}}(\underline{x}) = \prod_{t=1}^{T} \widetilde{\alpha}_t(x_t) = \sum_{\underline{\ell}\in([-M,M]\cap\mathbb{Z})^T} \prod_{t=1}^{T} \widehat{\alpha}_t(\ell_t)\,\mathbf{cos}(2\pi\underline{\ell}\diamond\underline{x}).$$

Plainly one observes

$$\boldsymbol{\mu}(I)=\prod_{t=1}^{T}\widetilde{\varphi}_t*F_t(0)=\prod_{t=1}^{T}\int_0^1\widetilde{\varphi}_t(x)F_t(-x)\,dx$$

(see the proof of [13, Lemma 4.2]). Then,

$$\Big|\sum_{n\le V}\Phi_{\underline{u},\underline{v}}(\underline{x}_n)-V\boldsymbol{\mu}(I)\Big|\le\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3, \tag{9.4}$$

where

$$\sum\nolimits_1=\sum_{n\le V}|\Phi_{\underline{u},\underline{v}}(\underline{x}_n)-\widetilde{\boldsymbol{\alpha}}(\underline{x}_n)|,$$

$$\sum\nolimits_2=\Big|\sum_{n\le V}\widetilde{\boldsymbol{\alpha}}(\underline{x})-V\prod_{t=1}^{T}\widetilde{\alpha}_t*F_t(0)\Big|,$$

$$\sum\nolimits_3=V\Big|\prod_{t=1}^{T}\widetilde{\alpha}_t*F_t(0)-\prod_{t=1}^{T}\widetilde{\varphi}_t*F_t(0)\Big|.$$

For $\sum_1$, we invoke [13, Proposition 1]:

$$|\Phi_{\underline{u},\underline{v}}(\underline{x})-\widetilde{\boldsymbol{\alpha}}(\underline{x})|\le B(\underline{x}), \tag{9.5}$$

where $B(\underline{x})=\sum_{t=1}^{T}\widetilde{\beta}_t(x_t)$. Every $\widetilde{\beta}_t$ is a non-negative trigonometric sum

$$\widetilde{\beta}_t(x)=(2M+2)^{-1}\sum_{|m|\le M}\widehat{\beta}_t(m)\cos(2\pi mx)$$

that satisfies $|\widehat{\beta}_t(m)|\le 4$ (by [13, (2.7)] again) and $\int_0^1\widetilde{\beta}_t(x)\,dx=2/(M+1)$. Thus, we have

$$\widetilde{\beta}_t*F_t(0)=(2M+2)^{-1}\sum_{|m|\le M}\widehat{\beta}_t(m)c_t(m), \tag{9.6}$$

$$|\widetilde{\beta}_t*F_t(x)|\le\frac{2}{M+1}\|F_t\|_\infty \tag{9.7}$$

and

$$\begin{aligned}\sum_{n\le V}\widetilde{\beta}_t(x_{tn})&\le\Big|\sum_{n\le V}\widetilde{\beta}_t(x_{tn})-V\widetilde{\beta}_t*F_t(0)\Big|+V|\widetilde{\beta}_t*F_t(0)|\\&\le\frac{2}{M+1}\sum_{|m|\le M}\Big|\sum_{n\le V}\cos(2\pi mx_{tn})-Vc_t(m)\Big|+\frac{2V}{M+1}\|F_t\|_\infty.\end{aligned}$$

Together with (9.5) we obtain

$$\sum\nolimits_1\le\frac{2T}{M+1}\max_{1\le t\le T}\sum_{1\le m\le M}\Delta_t(m,V)+\frac{2VT}{M+1}\max_{1\le t\le T}\|F_t\|_\infty.$$

Next likewise with (9.6) for $\widetilde{\alpha}_t * F_t(0)$, we infer that

$$\sum\nolimits_2 \le \sum_{\underline{m}\in([-M,M]\cap\mathbb{Z})^T} \prod_{t=1}^{T} |\widehat{\alpha}_t(m_t)|\, \boldsymbol{\Delta}(\underline{m},V).$$

For $\sum_3$, we apply the simple inequality

$$\Big|\prod_{t=1}^{T} A_t - \prod_{t=1}^{T} B_t\Big| \le \sum_{t=1}^{T} |A_t - B_t| \prod_{r\neq t} \max(|A_r|,|B_r|),$$

where $A_t, B_t \in \mathbb{C}$. Then, as both $0 \le \widetilde{\alpha}_t, \widetilde{\varphi}_t \le 1$ from [13, (2.8)], we deduce with (9.5) (specialized to $\widetilde{\varphi}_t$) and (9.7) that

$$\sum\nolimits_3 \le V \sum_{t=1}^{T} |(\widetilde{\alpha}_t - \widetilde{\varphi}_t) * F_t(0)| \prod_{r\neq t} \|F_r\|_\infty \le \frac{2VT}{M+1} \prod_{t=1}^{T} \|F_t\|_\infty,$$

as in the proof of [13, Lemma 4.2]. Inserting the above estimates for $\sum_i$ and (9.3) into (9.4), we get

$$\begin{aligned}(9.8)\quad &\Big|\sum_{n\le V} \Phi_{\underline{u},\underline{v}}(\underline{x}_n) - V\boldsymbol{\mu}(I)\Big| \\ &\le 2C_F \frac{VT}{M+1} + \frac{2T}{M+1} \sum_{1\le m\le M} \max_{1\le t\le T} \Delta_t(m,V) \\ &\quad + \sum_{\underline{m}\in([-M,M]\cap\mathbb{Z})^T} (2\pi)^T \prod_{t=1}^{T} \Big(\min\Big(\frac{1}{\pi|m_t|}, v_t - u_t\Big)\Big) \boldsymbol{\Delta}(\underline{m},V).\end{aligned}$$

Now consider the case that I contains a boundary point of $[0,1/2]^T$. For small positive ε, we replace a_t by ε when $a_t = 0$ and b_t by $1/2-\varepsilon$. Let I_ε be the new box. Then

$$D_I(V) \le D_{I_\varepsilon}(V) + \sum_{t=1}^{T} N_{t,\varepsilon}(V) + 4\varepsilon V C_F, \tag{9.9}$$

where

$$N_{t,\varepsilon}(V) = \#\{n \le V : x_{tn} \in [0,\varepsilon] \cup [1/2-\varepsilon, 1/2]\}.$$

As the value of $\widetilde{\varphi}_t$ equals $1/2$ at the end-point, we see that

$$N_{t,\varepsilon}(V) \le 2\sum_{n\le V} \widetilde{\varphi}_{0,\varepsilon}(x_{tn}) + 2\sum_{n\le V} \widetilde{\varphi}_{1/2-\varepsilon,1/2}(x_{tn}).$$

By (9.8), we have

$$\sum_{t=1}^{T} N_{t,\varepsilon}(V) \le 8C_F \frac{VT}{M+1} + \frac{8T}{M+1} \sum_{1\le m\le M} \max_{1\le t\le T} \Delta_t(m,V) + o(1) \tag{9.10}$$

as $\varepsilon \to 0+$. Finally we can replace $v_t - u_t$ in (9.8) by $b_t - a_t + 2/(M+1)$. The result follows from (9.8)–(9.10) and the discussion below (9.2).

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