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Sums of distances to the nearest integer and the discrepancy of digital nets

by

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1. Introduction. The concept of digital nets provides at the moment the most efficient method to generate point sets with small star-discrepancy D_N^*. For a set of points $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ in $[0,1)^d$ the *star-discrepancy* of the point set is defined by

$$D_N^* = \sup_B \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$

where the supremum is taken over all subintervals B of $[0,1)^d$ of the form $B = \prod_{i=1}^d [0, b_i)$, $0 < b_i \le 1$, $A_N(B)$ denotes the number of i with $\mathbf{x}_i \in B$ and λ is the Lebesgue measure.

It is known that for any set of N points in $[0,1)^2$ one has

$$\frac{N D_N^*}{\log N} \ge 0.06$$

(see for example [1]).

A *digital* $(0, s, 2)$-*net in base* 2 is a point set of $N = 2^s$ points $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ in $[0,1)^2$ which is generated as follows. Choose two $s \times s$-matrices C_1, C_2 over $\mathbb{Z}_2$ with the following property: For every integer k, $0 \le k \le s$, the system of the first k rows of C_1 together with the first $s-k$ rows of C_2 is linearly independent over $\mathbb{Z}_2$. Then to construct $\mathbf{x}_n := (x_n^{(1)}, x_n^{(2)})$ for $0 \le n \le 2^s - 1$, represent n in base 2:

$$n = n_{s-1} 2^{s-1} + \ldots + n_1 2 + n_0,$$

multiply C_i with the vector of digits:

$$C_i (n_0, \ldots, n_{s-1})^{\mathrm{T}} =: (y_1^{(i)}, \ldots, y_s^{(i)})^{\mathrm{T}} \in \mathbb{Z}_2^s$$

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and set

$$x_n^{(i)} := \sum_{j=1}^{s} \frac{y_j^{(i)}}{2^j}.$$

It was shown by Niederreiter [8] that for the star-discrepancy of any digital $(0, s, 2)$-net in base 2 we have

$$ND_N^* \leq \frac{1}{2}s + \frac{3}{2}, \tag{1}$$

hence

$$\limsup_{N\to\infty} \max \frac{ND_N^*}{\log N} \leq \frac{1}{2\log 2} = 0.7213\ldots \tag{2}$$

where the maximum is taken over all digital $(0, s, 2)$-nets in base 2 with $N = 2^s$ elements.

The simplest digital $(0, s, 2)$-net in base 2 is provided by choosing

$$C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

This gives the well-known Hammersley point set in base 2.

The star-discrepancy of this very special digital $(0, s, 2)$-net was studied by Halton and Zaremba [4], de Clerck [2] and Entacher [3]. The first two papers are very technical and very hard to read. Indeed in [4] an essential part of the proof (determining the extremal intervals) is not carried out in detail. [3] uses a new approach but also essentially relies on results from [4].

In this paper we study much more generally the star-discrepancy of digital $(0, s, 2)$-nets in base 2.

In Section 2 (see Theorem 1) we give a compact explicit formula for the discrepancy function of digital $(0, s, 2)$-nets in base 2. Our approach is via Walsh series analysis.

It turns out that this explicit formula is based on sums of distances to the nearest integer ($\|x\| := \min(x - [x], 1 - (x - [x]))$) of the form

$$\sum_{u=0}^{s-1} \|2^u \beta\| \varepsilon_u$$

with a real β and certain integer sequences $\varepsilon_u \in \{-1, 0, 1\}$.

In Section 3 we study such sums on their own and we give a certain "spectrum" result for $\sum_{u=0}^{s-1} \|2^u\beta\|$ (see Theorems 2 and 3), part of which will be needed in Section 4.

In Section 4 we use the above results to study the Hammersley point set once more, to give a simple and now self-contained proof for the exact value of the "discrete discrepancy" and of the star-discrepancy of this point set (Theorem 4). Further we show that it is the "worst distributed" digital $(0,s,2)$-net in base 2 with respect to star-discrepancy and we will get that for every digital $(0,s,2)$-net in base 2 we have the (essentially) best possible bound

$$ND_N^* \leq \frac{1}{3}s + \frac{19}{9}, \tag{3}$$

and that

$$\lim_{N\to\infty} \max \frac{ND_N^*}{\log N} = \frac{1}{3\log 2} = 0.4808\ldots \tag{4}$$

(the maximum is taken over all digital $(0,s,2)$-nets in base 2 with $N = 2^s$ elements) with equality for the Hammersley point sets, thereby improving the bounds (1) and (2) of Niederreiter (Theorem 5).

Numerical investigations suggest that the minimal value for

$$\limsup_{N\to\infty} \frac{ND_N^*}{\log N}$$

over all digital $(0,s,2)$-nets in base 2 is attained for the net generated by the matrices

$$C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

In Section 5 we give bounds for the star-discrepancy of this net and we show (Theorem 6) that for these nets

$$\frac{ND_N^*}{\log N} \geq \frac{1}{5\log 2} = 0.2885\ldots$$

holds for all N and that

$$\limsup_{N\to\infty} \frac{ND_N^*}{\log N} \leq 0.32654\ldots,$$

thereby answering a question of Entacher in [3, Section 4].

2. The discrepancy function of digital $(0,s,2)$-nets. For $0 \leq \alpha, \beta \leq 1$ we consider the discrepancy function

$$\Delta(\alpha,\beta) := A_N([0,\alpha) \times [0,\beta)) - N\alpha\beta$$

for digital $(0,s,2)$-nets $\mathbf{x}_0, \ldots, \mathbf{x}_{2^s-1}$ in base 2 (i.e. $N = 2^s$).

Since the generating matrices C_1, C_2 of a $(0,s,2)$-net must be regular, and since multiplying C_1, C_2 by a regular matrix A does not change the point set (only its order) we may assume in all the following that

$$C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} c_1^1 & c_1^2 & \dots & c_1^s \\ c_2^1 & c_2^2 & \dots & c_2^s \\ \cdots & \cdots & \cdots & \cdots \\ c_s^1 & c_s^2 & \dots & c_s^s \end{pmatrix} =: \begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \cdots \\ \vec{c}_s \end{pmatrix}.$$

We assume first that α and β are "s-bit", i.e.

$$\alpha = \frac{a_1}{2} + \ldots + \frac{a_s}{2^s}, \qquad \beta = \frac{b_1}{2} + \ldots + \frac{b_s}{2^s}.$$

For any s-bit number $\delta = d_1/2 + \ldots + d_s/2^s$ we write

$$\vec{\delta} := \begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix},$$

and for a non-negative integer $k = k_{s-1}2^{s-1} + \ldots + k_1 2 + k_0$ we write

$$\vec{k} := \begin{pmatrix} k_0 \\ \vdots \\ k_{s-1} \end{pmatrix}.$$

We need some further notation:

$$\vec{\gamma} := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{pmatrix} := C_2\vec{\alpha} + \vec{\beta}, \qquad \vec{\gamma}(u) := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_u \end{pmatrix},$$

$$C_2'(u) := \begin{pmatrix} c_1^{s-u+1} & \dots & c_u^{s-u+1} \\ \cdots & \cdots & \cdots \\ c_1^s & \dots & c_u^s \end{pmatrix}^{-1}.$$

($C_2'(u)$ exists since by the $(0, s, 2)$-net property the first $s-u$ rows of C_1 together with the first u rows of C_2 must form a linearly independent system, hence the matrix

$$C_2(u) := \begin{pmatrix} c_1^{s-u+1} & \dots & c_1^s \\ \cdots & \cdots & \cdots \\ c_u^{s-u+1} & \dots & c_u^s \end{pmatrix}$$

must be regular.) Note that $\gamma_u = (\vec{c}_u|\vec{\alpha}) + b_u$.

Further, for $0 \le u \le s-1$ let

$$m(u) := \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } (\vec{\gamma}(u)|C_2'\vec{e}_1) = 1, \\ \max\{1 \le j \le u : (\vec{\gamma}(u)|C_2'\vec{e}_i) = 0;\ i = 1, \ldots, j\} & \text{otherwise} \end{cases}$$

(here $(\cdot|\cdot)$ denotes the usual inner product in $\mathbb{Z}_2^u$, $\vec{e}_i$ is the ith unit vector in $\mathbb{Z}_2^u$, and $C_2' := C_2'(u)$).

Let $j(u) := u - m(u)$. Then we have

THEOREM 1. *For all α, β s-bit, for the discrepancy function $\Delta(\alpha,\beta)$ of the digital $(0,s,2)$-net in base 2 generated by*

$$C_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and C_2 we have

$$\Delta(\alpha,\beta) = \sum_{u=0}^{s-1} \|2^u\beta\| (-1)^{(\vec{c}_{u+1}|\vec{\alpha})} (-1)^{(\vec{\gamma}(u)|C_2'(u)(c_{u+1}^{s-u+1},\dots,c_{u+1}^{s})^{\mathrm{T}})} \times \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}$$

(*here for $u = 0$ we set $(\vec{\gamma}(u)|C_2'(u)(c_{u+1}^{s-u+1},\dots,c_{u+1}^{s})^{\mathrm{T}}) = 0$ and $a_{s+1} := 0$*).

Before we prove this result we give some remarks and examples.

REMARK 1. Note that $\Delta(\alpha,\beta)$ hence is of the form $\sum_{u=0}^{s-1} \|2^u\beta\|\varepsilon_u$ with some $\varepsilon_u \in \{-1,0,1\}$.

REMARK 2. Let $0 \le \alpha, \beta \le 1$ now be arbitrary (not necessarily s-bit). Since all the points of the digital net have coordinates $x_n^{(i)}$ of the form $a/2^s$ for some $a \in \{0,1,\dots,2^s-1\}$, we then have

$$\Delta(\alpha,\beta) = \Delta(\alpha(s),\beta(s)) + 2^s(\alpha(s)\beta(s) - \alpha\beta)$$

where $\alpha(s)$ (resp. $\beta(s)$) is the smallest s-bit number larger than or equal to α (resp. β).

EXAMPLE 1. Let C_2 be of triangular form

$$C_2 = \begin{pmatrix} c_1^1 & c_1^2 & \dots & c_1^{s-1} & 1 \\ c_2^1 & c_2^2 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{s-1}^1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then

$$C_2'(u) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & d_2^u \\ \dots & \dots & \dots & \dots & \dots \\ 1 & d_u^2 & \dots & d_u^{u-1} & d_u^u \end{pmatrix}$$

with certain $d_i^j \in \mathbb{Z}_2$. Hence

$$C_2'(u)\vec{e}_i = (0,\dots,0,1,d_{u+2-i}^i,\dots,d_u^i)^{\mathrm{T}},$$

and

$$(\vec{\gamma}(u)|C_2'(u)\vec{e}_i) = \gamma_{u+1-i} + \gamma_{u+2-i} d^i_{u+2-i} + \ldots + \gamma_u d^i_u.$$

Therefore

$$\max\{1 \le j \le u : (\vec{\gamma}(u)|C_2'(u)\vec{e}_i) = 0;\ i = 1, \ldots, j\} \\ = \max\{1 \le j \le u : \gamma_{u+1-i} = 0;\ i = 1, \ldots, j\},$$

hence $\gamma_u = \ldots = \gamma_{u+1-m(u)} = 0$, $\gamma_{u-m(u)} = 1$, so that

$$j(u) = u - m(u) = \max\{j \le u : \gamma_j = 1\} = \max\{j \le u : (\vec{c}_j|\vec{\alpha}) \ne b_j\}.$$

Respectively

$$j(u) = \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } (\vec{c}_j|\vec{\alpha}) = b_j \text{ for } j = 1, \ldots, u. \end{cases}$$

Further $(c^{s-u+1}_{u+1}, \ldots, c^s_{u+1}) = (0, \ldots, 0)$, and so for α, β s-bit we have

$$\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u\beta\| (-1)^{(\vec{c}_{u+1}|\vec{\alpha})} \, \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}.$$

Example 2. For the discrepancy function of the Hammersley point set, i.e. for the $(0, s, 2)$-net generated by

$$C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

because of $(\vec{c}_j|\vec{\alpha}) = a_{s+1-j}$ we obtain (for α, β s-bit)

$$\Delta(\alpha, \beta) = \sum_{u=0}^{s-1} \|2^u\beta\| \, \frac{1 - (-1)^{a_{s-u} + a_{s+1-j(u)}}}{2} \\ = \sum_{u=0}^{s-1} \|2^u\beta\| (a_{s-u} \oplus a_{s+1-j(u)})$$

(where $\oplus$ denotes addition modulo 2).

Example 3. For the discrepancy function of the $(0, s, 2)$-net generated by

$$C_1 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

because of $(\vec{c}_j|\vec{\alpha}) = a_1 \oplus \ldots \oplus a_{s+1-j}$ we obtain (for α, β s-bit)

$$\Delta(\alpha,\beta) = \sum_{u=0}^{s-1} \|2^u\beta\|(-1)^{a_1+\ldots+a_{s-u}} \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}.$$

For the proof of the Theorem 1 we need two auxiliary results.

LEMMA 1. *Let z be of the form $z = p/2^s$, $p \in \{0,\ldots,2^s-1\}$. Then for the characteristic function $\chi_{[0,z)}$ of the interval $[0,z)$ we have*

$$\chi_{[0,z)}(x) = \sum_{k=0}^{2^s-1} c_k(z)\mathrm{wal}_k(x),$$

where wal_k *denotes the kth Walsh function in base* 2 (*see Remark* 3),

$$c_k(z) = \begin{cases} z & \text{if } k = 0, \\ \mathrm{wal}_k(z)\,\dfrac{1}{2^{\upsilon(k)}}\,\psi(2^{\upsilon(k)}z) & \text{if } k \neq 0, \end{cases}$$

$\psi(x)$ is periodic with period 1 *and*

$$\psi(x) = \begin{cases} x & \text{if } 0 \le x < 1/2, \\ x-1 & \text{if } 1/2 \le x < 1, \end{cases}$$

and $\upsilon(k) = r$ if $2^r \le k < 2^{r+1}$.

REMARK 3. Recall that Walsh functions in base 2 can be defined as follows: For a non-negative integer k with base 2 representation $k = k_m 2^m + \ldots + k_1 2 + k_0$ and a real x with (canonical) base 2 representation $x = x_1/2 + x_2/2^2 + \ldots$ we have

$$\mathrm{wal}_k(x) = (-1)^{x_1k_0+x_2k_1+\ldots+x_{m+1}k_m} = (-1)^{(\vec{k}|\vec{x})}.$$

Proof of Lemma 1. This is a simple calculation, a proof can be found for example in [6, Lemma 2]. ∎

LEMMA 2. *Let ψ be as in Lemma* 1. *Then*

$$\psi(2^{l+1}\beta) - \sum_{i=0}^{l} \psi(2^i\beta) = \{\beta\} - b_{l+2}.$$

(*Here* $\{\beta\} = \beta - [\beta]$.)

Proof. Let $\{\beta\} = \sum_{j=1}^{\infty} b_j 2^{-j}$. Then

$$\psi(2^i\beta) = \sum_{j=i+1}^{\infty} b_j 2^{i-j} - b_{i+1}$$

and therefore

$$\begin{aligned}\sum_{i=0}^{l} \psi(2^i\beta) &= \sum_{i=0}^{l} \Big(\Big(\sum_{j=i+1}^{\infty} b_j 2^{i-j}\Big) - b_{i+1}\Big) \\ &= \sum_{j=1}^{l+1} b_j 2^{-j} \sum_{i=0}^{j-1} 2^i + \sum_{j=l+2}^{\infty} b_j 2^{-j} \sum_{i=0}^{l} 2^i - \sum_{i=0}^{l} b_{i+1} \\ &= \sum_{j=l+2}^{\infty} b_j 2^{(l+1)-j} - \sum_{j=1}^{\infty} b_j 2^{-j} = \psi(2^{l+1}\beta) - \{\beta\} + b_{l+2}. \ \blacksquare\end{aligned}$$

Proof of Theorem 1. Let $I := [0,\alpha) \times [0,\beta)$. Then for $\mathbf{y} = (y^{(1)}, y^{(2)}) \in [0,1)^2$ by Lemma 1 we have

$$\begin{aligned}\chi_I(\mathbf{y}) - \lambda(I) &= \chi_{[0,\alpha)}(y^{(1)})\chi_{[0,\beta)}(y^{(2)}) - \alpha\beta \\ &= \sum_{\substack{k,l=0 \\ (k,l)\neq(0,0)}}^{2^s-1} c_k(\alpha)c_l(\beta)\mathrm{wal}_k(y^{(1)})\mathrm{wal}_l(y^{(2)}) \\ &= \alpha \sum_{l=1}^{2^s-1} \mathrm{wal}_l(\beta)\,\frac{1}{2^{v(l)}}\,\psi(2^{v(l)}\beta)\mathrm{wal}_l(y^{(2)}) \\ &\quad + \beta \sum_{k=1}^{2^s-1} \mathrm{wal}_k(\alpha)\,\frac{1}{2^{v(k)}}\,\psi(2^{v(k)}\alpha)\mathrm{wal}_k(y^{(1)}) \\ &\quad + \sum_{k,l=1}^{2^s-1} \mathrm{wal}_k(\alpha)\mathrm{wal}_l(\beta)\,\frac{1}{2^{v(k)+v(l)}}\,\psi(2^{v(k)}\alpha)\psi(2^{v(l)}\beta) \\ &\quad \times \mathrm{wal}_k(y^{(1)})\mathrm{wal}_l(y^{(2)}).\end{aligned}$$

Hence

$$\begin{aligned}\Delta(\alpha,\beta) &= \alpha \sum_{l=1}^{2^s-1} \mathrm{wal}_l(\beta)\,\frac{1}{2^{v(l)}}\,\psi(2^{v(l)}\beta) \sum_{i=0}^{2^s-1} \mathrm{wal}_l(y_i) \\ &\quad + \beta \sum_{k=1}^{2^s-1} \mathrm{wal}_k(\alpha)\,\frac{1}{2^{v(k)}}\,\psi(2^{v(k)}\alpha) \sum_{i=0}^{2^s-1} \mathrm{wal}_k(x_i) \\ &\quad + \sum_{k,l=1}^{2^s-1} \mathrm{wal}_k(\alpha)\mathrm{wal}_l(\beta)\,\frac{\psi(2^{v(k)}\alpha)\psi(2^{v(l)}\beta)}{2^{v(k)+v(l)}} \sum_{i=0}^{2^s-1} \mathrm{wal}_k(x_i)\mathrm{wal}_l(y_i).\end{aligned}$$

(Here the net consists of the points $\mathbf{x}_i$, $i = 0,\ldots,2^s-1$, with $\mathbf{x}_i := (x_i, y_i)$.)

Since $\mathbf{x}_i$, $i = 0,\ldots,2^s-1$, is a digital $(0,s,2)$-net, for all $0 < k,l < 2^s$ we have

$$\sum_{i=0}^{2^s-1} \mathrm{wal}_k(x_i) = \sum_{i=0}^{2^s-1} \mathrm{wal}_l(y_i) = 0$$

(see for example [5, Lemma 2]).

We now consider $\sum_{i=0}^{2^s-1} \mathrm{wal}_k(x_i)\mathrm{wal}_l(y_i)$ with $x_i := x_i^{(1)}/2 + \ldots + x_i^{(s)}/2^s$ and $y_i := y_i^{(1)}/2 + \ldots + y_i^{(s)}/2^s$. We identify (x_i, y_i) with

$$(x_i^{(1)}, \ldots, x_i^{(s)}, y_i^{(1)}, \ldots, y_i^{(s)})^{\mathrm{T}} \in (\mathbb{Z}_2)^{2s}$$

and we define

$$(x_i, y_i) \oplus (x_i', y_i') := (x_i^{(1)} + x_i'^{(1)}, \ldots, y_i^{(s)} + y_i'^{(s)}).$$

Further $\mathrm{wal}_{k,l}(x_i, y_i) := \mathrm{wal}_k(x_i)\mathrm{wal}_l(y_i)$, hence

$$\mathrm{wal}_{k,l}((x_i, y_i) \oplus (x_i', y_i')) = \mathrm{wal}_{k,l}(x_i, y_i)\mathrm{wal}_{k,l}(x_i', y_i'),$$

i.e. $\mathrm{wal}_{k,l}$ is a character on $((\mathbb{Z}_2)^{2s}, \oplus)$.

The digital net $\mathbf{x}_0, \ldots, \mathbf{x}_{2^s-1}$ is a subgroup of $((\mathbb{Z}_2)^{2s}, \oplus)$, hence

$$\sum_{i=0}^{2^s-1} \mathrm{wal}_k(x_i)\mathrm{wal}_l(y_i) = \begin{cases} 2^s & \text{if } \mathrm{wal}_{k,l}(x_i, y_i) = 1 \text{ for all } i = 0, \ldots, 2^s - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(For more details see [5] or [7].)

Now $\mathrm{wal}_{k,l}(x_i, y_i) = (-1)^{(\vec{k}\,|\vec{x}_i) + (\vec{l}\,|\vec{y}_i)} = 1$ for all $i = 0, \ldots, 2^s - 1$ iff

$$(\vec{k}\,|\vec{x}_i) = (\vec{l}\,|\vec{y}_i) \quad \text{for all } i = 0, \ldots, 2^s - 1,$$

(by the definition of the net) this means

$$(\vec{k}\,|\vec{i}\,) = (\vec{l}\,|C_2\vec{i}\,) \quad \text{for all } i = 0, \ldots, 2^s - 1,$$

and this is satisfied if and only if

$$\vec{k} = C_2^{\mathrm{T}}\vec{l} =: \vec{k}(l).$$

Further

$$\mathrm{wal}_{k(l)}(\alpha)\mathrm{wal}_l(\beta) = (-1)^{(\vec{k}(l)|\vec{\alpha}) + (\vec{l}\,|\vec{\beta})} = (-1)^{(\vec{l}\,|C_2\vec{\alpha} + \vec{\beta})} = \mathrm{wal}_l(\gamma)$$

(see notations).

So

$$\begin{aligned} \Delta(\alpha, \beta) &= 2^s \sum_{u=0}^{s-1} \frac{\psi(2^u\beta)}{2^u} \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0\gamma_1 + \ldots + l_{u-1}\gamma_u + \gamma_{u+1}} \frac{\psi(2^{v(k(l))}\alpha)}{2^{v(k(l))}} \\ &= 2^s \sum_{u=0}^{s-1} \|2^u\beta\| (-1)^{(\vec{c}_{u+1}|\vec{\alpha})} \\ &\quad \times \frac{1}{2^u} \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0\gamma_1 + \ldots + l_{u-1}\gamma_u} \frac{\psi(2^{v(k(l))}\alpha)}{2^{v(k(l))}} \end{aligned}$$

(here $l := l_0 + l_1 2 + \ldots + l_u 2^u$; note that $(-1)^{\gamma_{u+1}} = (-1)^{(\vec{c}_{u+1}|\vec{\alpha})}(-1)^{b_{u+1}}$ and $\psi(2^u\beta)(-1)^{b_{u+1}} = \|2^u\beta\|$).

We now consider

$$\Sigma_1 := \frac{1}{2^u} \sum_{l=2^u}^{2^{u+1}-1} (-1)^{l_0\gamma_1+\ldots+l_{u-1}\gamma_u} \frac{\psi(2^{\upsilon(k(l))}\alpha)}{2^{\upsilon(k(l))}}$$
$$= \frac{1}{2^u} \sum_{w=0}^{s-1} \frac{\psi(2^w\alpha)}{2^w} \sum_{\substack{l=2^u \\ \upsilon(k(l))=w}}^{2^{u+1}-1} (-1)^{l_0\gamma_1+\ldots+l_{u-1}\gamma_u}.$$

For $2^u \le l < 2^{u+1}$, the condition $\upsilon(k(l)) = w$ means that there are $k_0, \ldots, k_{w-1} \in \mathbb{Z}_2$ such that

$$C_2^{\mathrm{T}} \vec{l} = (k_0, \ldots, k_{w-1}, 1, 0, \ldots, 0)^{\mathrm{T}},$$

that is,

$$\vec{c}_1 l_0 + \ldots + \vec{c}_u l_{u-1} + \vec{c}_{u+1} = k_0 \vec{e}_1 + \ldots + k_{w-1}\vec{e}_w + \vec{e}_{w+1} \tag{5}$$

where $\vec{e}_i$ is the ith unit vector in $\mathbb{Z}_2^s$.

Since $\vec{c}_1, \ldots, \vec{c}_{u+1}, \vec{e}_1, \ldots, \vec{e}_{w+1}$ by the $(0, s, 2)$-net property are linearly independent as long as $(u + 1) + (w + 1) \le s$ we must have $u + w \ge s - 1$. Hence

$$\Sigma_1 = \sum_{w=s-1-u}^{s-1} \frac{\psi(2^w\alpha)}{2^{u+w}} \sum_{\substack{l=2^u \\ \upsilon(k(l))=w}}^{2^{u+1}-1} (-1)^{l_0\gamma_1+\ldots+l_{u-1}\gamma_u}.$$

In the following we are concerned with evaluating the last sum in the above expression which equals

$$\Sigma_2 := \sum_{\substack{l=0 \\ \upsilon(k(l+2^u))=w}}^{2^u-1} \mathrm{wal}_l(\gamma) = \sum_{\substack{C_2' l=0 \\ \upsilon(k(C_2' l+2^u))=w}}^{2^u-1} \mathrm{wal}_{C_2' l}(\gamma)$$

(here C_2' stands for $C_2'(u)$; see notation). Now $\upsilon(k(C_2' l + 2^u)) = w$ means

$$C_2^{\mathrm{T}} \begin{pmatrix} C_2' \vec{l} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for some $k_i \in \mathbb{Z}_2$. This is equivalent to

$$(6)\qquad \begin{pmatrix} & D & \\ & & \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \vec{l} = \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} c_{u+1}^1 \\ \vdots \\ c_{u+1}^s \end{pmatrix}$$

with

$$D = \begin{pmatrix} c_1^1 & \dots & c_u^1 \\ \multicolumn{3}{c}{\dotfill} \\ c_1^{s-u} & \dots & c_u^{s-u} \end{pmatrix} C_2',$$

i.e. an $(s-u) \times u$-matrix.

Let $s - u = w + 1$. We first show that in this case equation (6) has a solution $\vec{l}$. This is equivalent to showing that system (5) has a solution, i.e., that there are $l_0, \dots, l_{u-1}, k_0, \dots, k_{w-1}$ in $\mathbb{Z}_2$ such that

$$\vec{c}_1 l_0 + \dots + \vec{c}_u l_{u-1} + \vec{c}_{u+1} + \vec{e}_1 k_0 + \dots + \vec{e}_w k_{w-1} + \vec{e}_{w+1} = 0.$$

Since $s = u + w + 1$ the vectors $\vec{c}_1, \dots, \vec{c}_{u+1}, \vec{e}_1, \dots, \vec{e}_{w+1}$ are linearly dependent, and hence we can find $l_0, \dots, l_{u-1}, l_u, k_0, \dots, k_{w-1}, k_w$ in $\mathbb{Z}_2$ not all zero such that

$$\vec{c}_1 l_0 + \dots + \vec{c}_u l_{u-1} + \vec{c}_{u+1} l_u + \vec{e}_1 k_0 + \dots + \vec{e}_w k_{w-1} + \vec{e}_{w+1} k_w = 0.$$

Assume that $l_u = 0$. Then $\vec{c}_1, \dots, \vec{c}_u, \vec{e}_1, \dots, \vec{e}_{w+1}$ are linearly dependent. But this contradicts the $(0, s, 2)$-net property since $\vec{c}_1, \dots, \vec{c}_u$ are the first u rows of the matrix C_2 and $\vec{e}_1, \dots, \vec{e}_{w+1}$ are the first $w+1$ rows of the matrix C_1 and $u + w + 1 = s$. Hence $l_u = 1$. In the same way one can show that $k_w = 1$. This shows that system (5), and hence also (6), has a solution.

Now the unique solution $\vec{l}$ of (6) is given by

$$\vec{l} = (c_{u+1}^{s-u+1}, \dots, c_{u+1}^s)^{\mathrm{T}}.$$

If $s - u \le w$, then the 2^{u+w-s} solutions therefore are given by

$$\vec{l} = (l_0, \dots, l_{u+w-(s+1)}, c_{u+1}^{w+1} \oplus 1, c_{u+1}^{w+2}, \dots, c_{u+1}^s)^{\mathrm{T}}$$

with $l_0, \dots, l_{u+w-(s+1)}$ arbitrary in $\mathbb{Z}_2$.

Hence for $w \ge s - u$ we have

$$\begin{aligned} \Sigma_2 &= \sum_{l_0, \dots, l_{u+w-(s+1)} \in \mathbb{Z}_2} (-1)^{(\vec{\gamma}(u) | C_2'(l_0, \dots, l_{u+w-(s+1)}, c_{u+1}^{w+1} \oplus 1, c_{u+1}^{w+2}, \dots, c_{u+1}^s)^{\mathrm{T}})} \\ &= (-1)^{(C_2'^{\mathrm{T}} \vec{\gamma}(u) | (0, \dots, 0, c_{u+1}^{w+1} \oplus 1, c_{u+1}^{w+2}, \dots, c_{u+1}^s)^{\mathrm{T}})} \sum_{l=0}^{2^{u+w-s}-1} \mathrm{wal}_l(C_2'^{\mathrm{T}} \gamma(u)). \end{aligned}$$

The last sum is a sum over all characters of $((\mathbb{Z}_2)^{u+w-s}, \oplus)$, and is therefore 2^{u+w-s} if $(C_2'^{\mathrm{T}}\vec{\gamma}(u)|\vec{e}_i) = 0$ for all $i = 1, \dots, u+w-s$ ($\vec{e}_i$ is the ith unit vector in $\mathbb{Z}_2^u$) and it is 0 otherwise.

Further, if $(C_2'^{\mathrm{T}}\vec{\gamma}(u)|\vec{e}_i) = 0$ for all $i = 1, \dots, u+w-s$ (we will call this the *condition* $*_u$), then

$$\begin{aligned}(C_2'^{\mathrm{T}}\vec{\gamma}(u)|(0,\dots,0,c_{u+1}^{w+1}\oplus 1, c_{u+1}^{w+2},\dots,c_{u+1}^{s})^{\mathrm{T}})\\ = (\vec{\gamma}(u)|C_2'(c_{u+1}^{s-u+1},\dots,c_{u+1}^{s})^{\mathrm{T}}) + (\vec{\gamma}(u)|C_2'\vec{e}_{u+w-s+1}),\end{aligned}$$

so that altogether we have

$$\Sigma_1 = \frac{1}{2^s}(-1)^{(\vec{\gamma}(u)|C_2'(c_{u+1}^{s-u+1},\dots,c_{u+1}^{s})^{\mathrm{T}})} f(u),$$

where

$$\begin{aligned}f(u) &:= 2\psi(2^{s-u-1}\alpha)\\ &\quad + \begin{cases}\sum_{w=s-u}^{s-1}\psi(2^w\alpha)(-1)^{(\vec{\gamma}(u)|C_2'\vec{e}_{u+w-s+1})} & \text{if } *_u \text{ holds},\\ 0 & \text{otherwise},\end{cases}\end{aligned}$$

and therefore

$$\Delta(\alpha,\beta) = \sum_{u=0}^{s-1}\|2^u\beta\|(-1)^{(\vec{c}_{u+1}|\vec{\alpha})}(-1)^{(\vec{\gamma}(u)|C_2'(c_{u+1}^{s-u+1},\dots,c_{u+1}^{s})^{\mathrm{T}})} f(u).$$

It remains to show that

$$f(u) = \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}.$$

By the definition of $m(u)$ we have $(\vec{\gamma}(u)|C_2'\vec{e}_1) = \dots = (\vec{\gamma}(u)|C_2'\vec{e}_{m(u)}) = 0$ and $(\vec{\gamma}(u)|C_2'\vec{e}_{m(u)+1}) = 1$, hence $*_u$ holds iff $u+w-s \le m(u)$. So finally

$$\begin{aligned}f(u) &= 2\psi(2^{s-u-1}\alpha) + \sum_{w=s-u}^{s-u+m(u)}\psi(2^w\alpha)(-1)^{(\vec{\gamma}(u)|C_2'\vec{e}_{u+w-s+1})}\\ &= 2\psi(2^{s-u-1}\alpha) + \sum_{w=s-u}^{s-u+m(u)-1}\psi(2^w\alpha) - \psi(2^{s-u+m(u)}\alpha)\\ &= \psi(2^{s-u-1}\alpha) - \sum_{w=0}^{s-u-2}\psi(2^w\alpha) + \sum_{w=0}^{s-u+m(u)-1}\psi(2^w\alpha) - \psi(2^{s-u+m(u)}\alpha)\\ &= \alpha - a_{s-u} - \alpha + a_{s+1-(u-m(u))} = a_{s+1-(u-m(u))} - a_{s-u}\\ &= \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2}\end{aligned}$$

where we used Lemma 2 and $j(u) = u - m(u)$. The result follows. ∎

3. A spectrum result for sums of distances to the nearest integer. Here we study sums of the form $\sum_{u=0}^{s-1}\|2^u\beta\|$ for $\beta\in\mathbb{R}$, especially for s-bit β, and we derive results which are of independent interest and/or will be used in Section 4.

The essential technical tool is provided by

LEMMA 3. *Assume that $\beta = 0.b_1b_2\ldots$ (this always means base 2 representation) has two equal consecutive digits b_ib_{i+1} with $i\le s-1$ and let i be minimal with this property, i.e.*

$$\begin{aligned}\beta &= 0.01\ldots0100b_{i+2}\ldots \quad &&\textit{or}\\ \beta &= 0.10\ldots0100b_{i+2}\ldots &&\textit{or}\\ \beta &= 0.01\ldots1011b_{i+2}\ldots &&\textit{or}\\ \beta &= 0.10\ldots1011b_{i+2}\ldots\end{aligned}$$

Replace β by

$$\begin{aligned}\gamma &= 0.10\ldots1010b_{i+2}\ldots \quad &&\textit{resp.}\\ \gamma &= 0.01\ldots1010b_{i+2}\ldots &&\textit{resp.}\\ \gamma &= 0.10\ldots0101b_{i+2}\ldots &&\textit{resp.}\\ \gamma &= 0.01\ldots0101b_{i+2}\ldots\end{aligned}$$

Then

$$\sum_{u=0}^{s-1}\|2^u\gamma\| = \sum_{u=0}^{s-1}\|2^u\beta\| + \begin{cases}\frac{1}{3}(1-(-1)^i/2^i)(1-\tau) & \textit{in the first two cases},\\ \frac{1}{3}(1-(-1)^i/2^i)\tau & \textit{in the last two cases},\end{cases}$$

where $\tau := 0.b_{i+2}b_{i+3}\ldots$

REMARK 4. In any case we have $\sum_{u=0}^{s-1}\|2^u\gamma\| \ge \sum_{u=0}^{s-1}\|2^u\beta\|$ with equality iff $\tau=1$ in the first two cases and iff $\tau=0$ in the last two cases.

Proof of Lemma 3. This is simple calculation. We just handle the first case here:

$$\begin{aligned}&\sum_{u=0}^{s-1}(\|2^u\gamma\|-\|2^u\beta\|)\\ &\quad= \|\gamma\| - \|2^i\beta\| + \left(\left(\frac{\tau}{2}-\frac{\tau}{4}\right)-\left(\frac{\tau}{4}-\frac{\tau}{8}\right)\pm\ldots+\left(\frac{\tau}{2^i}-\frac{\tau}{2^{i+1}}\right)\right)\\ &\quad= \left(\frac{1}{3}\left(1+\frac{1}{2^i}\right)-\frac{\tau}{2^{i+1}}\right)-\frac{\tau}{2}+\frac{1}{6}\left(1+\frac{1}{2^i}\right)\tau\\ &\quad= \frac{1}{3}\left(1+\frac{1}{2^i}\right)(1-\tau).\end{aligned}$$

The other cases are calculated in the same way. ∎

We immediately obtain a corollary which is useful in Section 4.

Corollary 1. *Assume that* $\beta = 0.1b_2b_3\ldots$ *has two equal consecutive digits* b_ib_{i+1} *with* $2 \le i \le s-1$ *and let* i *be the minimal index with this property, i.e.*

$$\begin{aligned}\beta &= 0.101\ldots0100b_{i+2}\ldots \quad &\text{or}\\ \beta &= 0.110\ldots0100b_{i+2}\ldots \quad &\text{or}\\ \beta &= 0.101\ldots1011b_{i+2}\ldots \quad &\text{or}\\ \beta &= 0.110\ldots1011b_{i+2}\ldots\end{aligned}$$

Replace β *by*

$$\begin{aligned}\gamma &= 0.110\ldots1010b_{i+2}\ldots \quad &\text{resp.}\\ \gamma &= 0.101\ldots1010b_{i+2}\ldots \quad &\text{resp.}\\ \gamma &= 0.110\ldots0101b_{i+2}\ldots \quad &\text{resp.}\\ \gamma &= 0.101\ldots0101b_{i+2}\ldots\end{aligned}$$

Then

$$\gamma + \sum_{u=0}^{s-1} \|2^u\gamma\| = \beta + \sum_{u=0}^{s-1}\|2^u\beta\| + \begin{cases} \frac{1}{3}(1-(-1)^{i-1}/2^{i-1})(1-\tau) & \text{in the first two cases,}\\ \frac{1}{3}(1-(-1)^{i-1}/2^{i-1})\tau & \text{in the last two cases,}\end{cases}$$

where $\tau := 0.b_{i+2}b_{i+3}\ldots$

Proof. This follows from $\beta + \|\beta\| = \gamma + \|\gamma\| = 1$, by applying Lemma 3 to $\beta' := 0.b_2b_3\ldots$ ∎

We obtain

Theorem 2. *Consider* $\beta \in \mathbb{R}$ *with the canonical base* 2 *representation* (*i.e. with infinitely many digits equal to zero*). *Then there exists*

$$\max_{\beta} \sum_{u=0}^{s-1} \|2^u\beta\| = \frac{s}{3} + \frac{1}{9} - (-1)^s\,\frac{1}{9\cdot 2^s}$$

and it is attained if and only if β *is of the form* β_0 *with*

$$\beta_0 = \frac{2}{3}\left(1-\left(-\frac{1}{2}\right)^{s+1}\right) \quad \text{or} \quad \beta_0 = \frac{1}{3}\left(1-\left(-\frac{1}{2}\right)^{s}\right).$$

Remark 5. Note that

$$\frac{2}{3}\left(1-\left(-\frac{1}{2}\right)^{s+1}\right) = \begin{cases} 0.1010\ldots101 & \text{if } s \text{ is odd,}\\ 0.1010\ldots011 & \text{if } s \text{ is even,}\end{cases}$$

$$\frac{1}{3}\left(1-\left(-\frac{1}{2}\right)^{s}\right) = \begin{cases} 0.0101\ldots011 & \text{if } s \text{ is odd,}\\ 0.0101\ldots101 & \text{if } s \text{ is even.}\end{cases}$$

Proof of Theorem 2. For any $\gamma = 0.c_1c_2\ldots c_sc_{s+1}\ldots$ with fixed $c_1,\ldots,c_s$ the sum $\sum_{u=0}^{s-1}\|2^u\gamma\|$ obviously becomes maximal if $c_s = 0$ and $c_{s+1} =$

$c_{s+2} = \ldots = 1$, or if $c_s = 1$ and $c_{s+1} = c_{s+2} = \ldots = 0$. Hence by Lemma 3 the supremum

$$\sup_{\beta} \sum_{u=0}^{s-1} \|2^u \beta\|$$

can only be attained, respectively approached by

$$\begin{aligned} \beta_1 &= 0.1010\ldots10\,111\ldots \quad \text{or} \\ &\quad (b_s \text{ is the last zero}) \\ \beta_2 &= 0.0101\ldots01 \quad \text{or} \\ \beta_3 &= 0.1010\ldots11 \\ &\quad (b_s \text{ is the last one}) \end{aligned}$$

if s is even, and by

$$\begin{aligned} \beta_4 &= 0.0101\ldots10\,111\ldots \quad \text{or} \\ \beta_5 &= 0.1010\ldots01 \quad \text{or} \\ \beta_6 &= 0.0101\ldots11 \end{aligned}$$

if s is odd.

Now we check easily that

$$\sum_{u=0}^{s-1} \|2^u \beta_i\| = \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s}$$

for $i = 1, \ldots, 6$ and the result follows. ■

The next theorem gives the result which we call the "spectrum" result (see Remark 6).

THEOREM 3. (a) *The maximum*

$$\max_{\beta\ s\text{-bit}} \sum_{u=0}^{s-1} \|2^u \beta\| = \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s}$$

is attained if and only if β *is one of the* β_0 *from Theorem* 2.

(b) *We have*

$$\max_{\substack{\beta\ s\text{-bit} \\ \beta \neq \beta_0}} \sum_{u=0}^{s-1} \|2^u \beta\| = \frac{s}{3} + \frac{1}{36} - (-1)^s \frac{7}{9 \cdot 2^s}$$

and this second successive maximum is attained if and only if β *is of the form* β' *with*

$$\beta' = \begin{cases} 0.01101010\ldots101 & \text{or} \\ 0.010101\ldots01101 & \text{or} \\ 0.10010101\ldots011 \end{cases}$$

if s is odd and

$$\beta' = \begin{cases} 0.100101010\ldots101 & or \\ 0.010101\ldots010011 & or \\ 0.101010\ldots101101 & or \\ 0.011010\ldots101011 \end{cases}$$

if s is even.

REMARK 6. Let

$$\max_{\beta\ s\text{-bit}} \sum_{u=0}^{s-1} \|2^u\beta\| =: \sum_{u=0}^{s-1} \|2^u\beta_0(s)\|.$$

Then by Theorem 3 we have

$$\lim_{s\to\infty}\Bigg(\sum_{u=0}^{s-1}\|2^u\beta_0(s)\| - \max_{\substack{\beta\ s\text{-bit}\\ \beta\neq\beta_0(s)}}\sum_{u=0}^{s-1}\|2^u\beta\|\Bigg) = \frac{1}{12}.$$

So one may ask the further usual "spectrum questions".

Proof of Theorem 3. (a) follows from Theorem 2.

Concerning part (b) it follows from Lemma 3 that it must be possible to reach one of the β_0 by applying a single transformation of Lemma 3 to β'.

For s odd this means (s even is handled quite analogously) that

$$\beta' \to 0.1010\ldots101$$

by the first or third transformation, i.e.

$$\beta' = 0.0101\ldots01\,00\,1010\ldots10101 \quad \text{or}$$
$$\beta' = 0.0101\ldots10\,11\,0101\ldots10101,$$

or that

$$\beta' \to 0.0101\ldots011$$

by the second or fourth transformation, i.e.

$$\beta' = 0.1010\ldots01\,00\,10\ldots1011 \quad \text{or}$$
$$\beta' = 0.1010\ldots10\,11\,01\ldots1011.$$

Further the double blocks b_ib_{i+1} must be placed so that the "error term" in Lemma 3 becomes minimal. We carry this out for the two transformations yielding

$$\beta' \to 0.1010\ldots101$$

(the second case is treated quite analogously).

If

$$\beta' = 0.0101\ldots01\,00\,1010\ldots10101$$

then the "error term" has the form

$$\frac{1}{3}\left(1-\frac{(-1)^i}{2^i}\right)(1-\tau) =: E(i)$$

with

$$\tau = 0.1010\ldots101 = \frac{2}{3}\left(1 - \frac{1}{2^{s-i}}\right),$$

and i is odd. Hence

$$E(i) = \frac{1}{9}\left(1 + \frac{1}{2^i}\right)\left(1 + \frac{1}{2^{s-1-i}}\right),$$

which becomes minimal for $i = (s-1)/2$, with value

$$E = \frac{1}{9}\left(1 + \frac{1}{2^{(s-1)/2}}\right)^2.$$

If

$$\beta' = 0.0101\ldots10\ 11\ 0101\ldots10101$$

then

$$E(i) = \frac{1}{3}\left(1 - \frac{(-1)^i}{2^i}\right)\tau$$

with

$$\tau = 0.0101\ldots10101 = \frac{1}{3}\left(1 - \frac{1}{2^{s-i-1}}\right),$$

and i is even. Hence

$$E(i) = \frac{1}{9}\left(1 - \frac{1}{2^i}\right)\left(1 - \frac{1}{2^{s-1-i}}\right),$$

which becomes minimal for $i = 2$ and for $i = s-3$ (note that $i = s-1$ would give one of the β_0 and $E(i) = 0$), with value

$$E = \frac{1}{12}\left(1 - \frac{8}{2^s}\right),$$

which is smaller than the E above.

By also dealing with the second case we find that this is the minimal possible value for E and we have found the first two values of β'. The third value for β' is found by treating the second case.

The minimal error term E also determines the value for

$$\sum_{u=0}^{s-1} \|2^u\beta'\| = \sum_{u=0}^{s-1} \|2^u\beta_0\| - E = \frac{s}{3} + \frac{1}{9} + \frac{1}{9\cdot 2^s} - \frac{1}{12}\left(1 - \frac{8}{2^s}\right)$$
$$= \frac{s}{3} + \frac{1}{36} + \frac{7}{9\cdot 2^s}.$$

The case of s even is dealt with quite analogously. ■

We again obtain a corollary:

COROLLARY 2. *The maximum*

$$\max_{\beta\, s\text{-}bit}\Big(\beta+\sum_{u=0}^{s-1}\|2^u\beta\|\Big)=\frac{s}{3}+\frac{7}{9}+(-1)^s\,\frac{1}{9\cdot 2^{s-1}}$$

is attained if and only if β is of the form

$$\beta_0=\frac{2}{3}\bigg(1-\Big(-\frac{1}{2}\Big)^{s+1}\bigg)\quad \textit{or}\quad \beta_0=\frac{5}{6}-\frac{1}{3}\Big(-\frac{1}{2}\Big)^{s}.$$

REMARK 7. Note that here

$$\beta_0=0.110101\ldots101\quad\text{or}\quad\beta_0=0.101010\ldots011$$

if s is even and

$$\beta_0=0.101010\ldots101\quad\text{or}\quad\beta_0=0.110101\ldots011$$

if s is odd.

Proof of Corollary 2. If $\beta<1/2$ then we replace β by $\beta+1/2$ and we obtain a larger value for the sum in question. So we can assume $\beta=0.1b_2b_3\ldots b_s$, and we note that $\beta+\|\beta\|=1$ always. So we have to maximize $\sum_{u=0}^{s-2}\|2^u(2\beta)\|$. By Theorem 3(a) the result follows. ∎

For later use (proof of Theorem 4(a)) we need a further type of "spectrum" result, namely Lemma 5. To prove it, we will use Lemma 4.

LEMMA 4. *Let $0\le\kappa<1$. Then*

(a) *The maximum*

$$\max_{\beta\, s\text{-}bit}\Big(\kappa\beta+\sum_{u=0}^{s-1}\|2^u\beta\|\Big)=:\Sigma_s^{\kappa}$$

is attained by

$$\beta=\begin{cases}0.1010\ldots1011 & \textit{for } s \textit{ even},\\ 0.1010\ldots101 & \textit{for } s \textit{ odd}.\end{cases}$$

(b) *The maximum*

$$\max_{\beta\, s\text{-}bit}\Big(-\kappa\beta+\sum_{u=0}^{s-1}\|2^u\beta\|\Big)=:\Sigma_s^{-\kappa}$$

is attained by

$$\beta=\begin{cases}0.0101\ldots0101 & \textit{for } s \textit{ even},\\ 0.0101\ldots011 & \textit{for } s \textit{ odd}.\end{cases}$$

Proof. (a) We must have $b_1=1$, otherwise $1-\beta$ gives a larger value than β. We proceed by induction on s. For $s=1,2,3$ the assertion is easily checked. Now (since $b_1 = 1$)

$$\Sigma^{\kappa}_{s+2} = \max_{\beta\ s+2\text{-bit}} \Big(\kappa\beta + \sum_{u=0}^{s+1} \|2^u\beta\|\Big)$$
$$= \max_{\beta'\ s+1\text{-bit}} \bigg(\frac{\kappa+1}{2} + \beta'\Big(\frac{\kappa-1}{2}\Big) + \sum_{u=0}^{s} \|2^u\beta'\|\bigg).$$

Now $(\kappa-1)/2 < 0$, so b'_1 must be zero, otherwise $1-\beta'$ would give a larger value. Hence $\beta' = \beta''/2$ with β'' s-bit, and therefore

$$\Sigma^{\kappa}_{s+2} = \frac{\kappa+1}{2} + \max_{\beta''\ s\text{-bit}} \bigg(\beta''\Big(\frac{\kappa+1}{4}\Big) + \sum_{u=0}^{s-1} \|2^u\beta''\|\bigg).$$

By the induction hypothesis the result follows.

(b) Set $\gamma = 1-\beta$. Then

$$-\gamma\kappa + \sum_{u=0}^{s-1} \|2^u\gamma\| = -\kappa + \kappa\beta + \sum_{u=0}^{s-1} \|2^u\beta\|$$

and by part (a) the result follows. ■

The next lemma is of independent interest. Note for example that $1/4$ is the "average value" for $\|x\|$.

Lemma 5. $$\max_{\substack{\beta\ s\text{-}bit\\ 0\le u_0\le s-1}} \sum_{\substack{u=0\\ u\ne u_0}}^{s-1} \|2^u\beta\| = \max_{\beta\ s\text{-}bit} \sum_{u=0}^{s-1} \|2^u\beta\| - \frac{1}{4}.$$

Proof. For u_0 fixed let

$$\Sigma_{u_0}(\beta) := \sum_{\substack{u=0\\ u\ne u_0}}^{s-1} \|2^u\beta\| \quad \text{and} \quad \Sigma_{u_0}(\beta_0) := \max_{\beta\ s\text{-bit}} \Sigma_{u_0}(\beta).$$

By Lemma 3, β_0 must be of the form

$$\beta_0 = 0.0101\ldots b_{u_0+1}b_{u_0+2}\ldots b_s \quad \text{or} \quad \beta_0 = 0.1010\ldots b_{u_0+1}b_{u_0+2}\ldots b_s.$$

Let

$$\overline{\beta}_0 := 0.b_1\ldots b_{u_0+1} \quad \text{and} \quad \widetilde{\beta}_0 := 0.b_{u_0+2}\ldots b_s.$$

Then

$$\Sigma_{u_0}(\beta_0) = \sum_{u=0}^{u_0-1} \|2^u\overline{\beta}_0\| + \kappa\widetilde{\beta}_0 + \sum_{u=u_0+1}^{s-1} \|2^u\beta_0\|$$
$$= \sum_{u=0}^{u_0-1} \|2^u\overline{\beta}_0\| + \kappa\widetilde{\beta}_0 + \sum_{u=0}^{s-u_0-2} \|2^u\widetilde{\beta}_0\|$$

with $\kappa = \sum_{i=1}^{u_0} (-1)^{b_i}/2^{u_0+2-i}$. If $b_{u_0} = 0$ then $\kappa > 0$, if $b_{u_0} = 1$ then $\kappa < 0$.

So by Lemma 3 (see also Theorem 3) the form of $\overline{\beta}_0$, and by Lemma 4 and by b_{u_0} the form of $\widetilde{\beta}_0$ is determined (note that the form of b_{u_0+1} must be different from b_{u_0} and hence is 0 in any case).

We have

$$\widetilde{\beta}_0 = \frac{1}{3}\left(1 - \frac{(-1)^{s-u_0-1}}{2^{s-u_0-1}}\right)$$

and

$$\kappa = -\frac{1}{6}\left(1 - \frac{(-1)^{u_0}}{2^{u_0}}\right) \quad \text{or} \quad \kappa = -\frac{1}{3}\left(1 + \frac{(-1)^{u_0}}{2^{u_0+1}}\right)$$

according to which value for $\overline{\beta}_0$ is chosen from Theorem 3.

Since we want to maximize

$$\Sigma_{u_0}(\beta_0) = \sum_{u=0}^{u_0-1} \|2^u \overline{\beta}_0\| + \kappa\widetilde{\beta}_0 + \sum_{u=0}^{s-u_0-2} \|2^u \widetilde{\beta}_0\|,$$

only the larger first value for κ is of relevance. Inserting it yields

$$\max_{\beta\, s\text{-bit}} \left(\sum_{u=0}^{s-1} \|2^u \beta\| - \Sigma_{u_0}(\beta_0)\right) = \frac{1}{18}\left(5 + \frac{(-1)^{u_0}}{2^{u_0}} + \frac{(-1)^{s-u_0-1}}{2^{s-u_0-1}} + \frac{(-1)^{s-1}}{2^{s-2}}\right),$$

which attains its minimal value $1/4$ for $u_0 = s-2$ if s is odd, and for $u_0 = 1$ if s is even. ∎

4. The discrepancy of the Hammersley net and an improved upper bound for the discrepancy of digital $(0, s, 2)$-nets. In Theorem 1 for α, β s-bit we have given an explicit formula for the discrepancy function

$$\Delta(\alpha, \beta) = A_{2^s}([0, \alpha) \times [0, \beta)) - 2^s \alpha\beta$$

of a digital $(0, s, 2)$-net in base 2.

Take now arbitrary α', β' with

$$\alpha - \frac{1}{2^s} < \alpha' \le \alpha \quad \text{and} \quad \beta - \frac{1}{2^s} < \beta' \le \beta.$$

Then (since all coordinates of the points of a digital net are s-bit) we have

$$\Delta(\alpha', \beta') = \Delta(\alpha, \beta) - 2^s(\alpha'\beta' - \alpha\beta),$$

hence for the star-discrepancy D_N^* of the net we have

$$\left| D_N^* - \frac{1}{N} \max_{\alpha,\beta\, s\text{-bit}} \Delta(\alpha, \beta) \right| < \frac{2}{N} - \frac{1}{N^2}$$

(note that $N = 2^s$).

We will call

$$\frac{1}{N}\max_{\alpha,\beta\, s\text{-bit}} \Delta(\alpha,\beta) =: D_N^{\mathrm{d}}$$

the *discrete discrepancy* of the net. D_N^{d} differs from D_N^* at most by the almost negligible quantity $2/N$ and seems for nets to be the more natural measure for the irregularities of distribution.

For a sequence of digital $(0,s,2)$-nets, $s=1,2,\ldots$, $N=2^s$, we have

$$\limsup_{N\to\infty}\frac{ND_N^*}{\log N} = \limsup_{N\to\infty}\frac{ND_N^{\mathrm{d}}}{\log N}$$

(the same holds for lim inf and for lim if it exists).

But if we want to obtain "exact results" the quantity D_N^{d} in spite of the minimal difference is much easier to handle than D_N^*.

This is clearly illustrated by the proof of the following theorem, in which we give the exact value of D_N^{d} and of D_N^* for the Hammersley net and the exact places where they are attained. For D_N^{d} we moreover give the "second successive maxima" and the exact places where they are attained. The proof for D_N^{d} is much shorter than the one for D_N^*.

In [4] Halton and Zaremba claim that they give the exact value of D_N^*, but they only give a vague hint on how to prove the extremality of the extremal intervals. Entacher [3] uses their result.

THEOREM 4. (a) *For the discrete discrepancy D_N^{d} of the Hammersley net with $N=2^s$ points we have*

$$ND_N^{\mathrm{d}} = \max_{\alpha,\beta\, s\text{-bit}} \Delta(\alpha,\beta) = \frac{s}{3}+\frac{1}{9}-\frac{(-1)^s}{9\cdot 2^s}$$

and the maximum will be attained if and only if α,β are of the form α_0,β_0 with:

- *for s odd*,

$$\alpha_0 = 0.0101\ldots1011,\qquad \beta_0 = 0.1010\ldots0101$$

or

$$\alpha_0 = 0.1010\ldots0101,\qquad \beta_0 = 0.0110\ldots1011,$$

- *for s even*,

$$\alpha_0=\beta_0=0.1010\ldots1011\quad \text{or}\quad \alpha_0=\beta_0=0.0101\ldots0101.$$

The second successive maximum for $\Delta(\alpha,\beta)$ (α,β s-bit) is given by

$$\max_{\substack{\alpha,\beta\, s\text{-bit}\\ (\alpha,\beta)\neq(\alpha_0,\beta_0)}} \Delta(\alpha,\beta) = \frac{s}{3}+\frac{1}{36}-(-1)^s\,\frac{7}{9\cdot 2^s}$$

and the places where this is attained can easily be obtained from the proof and from Theorem 3(b).

(b) *For the star-discrepancy D_N^* of the Hammersley net with $N = 2^s$ points we have*

$$ND_N^* = \frac{s}{3} + \frac{13}{9} - (-1)^s \frac{4}{9 \cdot 2^s}$$

and the maximum is attained if and only if α, β are of the form α_0, β_0 with:

- *for s odd,*

$$\alpha_0 = 0.1010\ldots10111, \quad \beta_0 = 0.1101\ldots01011$$

or

$$\alpha_0 = 0.1101\ldots01011, \quad \beta_0 = 0.1010\ldots10111,$$

- *for s even,*

$$\alpha_0 = \beta_0 = 0.1010\ldots01011 \quad \text{or} \quad \alpha_0 = \beta_0 = 0.1101\ldots10111$$

for $s \geq 4$. For $s \leq 3$ the extremal values (α_0, β_0) are $(1/2, 1/2)$ $(s = 1)$, $(3/4, 3/4)$ $(s = 2)$ and $(7/8, 7/8)$ $(s = 3)$.

Let us first draw a further consequence from the result and let us defer the proof of Theorem 4 to the end of this section.

As an almost immediate consequence we get the following bound for the discrepancy of digital $(0, s, 2)$-nets in base 2, which improves the bounds (1) and (2).

THEOREM 5. *For the star-discrepancy D_N^* of a digital $(0, s, 2)$-net in base 2 we have*

$$ND_N^* \leq \frac{s}{3} + \frac{19}{9}.$$

This bound is (by Theorem 4(b)) up to the summand 19/9 (which could be improved to 15/9) best possible.

In particular,

$$\lim_{N\to\infty} \max \frac{ND_N^*}{\log N} = \frac{1}{3\log 2} = 0.4808\ldots$$

where the maximum is taken over all digital $(0, s, 2)$-nets in base 2.

The value $1/(3\log 2)$ is attained for example for the sequence of Hammersley nets.

Proof. We have

$$D_N^* \leq D_N^{\mathrm{d}} + \frac{2}{N} - \frac{1}{N^2},$$

hence by Theorems 1 and 3,

$$ND_N^* \leq 2 + \max_{\beta\, s\text{-bit}} \sum_{u=0}^{s-1} \|2^u\beta\| - \frac{1}{2^s} \leq \frac{s}{3} + \frac{19}{9}.$$

From this and from Theorem 4,

$$\lim_{N\to\infty} \max \frac{ND_N^*}{\log N} = \frac{1}{3\log 2}. \ \blacksquare$$

For the proof of part (b) of Theorem 4 we need some notation:

REMARK 8. For

$$\alpha = 0.a_1 \ldots a_t \ldots a_s, \qquad \beta = 0.b_1 \ldots b_{s-t} \ldots b_s$$

we define

$$\alpha_t := 0.a_1 \ldots a_t, \qquad \beta_t := 0.b_{s+1-t} \ldots b_s,$$
$$\overline{\alpha}_t := 0.a_{t+1} \ldots a_s, \qquad \overline{\beta}_t := 0.b_1 \ldots b_{s-t}.$$

Further, set

$$\Sigma_s(\alpha,\beta) := \sum_{u=0}^{s-1} \|2^u\beta\|\sigma(u) \quad \text{with} \quad \sigma(u) := a_{s-u} \oplus a_{s+1-j(u)}.$$

In $\sigma(u)$ we usually set $a_{s+1-j(u)} = 0$ as long as $j(u) = 0$. If in this case we alternatively set $a_{s+1-j(u)} := 1$ then we denote the corresponding sum by $\Sigma_s^1(\alpha,\beta)$.

Further we define

$$T_s(\alpha,\beta) := \alpha + \beta + \Sigma_s(\alpha,\beta).$$

For $\kappa,\tau \in \mathbb{R}$ we more generally define

$$T_s^{\tau,\kappa}(\alpha,\beta) := \tau\alpha + \kappa\beta + \Sigma_s(\alpha,\beta).$$

Now

$$\begin{aligned} T_s(\alpha,\beta) &= \alpha + \beta + \Sigma_s(\alpha,\beta) \\ &= \alpha + \beta + \Sigma_{s-t}(\overline{\alpha}_t,\overline{\beta}_t) + \beta_t \sum_{u=0}^{s-t-1} \frac{(-1)^{b_{u+1}}}{2^{s-t-u}}\,\sigma(u) + \widetilde{\Sigma}_t(\alpha_t,\beta_t). \end{aligned}$$

Here $\widetilde{\Sigma}_t(\alpha_t,\beta_t)$ is either $\Sigma_t(\alpha_t,\beta_t)$ or $\Sigma_t^1(\alpha_t,\beta_t)$.

Since $\alpha = \alpha_t + \frac{1}{2^t}\overline{\alpha}_t$ and $\beta = \overline{\beta}_t + \frac{1}{2^{s-t}}\beta_t$ we get

$$T_s(\alpha,\beta) = T_{s-t}^{\tau,1}(\overline{\alpha}_t,\overline{\beta}_t) + \widetilde{T}_t^{1,\kappa_t}(\alpha_t,\beta_t),$$

where $\widetilde{T}$ is defined via $\widetilde{\Sigma}$ instead of Σ, and $\tau = 1/2^t$, and

$$\kappa_t = \frac{1}{2^{s-t}} + \sum_{u=0}^{s-t-1} \frac{(-1)^{b_{u+1}}}{2^{s-t-u}}\,\sigma(u).$$

Here it is important to note that κ only depends on the form of $\overline{\alpha}_t$ and $\overline{\beta}_t$.

Let us consider for example $t = 6$. Then it is an easy task to show with the help of MATHEMATICA that for all $d \in \{0,\ldots,2^6-1\}$ we have

$$\Big|\max_{\alpha_6,\beta_6} T_6^{1,d/2^6}(\alpha_6,\beta_6) - \max_{\alpha_6,\beta_6} \widetilde{T}_6^{1,d/2^6}(\alpha_6,\beta_6)\Big| \le 1/2^6.$$

Hence for all κ

$$|\max_{\alpha_6,\beta_6} T_6^{1,\kappa}(\alpha_6,\beta_6) - \max_{\alpha_6,\beta_6} \widetilde{T}_6^{1,\kappa}(\alpha_6,\beta_6)| < 1/2^5.$$

Further we need the following lemma:

LEMMA 6. *If*

$$T_s(\alpha_0,\beta_0) = \max_{\alpha,\beta\, s\text{-}bit} T_s(\alpha,\beta),$$

then β_0 *has at most three consecutive equal digits* $b_i b_{i+1} b_{i+2}$, $i \geq 2$, *in its base* 2 *representation.*

Proof. First we note that the first digit of β_0 must be one, otherwise replacing β_0 by $\beta_0 + 1/2$ and choosing a suitable α_0 gives a larger value T.

Then we note that, as is easily calculated, the special choice

$$\alpha' = 0.101\ldots 1011, \qquad \beta' = 0.101\ldots 1011$$

if s is even and

$$\alpha' = 0.1101\ldots 1011, \qquad \beta' = 0.1010\ldots 0111$$

if s is odd gives the value

$$T_s(\alpha',\beta') = \frac{s}{3} + \frac{13}{9} + \frac{1}{2^s} - (-1)^s \frac{4}{9}\cdot\frac{1}{2^s}.$$

Assume now on the contrary that β_0 has at least four equal digits $b_i b_{i+1} b_{i+2} b_{i+3}$, $i \geq 2$, in its base 2 representation. Assume these are ones (the other case is handled in the same way). Then

$$T_s(\alpha_0,\beta_0) \leq 1 + \beta_0 + \sum_{u=0}^{s-1} \|2^u \beta_0\|.$$

Now we can apply some of the transformations from Corollary 1 to β_0 until $b_i b_{i+1}$ is the first block of equal digits (with $i \geq 2$). Therefore

$$\beta_0 + \sum_{u=0}^{s-1} \|2^u \beta_0\|$$

will not decrease. Now we can apply two times one of the last two transformations from Corollary 1 to $b_i b_{i+1}$ and then to $b_{i+1} b_{i+2}$. Note that $\tau \geq 3/4$ in the first application and $\tau \geq 1/2$ in the second. Therefore

$$\beta_0 + \sum_{u=0}^{s-1} \|2^u \beta_0\|$$

increases at least by

$$\frac{1}{3}\cdot\frac{3}{4}\left(1 - \frac{(-1)^{i-1}}{2^{i-1}}\right) + \frac{1}{3}\cdot\frac{1}{2}\left(1 - \frac{(-1)^i}{2^i}\right) = \frac{5}{12} + \frac{(-1)^i}{3\cdot 2^i} \geq \frac{3}{8}.$$

Hence we have, by the remark at the beginning of this proof and by Corollary 2,

$$
\begin{aligned}
\frac{s}{3}+\frac{13}{9}+\frac{1}{2^s}-(-1)^s\,\frac{4}{9}\cdot\frac{1}{2^s} &\le T_s(\alpha_0,\beta_0)\\
&\le 1+\max_{\beta\ s\text{-bit}}\Big(\beta+\sum_{u=0}^{s-1}\|2^u\beta\|\Big)-\frac{3}{8}\\
&=\frac{5}{8}+\frac{s}{3}+\frac{7}{9}+(-1)^s\,\frac{2}{9\cdot 2^s},
\end{aligned}
$$

hence

$$
\frac{1}{24}+\frac{1}{2^s}\Big(1-\frac{2}{3}(-1)^s\Big)\le 0,
$$

a contradiction. ∎

Remark 9. It is easy to show with the help of a C^{++} program that the assertion of Theorem 4(b) holds for $s \le 11$.

In fact it is not difficult to prove (with the help of Lemmas 5 and 6) that the extremal values α_0, β_0 from Theorem 4(b) must have the property that $a_{s-u} \oplus a_{s+1-j(u)} = 1$ for all $u = 0, \ldots, s-1$. Hence for every β_0 there is only one possible α_0. So it was easily possible to carry out the numerical calculation with Mathematica.

Proof of Theorem 4. (a) We use Example 2. For a given β the value

$$
\Delta(\alpha,\beta)=\sum_{u=0}^{s-1}\|2^u\beta\|(a_{s-u}\oplus a_{s+1-j(u)})
$$

always becomes maximal if α is chosen such that $a_{s-u} \oplus a_{s+1-j(u)} = 1$ for all u. Hence D_N^{d} is attained for the β maximizing

$$
\sum_{u=0}^{s-1}\|2^u\beta\|
$$

(those are provided by Theorem 3) and the corresponding α. This gives the values claimed in the result.

For the second successive maximum there are principally two possible cases: either $a_{s-u} \oplus a_{s+1-j(u)} = 1$ for all u, and then β must be of the form from Theorem 3(b), or $a_{s-u} \oplus a_{s+1-j(u)} = 0$ for some u. But comparing Theorem 3 and Lemma 5 shows that only the first case can give the second successive maximum.

(b) For α, β s-bit $\Delta(\alpha,\beta)$ always is positive by Example 2. Hence D_N^* will certainly be attained for intervals of the form

$$
[0,\alpha-1/2^s]\times[0,\beta-1/2^s]
$$

with α, β s-bit, and therefore

$$
ND_N^*=\max_{\alpha,\beta\ s\text{-bit}}(\Delta(\alpha,\beta)+\alpha+\beta)-1/2^s
$$

(see Remark 2). By Remark 9 it suffices to assume that $s \geq 12$. Let $\alpha^{(0)}, \beta^{(0)}$ be such that

$$T_s(\alpha^{(0)}, \beta^{(0)}) = \max_{\alpha,\beta\ s\text{-bit}} T_s(\alpha, \beta).$$

By Lemma 6, $\beta^{(0)}$ has at most three consecutive equal digits (after the first place) and the first digit b_1 of $\beta^{(0)}$ is 1. Assume there is a $u \leq s-12$ with $\sigma(u) = 0$ (see Remark 8 for the notations here and in the following), and let u_0 be maximal with this property. Then change $a_{s-u_0}, \ldots, a_7$ so that $\sigma(u_0)$ becomes 1 and $\sigma(u_0+1), \ldots, \sigma(s-7)$ remain unchanged. Thereby κ_6 changes at most by $1/2^{s-6-u_0} \leq 1/2^6$. Finally choose $a_6, \ldots, a_1$ and $b_{s-5}, \ldots, b_s$ so that $\widetilde{T}^{1,\kappa_6}(\alpha'_6, \beta'_6)$ becomes maximal for the new values α', β'. Then (see Remark 8),

$$\begin{aligned}
T_s(\alpha', \beta') &= T^{\tau,1}_{s-6}(\overline{\alpha}'_6, \overline{\beta}'_6) + \widetilde{T}^{1,\kappa'_6}_6(\alpha'_6, \beta'_6) \\
&\qquad \text{(note that we obtain a new summand of value at least } 1/4, \\
&\qquad \text{but } \alpha \text{ may decrease to almost zero)} \\
&\geq T^{\tau,1}_{s-6}(\overline{\alpha}^{(0)}_6, \overline{\beta}^{(0)}_6) + \frac{1}{4} - \tau - |\kappa'_6 - \kappa_6| + \widetilde{T}^{1,\kappa_6}_6(\alpha'_6, \beta'_6) \\
&\qquad \text{(by the numerical result in Remark 8; note that the tilde} \\
&\qquad \text{on } \widetilde{T} \text{ is here related to } \alpha', \beta' \text{ and in the following line to} \\
&\qquad \alpha^{(0)}, \beta^{(0)}) \\
&\geq T^{\tau,1}_{s-6}(\overline{\alpha}^{(0)}_6, \overline{\beta}^{(0)}_6) + \frac{1}{4} - \tau - \frac{1}{2^6} + \widetilde{T}^{1,\kappa'_6}_6(\alpha^{(0)}_6, \beta^{(0)}_6) - \frac{1}{2^5} \\
&> T_s(\alpha^{(0)}, \beta^{(0)}) + \frac{1}{2^4} - \frac{4}{2^6} \\
&= T_s(\alpha^{(0)}, \beta^{(0)}),
\end{aligned}$$

a contradiction. Hence

$$\begin{aligned}
T_s(\alpha^{(0)}, \beta^{(0)}) = \beta^{(0)} &+ \sum_{u=0}^{s-12} \|2^u \beta^{(0)}\| \\
&+ \frac{1}{2^{11}} \overline{\alpha}^{(0)}_{11} + \alpha^{(0)}_{11} + \widetilde{\Sigma}_{11}(\alpha^{(0)}_{11}, \beta^{(0)}_{11}).
\end{aligned}$$

Therefore by Corollary 1, $b^{(0)}_1, \ldots, b^{(0)}_{s-11}$ and $a^{(0)}_{12}, \ldots, a^{(0)}_s$ must be of the form (we concentrate on "s odd", "s even" being carried out quite analogously)

$$\overline{\beta}^{(0)}_{11} = 0.110101\ldots01, \qquad \overline{\alpha}^{(0)}_{11} = 0.0101\ldots0111$$

or

$$\overline{\beta}^{(0)}_{11} = 0.1010\ldots011, \qquad \overline{\alpha}^{(0)}_{11} = 0.0101\ldots011.$$

So it remains to maximize $\widetilde{T}^{1,\kappa}_{11}(\alpha_{11}, \beta_{11})$.

In the first case we have

$$\left|\kappa+\frac{1}{3}\left(1-\frac{1}{2^{12}}\right)\right|<\frac{1}{2^{13}},$$

in the second case we have

$$\left|\kappa-\frac{1}{3}\left(1-\frac{1}{2^{12}}\right)\right|<\frac{1}{2^{13}},$$

so it suffices to maximize

$$\widetilde{T}_{11}^{1,-\frac{1}{3}(1-1/2^{12})}(\alpha_{11},\beta_{11})\quad\text{respectively}\quad\widetilde{T}_{11}^{1,\frac{1}{3}(1-1/2^{12})}(\alpha_{11},\beta_{11}).$$

This is easily done with a MATHEMATICA program and the result follows. ∎

5. A class of nets with smaller star-discrepancy. We have seen in Theorem 5 that the Hammersley net essentially is the "worst" distributed digital $(0,s,2)$-net in base 2.

We will show here that the star-discrepancy of the nets generated by

$$C_1=\begin{pmatrix}1&0&\dots&0&0\\0&1&\dots&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\0&0&\dots&1&0\\0&0&\dots&0&1\end{pmatrix}\quad\text{and}\quad C_2=\begin{pmatrix}1&1&\dots&1&1\\1&1&\dots&1&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\1&1&\dots&0&0\\1&0&\dots&0&0\end{pmatrix}$$

is essentially smaller. Indeed it seems, by numerical experiments carried out by Entacher, that these nets are the essentially best distributed digital $(0,s,2)$-nets in base 2. We have

THEOREM 6. *For the star-discrepancy D_N^* of the digital net in base 2 generated by*

$$C_1=\begin{pmatrix}1&0&\dots&0&0\\0&1&\dots&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\0&0&\dots&1&0\\0&0&\dots&0&1\end{pmatrix}\quad\textit{and}\quad C_2=\begin{pmatrix}1&1&\dots&1&1\\1&1&\dots&1&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\1&1&\dots&0&0\\1&0&\dots&0&0\end{pmatrix}$$

we have

$$\frac{ND_N^*}{s}\ge 0.2 \tag{7}$$

for all N $(N=2^s)$ and

$$\limsup_{N\to\infty}\frac{ND_N^*}{s}\le 0.226341\ldots \tag{8}$$

REMARK 10. Hence for these nets we have

$$0.2885\ldots=\frac{1}{5\log 2}\le\liminf_{N\to\infty}\frac{ND_N^*}{\log N}\le\limsup_{N\to\infty}\frac{ND_N^*}{\log N}\le 0.32654\ldots$$

Indeed we conjecture that

$$\lim_{N\to\infty} \frac{ND_N^*}{\log N} = \frac{1}{5\log 2},$$

and that this is the best possible value at all, i.e.

$$\lim_{N\to\infty} \min \frac{ND_N^*}{\log N} = \frac{1}{5\log 2},$$

where the minimum is taken over all digital $(0,s,2)$-nets in base 2.

Proof of Theorem 6. We will show that the lower bound even holds for

$$\max_{\alpha,\beta\ s\text{-bit}} \Delta(\alpha,\beta),$$

and also for the upper bound it suffices to consider $\Delta(\alpha,\beta)$ for α,β s-bit. Recall from Example 3 that for α,β s-bit we have

$$\begin{aligned}\Delta(\alpha,\beta) &= \sum_{u=0}^{s-1} \|2^u\beta\|(-1)^{a_1+\ldots+a_{s-u}} \frac{(-1)^{a_{s-u}} - (-1)^{a_{s+1-j(u)}}}{2} \\ &= \sum_{u=0}^{s-1} \|2^u\beta\|(-1)^{a_1+\ldots+a_{s-u-1}}(a_{s-u}\oplus a_{s+1-j(u)}),\end{aligned}$$

where

$$j(u) := \begin{cases} 0 & \text{if } u=0, \\ 0 & \text{if } a_1\oplus\ldots\oplus a_{s+1-j} = b_j \text{ for } j=1,\ldots,u, \\ \max\{j\le u : a_1\oplus\ldots\oplus a_{s+1-j}\ne b_j\} & \text{otherwise.}\end{cases}$$

We set $\widetilde{a}_i := a_1\oplus\ldots\oplus a_{s+1-i}$ and $\widetilde{\alpha} := 0.\widetilde{a}_1\ldots\widetilde{a}_s$. Then

$$a_{s+1-i} = \widetilde{a}_i \oplus \widetilde{a}_{i+1}, \qquad a_{s+1-j(u)} = \widetilde{a}_{r(u)} \oplus \widetilde{a}_{r(u)+1},$$

where

$$r(u) := \begin{cases} 0 & \text{if } u=0, \\ 0 & \text{if } b_j = \widetilde{a}_j \text{ for } j=1,\ldots,u, \\ \max\{r\le u : b_j\ne \widetilde{a}_j\} & \text{otherwise,}\end{cases}$$

and where we have to set $\widetilde{a}_{r(u)}\oplus\widetilde{a}_{r(u)+1} := 0$ if $r(u)=0$ and $\widetilde{a}_{s+1} := 0$. Then

$$\Delta(\alpha,\beta) = \sum_{u=0}^{s-1} \|2^u\beta\|\varrho(u) =: \delta(\widetilde{\alpha},\beta),$$

where

$$\varrho(u) := (-1)^{\widetilde{a}_{u+2}}(\widetilde{a}_{u+1}\oplus\widetilde{a}_{u+2}\oplus\widetilde{a}_{r(u)}\oplus\widetilde{a}_{r(u)+1}).$$

To obtain the lower bound consider

$$\beta = 0.00100010001\ldots b_s, \qquad \widetilde{\alpha} = 0.10001000100\ldots a_s$$

with the exception that $b_s = 1$ instead of 0 if $s = 4l+1$ or $s = 4l+2$. Then

$$\varrho(u) = \begin{cases} -1 & \text{if } u = 4l+3, \\ 1 & \text{otherwise} \end{cases}$$

with the only exception that $\varrho(s-1) = 0$ if $s = 4l$. Then

$$\beta = \sum_{i=0}^{[s/4]-1} \frac{1}{2^{4i+3}} + \frac{b_s}{2^s},$$

hence

$$\|2^u\beta\| = \sum_{i=\lceil u/4-1/2\rceil}^{[s/4]-1} \frac{1}{2^{4i+3-u}} + \frac{b_s}{2^{s-u}}$$

for $u \neq 4l+2$ and it is 1 minus this quantity if $u = 4l+2$. So

$$\delta(\widetilde{\alpha},\beta) = \sum_{l=0}^{[(s-5)/4]} (\|2^{4l}\beta\| + \|2^{4l+1}\beta\| + \|2^{4l+2}\beta\| - \|2^{4l+3}\beta\|) + R,$$

with

$$R = \begin{cases} 1/2 & \text{if } s = 4l+1, \\ 3/4 & \text{if } s = 4l+2, \\ 7/8 & \text{else.} \end{cases}$$

Inserting for $\|2^u\beta\|$ and evaluating the resulting finite geometric series then yields

$$\delta(\widetilde{\alpha},\beta) = \frac{4}{5}\left[\frac{s-1}{4}\right] + \frac{16^{[(s-1)/4]}-1}{16^{[s/4]}} \cdot \begin{cases} 2/25 + 7/8 & \text{if } s = 4l, \\ (-11/50) + 1/2 & \text{if } s = 4l+1, \\ (-7/100) + 3/4 & \text{if } s = 4l+2, \\ 1/200 + 7/8 & \text{if } s = 4l+3. \end{cases}$$

Now it is a simple task to check that in each of the four cases $\delta(\widetilde{\alpha},\beta)/s$ is decreasing to $1/5$, and so the lower bound follows.

To obtain the upper bound consider for given $r \in \mathbb{N}$ the quantity

$$\delta_r := \sup_{\alpha,\beta} \sum_{u=0}^{r-1} \varrho(u)\|2^u\beta\|,$$

where the supremum is taken over all $\beta \in [0,1)$ and over all $r+1$-bit $\alpha = 0.a_1 \ldots a_{r+1}$. (Note that this means that a_{r+1} is not automatically set to 0 as is done for r-bit α.)

This supremum is obviously attained (respectively approached) in the following form: let u_0 be the largest index such that $\varrho(u_0) \neq 0$; then $\varrho(u_0) = 1$. Further the supremum is attained for some β with $b_{r+1} = b_{r+2} = \ldots = 0$ if $b_{u_0+1} = 1$ and it is approached by β with $b_{r+1} = b_{r+2} = \ldots = 1$ if $b_{u_0+1} = 0$.

So it can be shown for example with MATHEMATICA that

$$\delta_{11} = \frac{5099}{2048} = 2.48975\ldots,$$

and this value is attained with $b_{u_0+1} = 1$.

Now for s with $s = 11q + w$, $0 \le w \le 10$, for all $\widetilde{\alpha}, \beta$ we have $\delta(\widetilde{\alpha}, \beta) \le q\delta_{11} + w$, hence

$$\frac{\delta(\widetilde{\alpha}, \beta)}{s} \le \frac{1}{s}\left[\frac{s}{11}\right] \cdot 2.48975\ldots + \frac{10}{s},$$

which tends to $0.226341\ldots$ as $s \to \infty$, and the result follows. ■

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