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Artin L-functions and modular forms associated to quasi-cyclotomic fields

by

SUNGHAN BAE (Daejeon), YONG HU (Beijing), and LINSHENG YIN (Beijing)

1. Introduction. A quadratic extension of a cyclotomic field, which is non-abelian Galois over the rational number field $\mathbb{Q}$, is called a *quasi-cyclotomic field*. All quasi-cyclotomic fields are described explicitly in [8] following the work in [1] and [3]. Actually for any cyclotomic field $\mathbb{Q}(\zeta_n)$ we construct a canonical $\mathbb{Z}/2\mathbb{Z}$-basis of the quotient space of $\{\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2} \mid \mathbb{Q}(\zeta_n, \sqrt{\alpha})/\mathbb{Q}$ is Galois$\}$ modulo the subspace $\{\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2} \mid \mathbb{Q}(\zeta_n, \sqrt{\alpha})/\mathbb{Q}$ is abelian$\}$. The minimal quasi-cyclotomic field containing the square root of a special element of the basis is called a primary quasi-cyclotomic field. L. S. Yin and C. Zhang [7] have studied the arithmetic of any quasi-cyclotomic field. In this paper we determine all irreducible representations of primary quasi-cyclotomic fields. Our methods enable one to determine the irreducible representations of an arbitrary quasi-cyclotomic field. We also compute the Artin conductors of the representations and the Artin L-functions for a class of quasi-cyclotomic fields. They correspond to a series of normalized newforms of weight one by Deligne–Serre's theorem [6, Th. 2]. We describe these modular forms explicitly.

First we recall the construction of primary quasi-cyclotomic fields. Let S be the set consisting of -1 and all prime numbers. For $p \in S$, we put $\bar{p} = 4, 8, p$ and set $p^* = -1, 2, (-1)^{(p-1)/2}p$ if $p = -1, 2$ and an odd prime number, respectively. For prime numbers $p < q$, we define

$$v_{pq} = \prod_{i=0}^{(p-1)/2} \prod_{j=0}^{(q-1)/2} \frac{\sin \frac{iq+j}{pq}\pi}{\sin \frac{jp+i}{pq}\pi} \quad ((i,j) \neq (0,0),\ p > 2)$$

and

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$$v_{2q}=\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{4q}}\prod_{j=1}^{(q-1)/2}\frac{\sin\frac{j\pi}{2q}\cdot\sin\frac{2j-1}{4q}\pi}{2\sin\frac{4j+1}{4q}\pi\cdot\sin\frac{j}{q}\pi\cdot\sin\frac{2j-1}{2q}\pi}.$$

For $p<q\in S$, we put

$$u_{pq}=\begin{cases}\sqrt{q^*} & \text{if } p=-1,\\ v_{pq} & \text{if } p=2 \text{ or } p\equiv q\equiv 1 \bmod 4,\\ \sqrt{p}\cdot v_{pq} & \text{if } p\equiv 1,\ q\equiv 3 \bmod 4,\\ \sqrt{q}\cdot v_{pq} & \text{if } p\equiv 3,\ q\equiv 1 \bmod 4,\\ \sqrt{pq}\cdot v_{pq} & \text{if } p\equiv q\equiv 3 \bmod 4.\end{cases}$$

The canonical $\mathbb{Z}/2\mathbb{Z}$-basis of the quotient space mentioned above is a subset of $\{u_{pq}\mid p<q\in S\}$. For $p<q\in S$ let $K=\mathbb{Q}(\zeta_{\bar{p}q})$ be the cyclotomic field of conductor $\bar{p}q$ and let $\widetilde{K}=K(\sqrt{u_{pq}})$. Then $\widetilde{K}$ is the smallest quasi-cyclotomic fields containing $\sqrt{u_{pq}}$. We call these fields $\widetilde{K}$ *primary quasi-cyclotomic fields*. Let $G=\mathrm{Gal}(K/\mathbb{Q})$ and $\widetilde{G}=\mathrm{Gal}(\widetilde{K}/\mathbb{Q})$. We always denote by ε the unique non-trivial element of $\mathrm{Gal}(\widetilde{K}/K)$. If $(p,q)=(-1,2)$, then the group G is generated by two elements σ_{-1} and σ_2, where $\sigma_{-1}(\zeta_8)=\zeta_8^{-1}$ and $\sigma_2(\zeta_8)=\zeta_8^5$. If $p=-1$ and $q\neq 2$, or if $p>2$, then G is generated by two elements σ_p and σ_q, where $\sigma_p(\zeta_p)=\zeta_p^a$, $\sigma_p(\zeta_q)=\zeta_q$ and $\sigma_q(\zeta_p)=\zeta_p$, $\sigma_q(\zeta_q)=\zeta_q^b$, with a,b being generators of $(\mathbb{Z}/\bar{p}\mathbb{Z})^*$ and $(\mathbb{Z}/q\mathbb{Z})^*$ respectively. If $p=2$, then G is generated by three elements σ_{-1},σ_2 and σ_q, where σ_{-1}, σ_2 act on ζ_8 as above and on ζ_q trivially, and σ_q acts on ζ_q as above and on ζ_8 trivially.

Next we describe the group $\widetilde{G}$ by generators and relations. An element $\sigma\in G$ has two lifts in $\widetilde{G}$. By [6, Sect. 3] the action of the two lifts on $\sqrt{u_{pq}}$ has the form $\pm\alpha\sqrt{u_{pq}}$ or $\pm\alpha\sqrt{u_{pq}}/\sqrt{-1}$ with $\alpha>0$. We fix the lift $\widetilde{\sigma}$ of σ to be the one with a positive sign. Then the other lift of σ is $\widetilde{\sigma}\varepsilon$. The group $\widetilde{G}$ is generated by $\varepsilon,\widetilde{\sigma}_p$ and $\widetilde{\sigma}_q$ (and $\widetilde{\sigma}_{-1}$ if $p=2$). Clearly ε commutes with the other generators. In addition, we have $\widetilde{\sigma}_p\widetilde{\sigma}_q=\widetilde{\sigma}_q\widetilde{\sigma}_p\varepsilon$ (and $\widetilde{\sigma}_{-1}$ commutes with $\widetilde{\sigma}_2$ and $\widetilde{\sigma}_q$ if $p=2$). For an element g of a group, we denote by $|g|$ the order of g in the group. Let $\log_{-1}:\{\pm1\}\to\mathbb{Z}/2\mathbb{Z}$ be the unique isomorphism. For an odd prime p and an integer a with $p\nmid a$, let $\left(\frac{a}{p}\right)$ be the quadratic residue symbol. We also define $\left(\frac{a}{2}\right)=\left(\frac{a}{-1}\right)=1$ for any a. Then we have (see [6, Th. 3])

$$|\widetilde{\sigma}_p|=\left(1+\log_{-1}\left(\frac{q^*}{p}\right)\right)|\sigma_p| \quad\text{and}\quad |\widetilde{\sigma}_q|=\left(1+\log_{-1}\left(\frac{p^*}{q}\right)\right)|\sigma_q|,$$

with the exception that $|\widetilde{\sigma}_2|=2|\sigma_2|$ when $(p,q)=(-1,2)$. If $p=2$, we have furthermore $|\widetilde{\sigma}_{-1}|=|\sigma_{-1}|$. Thus we have completely determined the group $\widetilde{G}$ by generators and relations.

2. Abelian subgroup of index 2. In this section we construct a special abelian subgroup of $\widetilde{G}$ of index 2 and determine its structure. We consider the following three cases separately:

Case A: $|\widetilde{\sigma}_p| = |\sigma_p|$ and $|\widetilde{\sigma}_q| = |\sigma_q|$;
Case B: $|\widetilde{\sigma}_p| = 2|\sigma_p|$, $|\widetilde{\sigma}_q| = |\sigma_q|$ or $|\widetilde{\sigma}_p| = |\sigma_p|$, $|\widetilde{\sigma}_q| = 2|\sigma_q|$;
Case C: $|\widetilde{\sigma}_p| = 2|\sigma_p|$ and $|\widetilde{\sigma}_q| = 2|\sigma_q|$.

All the three cases may happen: Case A if and only if $\left(\frac{p^*}{q}\right) = \left(\frac{q^*}{p}\right) = 1$; Case B if and only if $\left(\frac{p^*}{q}\right) \neq \left(\frac{q^*}{p}\right)$ or $(p,q) = (-1,2)$; Case C if and only if $\left(\frac{p^*}{q}\right) = \left(\frac{q^*}{p}\right) = -1$.

In Case A, we define the subgroup N of $\widetilde{G}$ to be

$$\text{(A2.1)} \qquad N = \begin{cases} \langle \widetilde{\sigma}_{-1}, \widetilde{\sigma}_2, \widetilde{\sigma}_q^2, \varepsilon\rangle & \text{if } p = 2, \\ \langle \widetilde{\sigma}_p, \widetilde{\sigma}_q^2, \varepsilon\rangle & \text{if } p \neq 2. \end{cases}$$

It is easy to see that the subgroup N is abelian of index 2 in $\widetilde{G}$ and is a direct sum of the cyclic groups generated by the above elements. Thus we have

$$\text{(A2.2)} \qquad N \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = -1, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p > 2. \end{cases}$$

In Case B, we define the subgroup N of $\widetilde{G}$ to be

$$\text{(B2.1)} \qquad N = \begin{cases} \langle \widetilde{\sigma}_{-1}, \widetilde{\sigma}_2, \widetilde{\sigma}_q^2\rangle & \text{if } p = 2, \\ \langle \widetilde{\sigma}_p, \widetilde{\sigma}_q^2\rangle & \text{if } p \neq 2 \text{ and } |\widetilde{\sigma}_q| = 2|\sigma_q|, \\ \langle \widetilde{\sigma}_p^2, \widetilde{\sigma}_q\rangle & \text{if } |\widetilde{\sigma}_p| = 2|\sigma_p|. \end{cases}$$

Again N is abelian and has index 2 in $\widetilde{G}$. In addition, we have

$$\text{(B2.2)} \qquad N \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } (p,q) = (-1,2), \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} & \text{if } p = -1,\ q > 2, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} & \text{if } p > 2. \end{cases}$$

In Case C, p, q are both odd prime numbers. Let $v_2(p-1)$ denote the power of 2 in $p-1$. We define the subgroup N of $\widetilde{G}$ to be

$$\text{(C2.1)} \qquad N = \begin{cases} \langle \widetilde{\sigma}_p^2, \widetilde{\sigma}_q\rangle & \text{if } v_2(p-1) \leq v_2(q-1), \\ \langle \widetilde{\sigma}_p, \widetilde{\sigma}_q^2\rangle & \text{if } v_2(p-1) > v_2(q-1). \end{cases}$$

Then N is an abelian subgroup of $\widetilde{G}$. When $v_2(p-1) \leq v_2(q-1)$, we have

$$|N| = \frac{|\widetilde{\sigma}_p^2| \cdot |\widetilde{\sigma}_q|}{|\langle \widetilde{\sigma}_p^2\rangle \cap \langle \widetilde{\sigma}_q\rangle|} = \frac{(p-1)\cdot 2(q-1)}{2},$$

thus $[\widetilde{G}:N]=2$ and N is a normal subgroup of $\widetilde{G}$. We have the same result when $v_2(p-1)>v_2(q-1)$. Although the subgroup $\langle\widetilde{\sigma}_p^2,\widetilde{\sigma}_q\rangle$ is always an abelian subgroup of $\widetilde{G}$ of index 2, when $v_2(p-1)>v_2(q-1)$ we are not able to get all irreducible representations of $\widetilde{G}$ from this subgroup. So we define N in two cases.

Next we determine the structure of the subgroup N in Case C. We consider the case $v_2(p-1)\le v_2(q-1)$ in detail. Let $d=\gcd((p-1)/2,q-1)$, $s=(p-1)/2d$ and $t=(q-1)/d$. Choose $u,v\in\mathbb{Z}$ such that $us+vt=1$. We have the relations

$$(\widetilde{\sigma}_p^2)^{p-1}=1,\qquad(\widetilde{\sigma}_p^2)^{(p-1)/2}=\varepsilon=\widetilde{\sigma}_q^{q-1}.$$

Let M be the free abelian group generated by two words α, β. Let

$$\alpha_1=(p-1)\alpha,\qquad\beta_1=\frac{p-1}{2}\alpha-(q-1)\beta,$$

and let M_1 be the subgroup of M generated by α_1,β_1. Then M_1 is the kernel of the homomorphism

$$M\to N,\qquad\alpha\mapsto\widetilde{\sigma}_p^2,\qquad\beta\mapsto\widetilde{\sigma}_q.$$

So we have $N\cong M/M_1$. Define the matrix

$$A=\begin{pmatrix}p-1&(p-1)/2\\0&1-q\end{pmatrix}.$$

Then $(\alpha_1,\beta_1)=(\alpha,\beta)\cdot A$. We determine the structure of M_1 by considering the standard form of A. Define

$$P=\begin{pmatrix}u&v\\-t&s\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}),\qquad Q=\begin{pmatrix}1&2tv-1\\-1&-2tv+2\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}).$$

Then

$$B=PAQ=\begin{pmatrix}d&0\\0&-2s(q-1)\end{pmatrix}$$

is the standard form of A. Let

$$(\tau,\mu)=(\alpha,\beta)P^{-1}\quad\text{and}\quad(\tau_1,\mu_1)=(\alpha_1,\beta_1)Q.$$

Then $(\tau_1,\mu_1)=(\tau,\mu)B$, $M=\mathbb{Z}\tau\oplus\mathbb{Z}\mu$ and $M_1=\mathbb{Z}d\tau\oplus\mathbb{Z}2s(q-1)\mu$. We thus have

$$N\cong M/M_1\cong\mathbb{Z}/d\mathbb{Z}\oplus\mathbb{Z}/2s(q-1)\mathbb{Z}.$$

By abuse of notation, we also write

$$(\tau,\mu)=(\widetilde{\sigma}_p^2,\widetilde{\sigma}_q)P^{-1}=(\widetilde{\sigma}_p^{2s}\widetilde{\sigma}_q^{t},\widetilde{\sigma}_p^{-2v}\widetilde{\sigma}_q^{u}).$$

Then τ,μ are of order $d,2s(q-1)$ respectively, and N is a direct sum of $\langle\tau\rangle$ and $\langle\mu\rangle$. We have $\widetilde{\sigma}_p^2=\tau^u\mu^{-t}$ and $\widetilde{\sigma}_q=\tau^v\mu^s$. When $v_2(p-1)>v_2(q-1)$, we get the structure of N in the same way. So in Case C we have

$$\text{(C2.2)} \qquad N \cong \begin{cases} \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z} & \text{if } v_2(p-1) \le v_2(q-1), \\ \mathbb{Z}/d'\mathbb{Z} \oplus \mathbb{Z}/2s'(p-1)\mathbb{Z} & \text{if } v_2(p-1) > v_2(q-1), \end{cases}$$

where $d = \gcd((p-1)/2, q-1)$, $s = (p-1)/2d$ and $d' = \gcd(p-1, (q-1)/2)$, $s' = (q-1)/2d'$.

Now we summarize our results in the following

PROPOSITION 2.1. *The abelian subgroup N of the group $\widetilde{G}$ of index* 2 *defined in* (A2.1), (B2.1) *and* (C2.1) *has the structure described in* (A2.2), (B2.2) *and* (C2.2) *in Cases A, B and C, respectively. In particular, every irreducible representation of $\widetilde{G}$ has dimension* 1 *or* 2*.*

3. 2-dimensional representations. We determine all irreducible representations of $\widetilde{G}$ in this section. We will freely use some basic facts from representation theory. For the details, see [5].

It is well-known that the 1-dimensional representations of $\widetilde{G}$ correspond bijectively to those of the maximal abelian quotient G of $\widetilde{G}$, which are Dirichlet characters. So we construct the 2-dimensional irreducible representations of $\widetilde{G}$. From the dimension formula for all irreducible representations, we see that $\widetilde{G}$ has $|G|/4$ irreducible representations of dimension 2, up to isomorphism. Let N be the subgroup of $\widetilde{G}$ defined in the previous section. Let $\widetilde{G} = N \cup \sigma N$ be the decomposition into cosets. If $\rho : N \to \mathbb{C}^*$ is a representation of N, the induced representation $\widetilde{\rho}$ of ρ is a representation of $\widetilde{G}$ of dimension 2. The space of the representation $\widetilde{\rho}$ is $V = \mathrm{Ind}_N^{\widetilde{G}}(\mathbb{C}) = \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}[N]} \mathbb{C}$ with basis $e_1 = 1 \otimes 1$ and $e_2 = \sigma \otimes 1$. The group homomorphism

$$\widetilde{\rho} : \widetilde{G} \to \mathrm{GL}(V) \simeq \mathrm{GL}_2(\mathbb{C})$$

is given by

$$\text{(3.1)} \qquad \widetilde{\rho}(\widetilde{\sigma}) = \begin{pmatrix} \rho(\widetilde{\sigma}) & \rho(\widetilde{\sigma}\sigma) \\ \rho(\sigma^{-1}\widetilde{\sigma}) & \rho(\sigma^{-1}\widetilde{\sigma}\sigma) \end{pmatrix}, \qquad \forall \widetilde{\sigma} \in \widetilde{G},$$

where $\rho(\widetilde{\sigma}) = 0$ if $\widetilde{\sigma} \notin N$. The representation $\widetilde{\rho}$ is irreducible if and only if $\rho \not\cong \rho^\tau$ for every $\tau \in \widetilde{G} \setminus N$, where ρ^τ is the conjugate representation of ρ defined by

$$\rho^\tau(x) = \rho(\tau^{-1} x \tau), \qquad \forall x \in N.$$

Since N is abelian, we only need to check $\rho \not\cong \rho^\sigma$.

Now we begin to construct all 2-dimensional irreducible representations of $\widetilde{G}$. As in the previous section, we consider the three cases separately. In addition, we consider the case when p and q are odd prime numbers in detail, and only state the results when $p = -1$ or 2.

3.1. Case A. Assume $p > 2$. In this case we have $N = \langle \widetilde{\sigma}_p, \widetilde{\sigma}_q^2, \varepsilon \rangle$ and

$$N \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Every irreducible representation of N can be written as $\rho_{ijk} : N \to \mathbb{C}^*$ with

$$\rho_{ijk}(\widetilde{\sigma}_p) = \zeta_{p-1}^i, \quad \rho_{ijk}(\widetilde{\sigma}_q^2) = \zeta_{q-1}^{2j}, \quad \rho_{ijk}(\varepsilon) = (-1)^k,$$

where $0 \le i < p-1$, $0 \le j < (q-1)/2$ and $k = 0, 1$. Since $\widetilde{G} = N \cup \widetilde{\sigma}_q N$ and $\rho_{ijk}^{\widetilde{\sigma}_q}(\widetilde{\sigma}_p) = \rho_{ijk}(\varepsilon)\rho_{ijk}(\widetilde{\sigma}_p) = (-1)^k \rho_{ijk}(\widetilde{\sigma}_p)$, we have

$$\rho_{ijk}^{\widetilde{\sigma}_q} \ncong \rho_{ijk} \iff k = 1.$$

Write $\rho_{ij} = \rho_{ij1}$. The representation $\widetilde{\rho}_{ij} : \widetilde{G} \to \mathrm{GL}_2(\mathbb{C})$ induced from ρ_{ij} is given by

$$\text{(A3.1)} \quad \widetilde{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} \zeta_{p-1}^i & 0 \\ 0 & -\zeta_{p-1}^i \end{pmatrix}, \quad \widetilde{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2j} \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\rho}_{ij}(\varepsilon) = -I,$$

where I is the identity matrix of degree 2. Since

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p^2) = \begin{pmatrix} \zeta_{p-1}^{2i} & 0 \\ 0 & \zeta_{p-1}^{2i} \end{pmatrix} \quad \text{and} \quad \widetilde{\rho}_{ij}(\widetilde{\sigma}_q^2) = \begin{pmatrix} \zeta_{q-1}^{2j} & 0 \\ 0 & \zeta_{q-1}^{2j} \end{pmatrix},$$

we see that the representations $\widetilde{\rho}_{ij}$ with $0 \le i < (p-1)/2$, $0 \le j < (q-1)/2$ are irreducible and are not isomorphic to each other, by considering the values of the characters of these representations at $\widetilde{\sigma}_p^2$ and $\widetilde{\sigma}_q^2$. The number of these representations is $\frac{p-1}{2} \cdot \frac{q-1}{2} = \frac{|G|}{4}$. So they are all the irreducible representations of $\widetilde{G}$ of dimension 2.

Similarly, when $p = -1$, all irreducible representations of $\widetilde{G}$ of dimension 2 are $\widetilde{\rho}_j$ with $0 \le j < (q-1)/2$, where

$$\text{(A3.2)} \quad \widetilde{\rho}_j(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}_j(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2j} \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\rho}(\varepsilon) = -I,$$

and when $p = 2$, all irreducible representations of $\widetilde{G}$ of dimension 2 are $\bar{\rho}_{ij}$ with $0 \le i \le 1$ and $0 \le j < (q-1)/2$, where $\bar{\rho}_{ij}(\varepsilon) = -I$ and

$$\text{(A3.3)} \quad \bar{\rho}_{ij}(\widetilde{\sigma}_{-1}) = (-1)^i I, \quad \bar{\rho}_{ij}(\widetilde{\sigma}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2j} \\ 1 & 0 \end{pmatrix}.$$

3.2. Case B. Assume $p > 2$ and $|\widetilde{\sigma}_q| = 2|\sigma_q|$. Then $N = \langle \widetilde{\sigma}_p, \widetilde{\sigma}_q^2 \rangle$, and

$$N \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z}.$$

Any irreducible representation of N has the form $\rho_{ij} : N \to \mathbb{C}^*$, where

$$\rho_{ij}(\widetilde{\sigma}_p) = \zeta_{p-1}^i, \quad \rho_{ij}(\widetilde{\sigma}_q^2) = \zeta_{q-1}^j, \quad \rho_{ij}(\varepsilon) = \rho_{ij}(\widetilde{\sigma}_q^2)^{(q-1)/2} = (-1)^j,$$

and $0 \le i < p-1,\ 0 \le j < q-1$. It is easy to check that

$$\rho_{ij}^{\widetilde{\sigma}_q} \not\cong \rho_{ij} \iff j \equiv 1 \bmod 2.$$

The representation $\widetilde{\rho}_{ij} : \widetilde{G} \to \mathrm{GL}_2(\mathbb{C})$ induced from ρ_{ij} with odd j is given by

$$\text{(B3.1)} \qquad \widetilde{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} \zeta_{p-1}^i & 0 \\ 0 & -\zeta_{p-1}^i \end{pmatrix}, \quad \widetilde{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix}.$$

Since

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p^2) = \begin{pmatrix} \zeta_{p-1}^{2i} & 0 \\ 0 & \zeta_{p-1}^{2i} \end{pmatrix} \quad \text{and} \quad \widetilde{\rho}_{ij}(\widetilde{\sigma}_q^2) = \begin{pmatrix} \zeta_{q-1}^j & 0 \\ 0 & \zeta_{q-1}^j \end{pmatrix},$$

we see that the representations $\widetilde{\rho}_{ij}$ with $0 \le i < (p-1)/2$ and $0 \le j < q-1$, $2 \nmid j$ are irreducible and are not isomorphic to each other. The number of these representations is $|G|/4$. So they are all the irreducible representations of $\widetilde{G}$ of dimension 2.

Similarly, when $(p,q) = (-1,2)$, there is only one irreducible representation $\widetilde{\rho}_0$ of dimension 2 defined by

$$\text{(B3.2)} \qquad \widetilde{\rho}_0(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}_0(\widetilde{\sigma}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $p = -1$ and $q > 2$, all irreducible representations of dimension 2 are $\widetilde{\rho}_j$ with $0 \le j < q-1$, $2 \nmid j$, where $\widetilde{\rho}_j$ is defined by

$$\text{(B3.3)} \qquad \widetilde{\rho}_j(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}_j(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix}.$$

When $p = 2$, all irreducible representations of dimension 2 are $\bar{\rho}_{ij}$ with $0 \le i \le 1$ and $0 \le j < q-1$, $2 \nmid j$, where $\bar{\rho}_{ij}$ is defined by

$$\text{(B3.4)} \quad \bar{\rho}_{ij}(\widetilde{\sigma}_{-1}) = (-1)^i I, \quad \bar{\rho}_{ij}(\widetilde{\sigma}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix}.$$

When $|\widetilde{\sigma}_p| = 2|\sigma_p|$, all irreducible representations of dimension 2 are $\hat{\rho}_{ij}$ with $0 \le i < p-1,\ 2 \nmid i$ and $0 \le j < (q-1)/2$, where $\hat{\rho}_{ij}$ is defined by

$$\text{(B3.5)} \qquad \hat{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} 0 & \zeta_{p-1}^i \\ 1 & 0 \end{pmatrix}, \quad \hat{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} \zeta_{q-1}^j & 0 \\ 0 & -\zeta_{q-1}^j \end{pmatrix}.$$

3.3. Case C. Assume $v_2(p-1) \le v_2(q-1)$. Let

$$d = \gcd\left(\frac{p-1}{2}, q-1\right), \quad s = \frac{p-1}{2d}, \quad t = \frac{q-1}{d}, \quad us + vt = 1$$

as before. Here t must be even and u odd. Let $\tau = \widetilde{\sigma}_p^{2s} \cdot \widetilde{\sigma}_q^t$ and $\mu = \widetilde{\sigma}_p^{-2v} \cdot \widetilde{\sigma}_q^u$. Then $N = \langle \widetilde{\sigma}_p^2, \widetilde{\sigma}_q \rangle = \langle \tau, \mu \rangle$ and

$$N \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z}.$$

Any irreducible representation $\rho_{ij} : N \to \mathbb{C}^*$ is of the form

$$\rho_{ij}(\tau) = \zeta_d^i = \zeta_{(p-1)(q-1)}^{2s(q-1)i}, \qquad \rho_{ij}(\mu) = \zeta_{2s(q-1)}^j = \zeta_{(p-1)(q-1)}^{dj}.$$

From $\widetilde{\sigma}_p^2 = \tau^u \mu^{-t}$ and $\widetilde{\sigma}_q = \tau^v \mu^s$, we have

$$\rho_{ij}(\widetilde{\sigma}_p^2) = \zeta_{p-1}^{2sui-j}, \quad \rho_{ij}(\widetilde{\sigma}_q) = \zeta_{2(q-1)}^{2tvi+j}, \quad \rho_{ij}(\varepsilon) = \rho_{ij}(\widetilde{\sigma}_p^2)^{(p-1)/2} = (-1)^j.$$

It is easy to show

$$\rho_{ij}^{\widetilde{\sigma}_p} \not\cong \rho_{ij} \;\Leftrightarrow\; j \equiv 1 \bmod 2.$$

The representation $\widetilde{\rho}_{ij} : \widetilde{G} \to \mathrm{GL}_2(\mathbb{C})$ induced from ρ_{ij} with odd j is given by

$$\widetilde{\rho}_{ij}(\tau) = \begin{pmatrix} \zeta_d^i & 0 \\ 0 & \zeta_d^i \end{pmatrix}, \qquad \widetilde{\rho}_{ij}(\mu) = \begin{pmatrix} \zeta_{2s(q-1)}^j & 0 \\ 0 & -\zeta_{2s(q-1)}^j \end{pmatrix}.$$

Here in the first equality we used the fact that t is even, and in the second equality we used the fact that u is odd. Furthermore, we have

$$\text{(C3.1)} \qquad \widetilde{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} 0 & \zeta_{p-1}^{2sui-j} \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} \zeta_{2(q-1)}^{2tvi+j} & 0 \\ 0 & -\zeta_{2(q-1)}^{2tvi+j} \end{pmatrix}.$$

By considering the values of the character of $\widetilde{\rho}_{ij}$ at τ and μ^2, we see that all the representations $\widetilde{\rho}_{ij}$ with $0 \le i < d$ and $0 \le j < s(q-1)$, $2 \nmid j$ are irreducible and are not isomorphic to each other. The number of these representations is $d \cdot s(q-1)/2 = |G|/4$. So they are all the irreducible representations of $\widetilde{G}$ of dimension 2.

Similarly, if $v_2(p-1) > v_2(q-1)$, we let

$$d' = \gcd\left(p-1, \frac{q-1}{2}\right), \qquad s' = \frac{p-1}{d}, \qquad t' = \frac{q-1}{2d}, \qquad u's' + v't' = 1.$$

Then all the irreducible representations of $\widetilde{G}$ of dimension 2 are $\hat{\rho}_{ij}$ with $0 \le i < d'$ and $0 \le j < t'(p-1)$, $2 \nmid j$, where $\hat{\rho}_{ij}$ is defined by

$$\text{(C3.2)} \quad \hat{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} \zeta_{2(p-1)}^{2s'u'i+j} & 0 \\ 0 & -\zeta_{2(p-1)}^{2s'u'i+j} \end{pmatrix}, \qquad \hat{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2t'v'i-j} \\ 1 & 0 \end{pmatrix}.$$

Let $\mathrm{R}^2(\widetilde{G})$ be the set of all irreducible representations, up to isomorphism, of $\widetilde{G}$ of dimension 2. To summarize, we have proved the following

THEOREM 3.1. *All 2-dimensional irreducible representations of $\widetilde{G}$ are induced from representations of N. In detail, we have:*

In Case A

$$\mathrm{R}^2(\widetilde{G}) = \begin{cases} \{\widetilde{\rho}_j \mid 0 \le j < (q-1)/2\} & \text{if } p=-1,\\ \{\bar{\rho}_{ij} \mid i=0,1,\ 0 \le j < (q-1)/2\} & \text{if } p=2,\\ \{\widetilde{\rho}_{ij} \mid 0 \le i < (p-1)/2,\ 0 \le j < (q-1)/2\} & \text{if } p>2,\end{cases}$$

where $\widetilde{\rho}_j$, $\bar{\rho}_{ij}$ *and* $\widetilde{\rho}_{ij}$ *are defined in* (A3.2), (A3.3) *and* (A3.1) *respectively.*

In Case B

$$\mathrm{R}^2(\widetilde{G}) = \begin{cases} \{\widetilde{\rho}_0\} & \text{if } (p,q)=(-1,2),\\ \{\widetilde{\rho}_j \mid 0 \le j < q-1,\ 2 \nmid j\} & \text{if } p=-1,\ q>2,\\ \{\bar{\rho}_{ij} \mid i=0,1,\ 0 \le j < q-1,\ 2 \nmid j\} & \text{if } p=2,\\ \{\hat{\rho}_{ij} \mid 0 \le i < p-1,\ 2 \nmid i,\ 0 \le j < (q-1)/2\} & \text{if } |\widetilde{\sigma}_p| = 2|\sigma_p|,\\ \{\widetilde{\rho}_{ij} \mid 0 \le i < (p-1)/2,\ 0 \le j < q-1,\ 2 \nmid j\} & \text{otherwise},\end{cases}$$

where $\widetilde{\rho}_0$, $\widetilde{\rho}_j$, $\bar{\rho}_{ij}$, $\hat{\rho}_{ij}$ *and* $\widetilde{\rho}_{ij}$ *are defined in* (B3.2), (B3.3), (B3.4), (B3.5) *and* (B3.1) *respectively.*

In Case C

$$\mathrm{R}^2(\widetilde{G}) = \begin{cases} \{\widetilde{\rho}_{ij} \mid 0 \le i < d,\ 0 \le j < s(q-1),\ 2 \nmid j\} & \text{if } v_2(p-1) \le v_2(q-1),\\ \{\hat{\rho}_{ij} \mid 0 \le i < d',\ 0 \le j < t'(p-1),\ 2 \nmid j\} & \text{otherwise},\end{cases}$$

where $\widetilde{\rho}_{ij}$ *and* $\hat{\rho}_{ij}$ *are defined in* (C3.1) *and* (C3.2) *respectively.*

4. The Frobenius maps. This section is a preparation for the next two sections where we will compute the Artin conductors of representations and the Artin L-functions of some quasi-cyclotomic fields $\widetilde{K}$. For a prime number ℓ, we say that ℓ is ramified (resp. inert, splitting) in the relative quadratic extension $\widetilde{K}/K$ if the prime ideals of K over ℓ are ramified (resp. inert, splitting) in $\widetilde{K}$. For a prime number ℓ which is unramified in $\widetilde{K}/K$, let I_ℓ (resp. $\widetilde{I}_\ell$) be the inert group of ℓ in the extension $K/\mathbb{Q}$ (resp. $\widetilde{K}/\mathbb{Q}$). Let Fr_ℓ be the Frobenius automorphism of ℓ in G/I_ℓ, and $\widetilde{\mathrm{Fr}}_\ell$ the Frobenius automorphism of ℓ in $\widetilde{G}/\widetilde{I}_\ell$ associated to some prime ideal over ℓ.

To compute the Artin conductors of representations, we need to construct a uniformizer in the completion of $\widetilde{K}$ at a prime ideal, in particular at a prime ideal over 2. Generally we are not able to get such a uniformizer, but we can do it in the case $p=-1$. In addition, to calculate the Artin L-functions of representations, we need to know $\widetilde{\mathrm{Fr}}_\ell$, in particular for $\ell = 2$, and so we need to know the decomposition of 2 in $\widetilde{K}$. For odd $p < q \in S$, we calculated some examples by computer which suggest that 2 is always unramified in $\widetilde{K}$. But we are not able to show this. Furthermore, we do not know when 2 splits in $\widetilde{K}/K$ and when 2 is inert in $\widetilde{K}/K$. But when $p=-1$, we can solve these problems (see below). So in this paper we only compute the Artin conductors and Artin L-functions of representations in the case $p=-1$.

From now on, we always assume that $p=-1$, so $K=\mathbb{Q}(\zeta_{4q})$ and $\widetilde{K}=K(\sqrt[4]{q^*})$. In this section we determine $\widetilde{\mathrm{Fr}}_\ell$ by Fr_ℓ for $\ell=2$. In [6, Sect. 5] the decomposition of some odd prime numbers in $\widetilde{K}/K$ was determined. Now we determine the decomposition of 2 in $\widetilde{K}/K$. The result below is a more explicit reformulation of Theorem 2 in [7].

Proposition 4.1. *If $q=2$, then 2 is ramified in $\widetilde{K}/K$. If q is odd, then 2 is unramified in $\widetilde{K}/K$ if and only if $\left(\frac{2}{q}\right)=1$, and in this case 2 splits in $\widetilde{K}/K$ if $q^*\equiv 1 \bmod 16$, and is inert in $\widetilde{K}/K$ otherwise.*

Proof. We first consider the case $q=2$. The unique prime ideal of K over 2 is the principal ideal generated by $\pi_2=1-\zeta_8$. Since the ramification degree of 2 in $K/\mathbb{Q}$ is 4 and $\sqrt{2}=\pi_2(\pi_2+2\zeta_8)\zeta_8$, we deduce that 2 is ramified in $\widetilde{K}/K$ if and only if $x^2\equiv\sqrt{2} \bmod \pi_2^8$ is not solvable in the ring O_K of integers of K by [7, Th. 2(1)], which is equivalent to $\left(1+\frac{2}{\pi_2}\zeta_8\right)\zeta_8$ not being a square modulo π_2^6. Since $2=u\pi_2^4$ for some unit u, we have

$$\left(1+\frac{2}{\pi_2}\zeta_8\right)\zeta_8\equiv\zeta_8\equiv(1-\pi_2) \bmod \pi_2^3,$$

hence $\left(1+\frac{2}{\pi_2}\zeta_8\right)\zeta_8$ is not a square modulo π_2^3. So 2 is ramified in $\widetilde{K}/K$.

Now we assume that q is odd. Let $\pi_2=1-\zeta_4$. Since the ramification degree of 2 in K is 2, we see that 2 is unramified in $\widetilde{K}/K$ if and only if $x^2\equiv\sqrt{q^*} \bmod \pi_2^4$ is solvable in O_K (see [7, Th. 2(1)]). Furthermore, 2 splits in $\widetilde{K}/K$ if and only if $x^2\equiv\sqrt{q^*} \bmod \pi_2^5$ is solvable in O_K. The explicit computation of the Gauss sum gives

$$\sqrt{q^*}=\sum_{a=1}^{q-1}\left(\frac{a}{q}\right)\zeta_q^a=1+2\sum_{(\frac{a}{q})=1}\zeta_q^a.$$

Let $\alpha=\sum_{(\frac{a}{q})=1}\zeta_q^a$, $\beta=\sum_{(\frac{a}{q})=1}\zeta_{2q}^a$, and $\gamma=\sum_{(\frac{a}{q})=1}\sum_{(\frac{b}{q})=1,\,a<b}\zeta_{2q}^{a+b}$, where in the summations a,b run over $1,\dots,q-1$. Then $\alpha=\beta^2-2\gamma$, which together with the equality $2=\pi_2^2-\pi_2^3$ gives

$$\begin{aligned}\sqrt{q^*}&=1+2\beta^2-4\gamma=1+\pi_2^2\beta^2-\pi_2^3\beta^2-4\gamma\\&\equiv(1+\pi_2\beta)^2-\pi_2^3(\beta+\beta^2)+\pi_2^4(\beta-\gamma)\\&\equiv(1+\pi_2\beta)^2-\pi_2^3(\alpha+\beta)+\pi_2^4(\beta+\gamma) \bmod \pi_2^5.\end{aligned}$$

Since $\zeta_{2q}=-\zeta_q^{-(q-1)/2}=-\zeta_q^t$, where t is the inverse of 2 in $(\mathbb{Z}/q\mathbb{Z})^*$, we see that $\beta=\sum_{(\frac{a}{q})=1}(-1)^a\zeta_q^{ta}\equiv\sum_{(\frac{a}{q})=1}\zeta_q^{ta} \bmod 2$. So if $\left(\frac{2}{q}\right)=1$ we have $\alpha\equiv\beta \bmod 2$ and thus 2 is unramified in $\widetilde{K}/K$, and if $\left(\frac{2}{q}\right)=-1$ we have $\alpha+\beta\equiv\sum_{a=1}^{q-1}\zeta_q^a=-1 \bmod 2$ and thus 2 is ramified in $\widetilde{K}/K$.

Now we assume $\left(\frac{2}{q}\right) = 1$. Then $\sqrt{q^*} \bmod \pi_2^5$ is a square if and only if $\pi_2 \mid \beta + \gamma$. We consider $2(\beta+\gamma)$. Since $\alpha \equiv \beta \bmod 2$, we have

$$2(\beta+\gamma) = 2\beta + \beta^2 - \alpha \equiv \alpha(\alpha+1) \bmod 4.$$

From $\sqrt{q^*} = 1 + 2\alpha$, we see that $\alpha(\alpha+1) = (q^*-1)/4$. Since $8 \mid q^* - 1$ under the assumption $\left(\frac{2}{q}\right) = 1$, we have $\beta + \gamma \equiv (q^*-1)/8 \bmod 2$. So $\pi_2 \mid \beta + \gamma$ if and only if $\pi_2 \mid (q^*-1)/8$, that is, $2 \mid (q^*-1)/8$. The proof is complete. ∎

Now we assume that 2 is unramified in $\widetilde{K}/K$. Let $\mathrm{Fr}_2 \in G$ be such that $\mathrm{Fr}_2(\zeta_4) = 1$ and $\mathrm{Fr}_2(\zeta_q) = \zeta_q^2$. It is a Frobenius element of 2 in G modulo I_2. We have $\mathrm{Fr}_2 = \sigma_2^{b_2}$ for some $b_2 \in \mathbb{Z}$ with $2 \mid b_2$ as $\left(\frac{2}{q}\right) = 1$. Thus $\widetilde{\mathrm{Fr}}_2 = \widetilde{\sigma}_2^{b_2}$ or $\widetilde{\mathrm{Fr}}_2 = \widetilde{\sigma}_2^{b_2}\varepsilon$. We need to determine $\widetilde{\mathrm{Fr}}_2$ completely. Since $\left(\frac{2}{q}\right) = 1$, we have

$$\sqrt{q^*} \equiv (1 + \pi_2\alpha)^2 + \pi_2^4(\beta+\gamma) \bmod \pi_2^5.$$

Write $u = 1 + \pi_2\alpha$ for simplicity. Since $\sqrt{q^*} \equiv u^2 \bmod \pi_2^4$, we see $(\sqrt[4]{q^*} - u)/2 \in O_{\widetilde{K}}$. Let $\wp$ be the prime ideal of $\widetilde{K}$ over 2 associated to $\widetilde{\mathrm{Fr}}_2$. By the definition, we have

$$\widetilde{\mathrm{Fr}}_2\left(\frac{\sqrt[4]{q^*} - u}{2}\right) \equiv \left(\frac{\sqrt[4]{q^*} - u}{2}\right)^2 \equiv (\beta+\gamma) + \frac{\sqrt[4]{q^*} - u}{2} \bmod \wp.$$

On the other hand, since $\widetilde{\sigma}_q^{b_2}(\sqrt[4]{q^*}) = (-1)^{b_2/2}\sqrt[4]{q^*}$ and $\widetilde{\sigma}_q^{b_2}(u) = u$ as $2 \mid b_2$, we have

$$\widetilde{\sigma}_q^{b_2}\left(\frac{\sqrt[4]{q^*} - u}{2}\right) = \frac{(-1)^{b_2/2}\sqrt[4]{q^*} - u}{2}$$

and

$$\widetilde{\sigma}_q^{b_2}\varepsilon\left(\frac{\sqrt[4]{q^*} - u}{2}\right) = \frac{(-1)^{b_2/2+1}\sqrt[4]{q^*} - u}{2}.$$

So if $2 \mid b_2/2$ we have $\widetilde{\mathrm{Fr}}_2 = \widetilde{\sigma}_2^{b_2}$ if and only if $\pi_2 \mid \beta + \gamma$ (that is, 2 splits in $\widetilde{K}/K$), and if $2 \nmid b_2/2$ we have $\widetilde{\mathrm{Fr}}_2 = \widetilde{\sigma}_2^{b_2}$ if and only if $\pi_2 \nmid \beta+\gamma$ (that is, 2 is inert in $\widetilde{K}/K$). In the case $q \equiv 3 \bmod 4$, we can always assume that $2 \nmid b_2/2$, since if $4 \mid b_2$, we may replace b_2 by $b_2 + (q-1)$. In the case $q \equiv 1 \bmod 4$, we have $2 \mid b_2/2$ iff $2^{(q-1)/4} \equiv 1 \bmod q$ iff q has the form $A^2 + 64B^2$ for $A, B \in \mathbb{Z}$, by Exercise 28 in Chap. 5 of [4]. So we get the following result:

Proposition 4.2. *Assume that* 2 *is unramified in* $\widetilde{K}/K$. *Let* $\mathrm{Fr}_2 = \sigma_2^{b_2}$. *Then* $2 \mid b_2$. *If* $q \equiv 3 \bmod 4$, *we always assume* $b_2 \equiv 2 \bmod 4$. *Let* P_0 *be the set of prime numbers of the form* $A^2 + 64B^2$ *with* $A, B \in \mathbb{Z}$. *Then*

$$\widetilde{\mathrm{Fr}}_2 = \begin{cases} \widetilde{\sigma}_2^{b_2} & \text{if } q \notin P_0,\ 16 \nmid q^*-1, \text{ or } q \in P_0,\ 16 \mid q^*-1, \\ \widetilde{\sigma}_2^{b_2}\varepsilon & \text{if } q \in P_0,\ 16 \nmid q^*-1, \text{ or } q \notin P_0,\ 16 \mid q^*-1. \end{cases}$$

The following lemma is useful in the computation of Artin L-functions.

Lemma 4.3. *We have* $\varepsilon \in \widetilde{I}_\ell$ *if and only if* ℓ *is ramified in* $\widetilde{K}/K$.

Proof. The canonical projection $\widetilde{G} \to G \simeq \widetilde{G}/\langle\varepsilon\rangle$ induces a surjective homomorphism $\widetilde{I}_\ell \to I_\ell$ which implies the isomorphism $\widetilde{I}_\ell/\langle\varepsilon\rangle \cap \widetilde{I}_\ell \cong I_\ell$. Thus ℓ is ramified in $\widetilde{K}/K$ iff $|\widetilde{I}_\ell| = 2|I_\ell|$ iff $|\widetilde{I}_\ell \cap \langle\varepsilon\rangle| = 2$ iff $\varepsilon \in \widetilde{I}_\ell$. ∎

5. The conductors of representations. In this section we compute the Artin conductors of all 2-dimensional irreducible representations of $\widetilde{G}$ in the case $p = -1$. First we recall the definition of the Artin conductor. For details, see [2, Chap. 6].

The notations are as before. Let ℓ be a prime number in $\mathbb{Q}$, and choose a prime ideal $\mathfrak{p}$ in $\widetilde{K}$ over ℓ. Let $\widetilde{G}_\ell = \widetilde{G}(\widetilde{K}_\mathfrak{p}/\mathbb{Q}_\ell)$ be the corresponding decomposition subgroup. Let v be the normalized valuation in $\widetilde{K}_\mathfrak{p}$. For $i \geq 0$, define the ramification groups

$$\widetilde{G}_{\ell,i} = \{\sigma \in \widetilde{G}_\ell \mid v(\sigma(x) - x) > i \text{ for all } x \in O_{\widetilde{K}_\mathfrak{p}}\}.$$

The group $\widetilde{G}_{\ell,0}$ is the inertia subgroup of $\widetilde{G}_\ell$. Let π be a uniformizer in $\widetilde{K}_\mathfrak{p}$. Then for $i > 0$,

$$\widetilde{G}_{\ell,i} = \{\sigma \in \widetilde{G}_\ell \mid v(\sigma(\pi) - \pi) > i\}.$$

For a representation ρ of $\widetilde{G}$ with character χ and representation space V, let

$$f(\chi, \ell) = f(\rho, \ell) = \sum_{i=0}^{\infty} \frac{|\widetilde{G}_{\ell,i}|}{|\widetilde{G}_{\ell,0}|}\,(\chi(1) - \chi(\widetilde{G}_{\ell,i})),$$

where $\chi(\widetilde{G}_{\ell,i}) = |\widetilde{G}_{\ell,i}|^{-1}\sum_{s\in\widetilde{G}_{\ell,i}}\chi(s)$. We have $f(\chi,\ell) = 0$ if ρ is unramified over ℓ, i.e. $V = V^{\widetilde{G}_{\ell,0}}$. The *Artin conductor* of the representation ρ is defined as

$$\mathfrak{f}(\chi) = \mathfrak{f}(\rho) = \prod_\ell \ell^{f(\chi,\ell)}.$$

From the result in the previous section, we know that ℓ is unramified in $\widetilde{K}/\mathbb{Q}$ if $\ell \neq 2, q$. Thus to compute the conductor $\mathfrak{f}(\chi)$, we only need to calculate $f(\chi, 2)$ and $f(\chi, q)$. We consider the cases $q = 2$ and q odd separately.

5.1. Case $q = 2$. In the case $(p, q) = (-1, 2)$, there is only one 2-dimensional irreducible representation $\widetilde{\rho}_0$ of $\widetilde{G}$. Let $\widetilde{\chi}_0$ be the character of $\widetilde{\rho}_0$. Since only 2 is ramified in $\widetilde{K}$, we only need to calculate $f(\widetilde{\chi}_0, 2)$.

As in the previous section, let $\pi_2 = 1 - \zeta_8$. Let $\wp$ be a prime ideal in $\widetilde{K}$ over 2 and let v be the normalized valuation in $\widetilde{K}_\wp$. From the proof of Proposition 4.1, we see that $\sqrt{2}/\pi_2^2 \equiv 1 - \pi_2 \bmod \pi_2^3$. Thus

$$v\left(\frac{\sqrt{2}}{\pi_2^2} - 1\right) = v\left(\frac{\sqrt[4]{2}}{\pi_2} - 1\right) + v\left(\frac{\sqrt[4]{2}}{\pi_2} + 1\right) = v(\pi_2) = 2.$$

We have $v(\sqrt[4]{2}/\pi_2-1) = v(\sqrt[4]{2}/\pi_2+1) = 1$. So $\pi = \sqrt[4]{2}/\pi_2-1$ is a uniformizer of $\widetilde{K}_\wp$. The group $\widetilde{G}$ is generated by $\widetilde{\sigma}_{-1}$ and $\widetilde{\sigma}_2$, and $\widetilde{\sigma}_{-1}(\sqrt[4]{2}) = \sqrt[4]{2}$ and $\widetilde{\sigma}_2(\sqrt[4]{2}) = \sqrt[4]{2}/\sqrt{-1}$. Clearly $G_{2,0} = \widetilde{G}$. Furthermore,

$$\begin{aligned} v(\widetilde{\sigma}_{-1}(\pi)-\pi) &= v\left(\frac{\sqrt[4]{2}}{1-\zeta_8^{-1}} - \frac{\sqrt[4]{2}}{1-\zeta_8}\right) = v\left(-\frac{1+\zeta_8}{1-\zeta_8}\sqrt[4]{2}\right) = 2,\\ v(\widetilde{\sigma}_2(\pi)-\pi) &= v\left(\frac{\sqrt[4]{2}/\sqrt{-1}}{1+\zeta_8} - \frac{\sqrt[4]{2}}{1-\zeta_8}\right) = v\left(-\frac{1-\zeta_8^3}{1-\zeta_8}\sqrt[4]{2}\right) = 2,\\ v(\varepsilon(\pi)-\pi) &= v(-2(\pi+1)) = 8. \end{aligned}$$

Thus $G_{2,1} = G_{2,0} = \widetilde{G}$ and $G_{2,2} = \cdots = G_{2,7} = \langle\varepsilon\rangle$. By an easy computation we get $\widetilde{\chi}_0(G_{2,0}) = \widetilde{\chi}_0(G_{2,1}) = \cdots = \widetilde{\chi}_0(G_{2,7}) = 0$, and $\widetilde{\chi}_0(G_{2,n}) = 2$ for $n \geq 8$. So we obtain

$$f(\widetilde{\chi}_0, 2) = 2 + 2 + \frac{1}{4}\cdot 2\cdot 6 = 7. \tag{5.1}$$

5.2. Case of q odd. To compute $f(\chi, q)$, we consider the cases $\left(\frac{-1}{q}\right) = 1$ and $\left(\frac{-1}{q}\right) = -1$ separately. Let $\wp$ be a prime ideal in $\widetilde{K}$ over q. Let v be the normalized valuation in $\widetilde{K}_\wp$.

5.2.1. *Assume* $\left(\frac{-1}{q}\right) = 1$. Then q is unramified in $\widetilde{K}/K$ but ramified in $K/\mathbb{Q}$. We see $\pi = 1-\zeta_q$ is a uniformizer of $\widetilde{K}_\wp$. Now all 2-dimensional irreducible representations of $\widetilde{G}$ are as in Case A. Let $\widetilde{\chi}_j$ be the character of $\widetilde{\rho}_j$. It is easy to see that $\widetilde{G}_{q,0} = \langle\widetilde{\sigma}_q\rangle$. Notice that $\varepsilon \notin \widetilde{G}_{q,0}$.

Let $1 \neq \widetilde{\sigma} \in \widetilde{G}_{q,0}$ and $\widetilde{\sigma}(\zeta_q) = \zeta_q^a$, $1 < a \leq q-1$. We have

$$v(\widetilde{\sigma}\pi - \pi) = v(\zeta_q - \zeta_q^a) = v(1-\zeta_q^{a-1}) = 1.$$

Thus $\widetilde{G}_{q,n} = \{1\}$ for $n \geq 1$. By an easy computation we get $\widetilde{\chi}_j(\widetilde{G}_{q,0}) = 0$ and $\widetilde{\chi}_j(\widetilde{G}_{q,n}) = 2$ for $n \geq 1$. We obtain

$$f(\widetilde{\chi}_j, q) = 2. \tag{5.2}$$

5.2.2. *Assume* $\left(\frac{-1}{q}\right) = -1$. Then q is ramified both in $\widetilde{K}/K$ and in $K/\mathbb{Q}$, and all 2-dimensional irreducible representations are as in Case B. Let $\widetilde{\chi}_j$ be the character of $\widetilde{\rho}_j$. Since $v(1-\zeta_q) = 2$ and $v(\sqrt[4]{-q}) = \frac{1}{4}(2(q-1)) = (q-1)/2$, we see that $\pi = \sqrt[4]{-q}/(1-\zeta_q)^{(q-3)/4}$ is a uniformizer of q in $\widetilde{K}$. It is obvious that $\widetilde{G}_{q,0} = \langle\widetilde{\sigma}_q\rangle$. Notice that in this case $\varepsilon \in \widetilde{G}_{q,0}$.

Let $1 \neq \widetilde{\sigma} \in \widetilde{G}_{q,0}$ and $\widetilde{\sigma}(\zeta_q) = \zeta_q^a$, $1 < a \leq q-1$. We have

$$\begin{aligned} v(\widetilde{\sigma}\pi - \pi) + v(\widetilde{\sigma}\varepsilon\pi - \pi) &= v(\widetilde{\sigma}\pi - \pi) + v(-\widetilde{\sigma}\pi - \pi) = v(\widetilde{\sigma}\pi^2 - \pi^2) \\ &= v\left(\frac{\left(\frac{a}{q}\right)\sqrt{-q}}{(1-\zeta_q^a)^{(q-3)/2}} - \frac{\sqrt{-q}}{(1-\zeta_q)^{(q-3)/2}}\right) \\ &= v(\pi^2) + v\left(\frac{1 - \left(\frac{a}{q}\right)\left(\sum_{i=0}^{a-1}\zeta_q^i\right)^{(q-3)/2}}{\left(\sum_{i=0}^{a-1}\zeta_q^i\right)^{(q-3)/2}}\right) \\ &= 2 + v\left(1 - \left(\frac{a}{q}\right)\left(\sum_{i=0}^{a-1}\zeta_q^i\right)^{(q-3)/2}\right). \end{aligned}$$

Let $t = v\big(1 - \left(\frac{a}{q}\right)\left(\sum_{i=0}^{a-1}\zeta_q^i\right)^{(q-3)/2}\big)$. We claim that $t = 0$. Otherwise $t > 0$. Since

$$\left(\sum_{i=0}^{a-1}\zeta_q^i\right)^{(q-3)/2} \equiv a^{(q-3)/2} \equiv \begin{cases} 1 \bmod (1-\zeta_q) & \text{if } \left(\frac{a}{q}\right) = 1, \\ -1 \bmod (1-\zeta_q) & \text{if } \left(\frac{a}{q}\right) = -1, \end{cases}$$

we always have $a \equiv 1 \bmod q$ and thus $a = 1$, which contradicts the assumption that $a > 1$. This shows the claim. Thus $v(\widetilde{\sigma}\pi - \pi) = v(\widetilde{\sigma}\varepsilon\pi - \pi) = 1$, as $v(\widetilde{\sigma}\pi - \pi + \widetilde{\sigma}\varepsilon\pi - \pi) = v(2\pi) = 1$. So we get $\widetilde{G}_{q,n} = \{1\}$ for $n \geq 1$. By an easy computation, $\widetilde{\chi}_j(\widetilde{G}_{q,0}) = 0$ and $\widetilde{\chi}_j(\widetilde{G}_{q,n}) = 2$ for $n \geq 1$. We obtain

$$f(\widetilde{\chi}_j, q) = 2. \tag{5.3}$$

Next we compute $f(\widetilde{\chi}_j, 2)$. We consider the cases $\left(\frac{2}{q}\right) = 1$ and $\left(\frac{2}{q}\right) = -1$ separately. Let $\wp$ be a prime ideal in $\widetilde{K}$ over 2. Let v be the normalized valuation in $\widetilde{K}_\wp$.

5.2.3. *Assume* $\left(\frac{2}{q}\right) = 1$. Then 2 is unramified in $\widetilde{K}/K$ but ramified in $K/\mathbb{Q}$, and $\pi = 1 - \zeta_4$ is a uniformizer in $\widetilde{K}_\wp$. It is easy to see that $\widetilde{G}_{2,0} = \langle\widetilde{\sigma}_{-1}\rangle$. Notice that in this case $\varepsilon \notin \widetilde{G}_{2,0}$. We have

$$v(\widetilde{\sigma}_{-1}\pi - \pi) = v(\zeta_4 - \zeta_4^{-1}) = v(2) = 2.$$

Thus $\widetilde{G}_{2,0} = \widetilde{G}_{2,1} = \langle\widetilde{\sigma}_{-1}\rangle$ and $\widetilde{G}_{2,n} = \{1\}$ for $n > 1$. By an easy computation, $\widetilde{\chi}_j(\widetilde{G}_{2,0}) = \widetilde{\chi}_j(\widetilde{G}_{2,1}) = 1$ and $\widetilde{\chi}_j(\widetilde{G}_{2,n}) = 2$ for $n > 1$. We obtain

$$f(\widetilde{\chi}_j, 2) = 1 + 1 = 2. \tag{5.4}$$

5.2.4. *Assume* $\left(\frac{2}{q}\right) = -1$. Now 2 is ramified both in $\widetilde{K}/K$ and in $K/\mathbb{Q}$. As in the previous section, let $\pi_2 = 1 - \zeta_4$, $\alpha = \sum_{(\frac{a}{q})=1}\zeta_q^a$ and $\beta = \sum_{(\frac{a}{q})=1}\zeta_{2q}^a$, where the summations are over $1 \leq a \leq q-1$. From the previous section we have

$$\sqrt{q^*} \equiv (1 + \pi_2\beta)^2 + \pi_2^3 \bmod \pi_2^4. \tag{5.5}$$

Let $\mu = 1 + \pi_2\beta$. We claim that $\pi = (\sqrt[4]{q^*} + \mu)/\pi_2$ is a uniformizer in $\widetilde{K}_\wp$. In fact, since

$$v(\sqrt[4]{q^*} + \mu) + v(\sqrt[4]{q^*} - \mu) = v(\sqrt{q^*} - \mu^2) = v(\pi_2^3) = 6$$

and $v((\sqrt[4]{q^*} + \mu) + (\sqrt[4]{q^*} - \mu)) = v(2\sqrt[4]{q^*}) = 4$, we must have

$$v(\sqrt[4]{q^*} + \mu) = v(\sqrt[4]{q^*} - \mu) = 3,$$

and thus $v((\sqrt[4]{q^*} + \mu)/\pi_2) = 1$.

It is obvious that $\widetilde{G}_{2,0} = \{1, \varepsilon, \widetilde{\sigma}_{-1}, \widetilde{\sigma}_{-1}\varepsilon\}$. Since $\widetilde{\sigma}_{-1}(\sqrt[4]{q^*}) = \sqrt[4]{q^*}$ and $\widetilde{\sigma}_{-1}\varepsilon(\sqrt[4]{q^*}) = -\sqrt[4]{q^*}$, we have

$$\begin{aligned} v(\widetilde{\sigma}_{-1}\pi - \pi) &= v\left(\widetilde{\sigma}_{-1}\frac{\sqrt[4]{q^*} + 1 + \pi_2\beta}{\pi_2} - \frac{\sqrt[4]{q^*} + 1 + \pi_2\beta}{\pi_2}\right) \\ &= v\left(\widetilde{\sigma}_{-1}\frac{\sqrt[4]{q^*} + 1}{\pi_2} - \frac{\sqrt[4]{q^*} + 1}{\pi_2}\right) \qquad (\text{since } \widetilde{\sigma}_{-1}\beta = \beta) \\ &= v\left(\frac{\sqrt[4]{q^*} + 1}{1 - \zeta_4^{-1}} - \frac{\sqrt[4]{q^*} + 1}{1 - \zeta_4}\right) = v(\sqrt[4]{q^*} + 1). \end{aligned}$$

To compute it, we first claim that $\pi_2 \nmid \beta$. Otherwise, $2 \mid \beta$ as $\beta \in \mathbb{Q}(\zeta_q)$. From the previous section, we have $\sqrt{q^*} = 1 + 2\alpha$ and $\alpha + \beta \equiv 1 \bmod 2$, thus $\sqrt{q^*} \equiv -1 + 2\beta \equiv -1 \bmod 4$ and so $q^* \equiv 1 \bmod 8$. This contradicts the assumption $\left(\frac{2}{q}\right) = -1$. We have shown the claim. Thus $v(\beta) = 0$. Since $v(\sqrt[4]{q^*} + 1 + \pi_2\beta) = 3$, we have $v(\sqrt[4]{q^*} + 1) = 2$, so $v(\widetilde{\sigma}_{-1}\pi - \pi) = 2$.

We now compute $v(\widetilde{\sigma}_{-1}\varepsilon\pi - \pi)$. We have

$$\begin{aligned} v(\widetilde{\sigma}_{-1}\varepsilon\pi - \pi) &= v\left(\widetilde{\sigma}_{-1}\varepsilon\,\frac{\sqrt[4]{q^*} + 1 + \pi_2\beta}{\pi_2} - \frac{\sqrt[4]{q^*} + 1 + \pi_2\beta}{\pi_2}\right) \\ &= v\left(\frac{-\sqrt[4]{q^*} + 1}{1 - \zeta_4^{-1}} - \frac{\sqrt[4]{q^*} + 1}{1 - \zeta_4}\right) = v(\sqrt[4]{q^*} + \zeta_4). \end{aligned}$$

Observe that

$$v(\sqrt[4]{q^*} + \zeta_4) + v(\sqrt[4]{q^*} - \zeta_4) = v(\sqrt{q^*} + 1) = v\left(2\,\frac{\sqrt{q^*} + 1}{2}\right) = 4,$$

since $\pi_2 \nmid (\sqrt{q^*} + 1)/2$. Furthermore, since

$$v((\sqrt[4]{q^*} + \zeta_4) + (\sqrt[4]{q^*} - \zeta_4)) = v(2\sqrt[4]{q^*}) = 4,$$

we must have $v(\sqrt[4]{q^*} + \zeta_4) = v(\sqrt[4]{q^*} - \zeta_4) = 2$, so $v(\widetilde{\sigma}_{-1}\varepsilon\pi - \pi) = 2$. In addition, we have

$$v(\varepsilon\pi - \pi) = v\left(\frac{-\sqrt[4]{q^*} + 1 + \pi_2\beta}{\pi_2} - \frac{\sqrt[4]{q^*} + 1 + \pi_2\beta}{\pi_2}\right) = 2.$$

By the discussion above we have $\widetilde{G}_{2,0} = \widetilde{G}_{2,1}$ and $\widetilde{G}_{2,n} = \{1\}$ for $n > 1$. By an easy computation, $\widetilde{\chi}_j(\widetilde{G}_{2,0}) = \widetilde{\chi}_j(\widetilde{G}_{2,1}) = 0$ and $\widetilde{\chi}_j(\widetilde{G}_{2,n}) = 2$ for $n > 2$. We obtain

$$f(\widetilde{\chi}_j, 2) = 2 + 2 = 4. \tag{5.6}$$

5.3. Global conductors. By the equalities (5.1)–(5.6) above, we get the following

THEOREM 5.1. *In the case $q = 2$, the conductor of the unique 2-dimensional irreducible representation $\widetilde{\rho}_0$ of $\widetilde{G}$ is equal to $\mathfrak{f}(\widetilde{\rho}_0) = 2^7$. In the case that q is odd, all the 2-dimensional irreducible representations $\widetilde{\rho}_j$ of $\widetilde{G}$ have the conductor $\mathfrak{f}(\widetilde{\rho}_j) = 2^{2(1+\log_{-1}(\frac{2}{q}))}q^2$.*

6. The Artin L-functions. In this section we compute the Artin L-functions of the quasi-cyclotomic fields $\widetilde{K} = \mathbb{Q}(\zeta_{4q}, \sqrt[4]{q^*})$.

The L-functions associated to the 1-dimensional representations of $\widetilde{G}$ are the well-known Dirichlet L-functions. So we compute the L-functions associated to the 2-dimensional irreducible representations of $\widetilde{G}$. Let $\varphi : \widetilde{G} \to \mathrm{GL}(V)$ be a 2-dimensional irreducible representation. The Artin L-function $L(\varphi, s)$ associated to φ is defined as the product

$$L(\varphi, s) = \prod_{\ell \text{ prime}} L_\ell(\varphi, s),$$

where the local factors are defined as $L_\ell(\varphi, s) = \det(1 - \varphi(\widetilde{\mathrm{Fr}}_\ell)\ell^{-s}|V^{\widetilde{I}_\ell})^{-1}$. Now we begin to compute them. First we notice that if ℓ is ramified in $\widetilde{K}/K$, then $V^{\widetilde{I}_\ell} = 0$ and $L_\ell(\varphi, s) = 1$, which is due to the facts that $\varepsilon \in \widetilde{I}_\ell$ by Lemma 4.3 and $\varphi(\varepsilon) = -I$ for any irreducible representation φ of $\widetilde{G}$ by Theorem 3.1.

6.1. Case $q = 2$. By Section 3, there is only one 2-dimensional representation $\widetilde{\rho}_0$ in this case, which is defined by

$$\widetilde{\rho}_0(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}_0(\widetilde{\sigma}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since 2 is ramified in $\widetilde{K}/K$, we have $L_2(\widetilde{\rho}_0, s) = 1$. Assume that ℓ is an odd prime number.

If $\ell \equiv 7 \bmod 8$, then $\mathrm{Fr}_\ell = \sigma_{-1}$ and thus $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_{-1}$ or $\widetilde{\sigma}_{-1}\varepsilon$. In any case we have

$$L_\ell(\widetilde{\rho}_0, s) = \det\left(I \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ell^{-s}\right)^{-1} = (1 - \ell^{-2s})^{-1}.$$

If $\ell \equiv 5 \bmod 8$, then $\mathrm{Fr}_\ell = \sigma_2$ and thus $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_2$ or $\widetilde{\sigma}_2\varepsilon$. We have

$$L_\ell(\widetilde{\rho}_0, s) = \det\left(I \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ell^{-s}\right)^{-1} = (1 + \ell^{-2s})^{-1}.$$

If $\ell \equiv 3 \bmod 8$, then $\mathrm{Fr}_\ell = \sigma_{-1}\sigma_2$ and thus $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_{-1}\widetilde{\sigma}_2$ or $\widetilde{\sigma}_{-1}\widetilde{\sigma}_2\varepsilon$. We have

$$L_\ell(\widetilde{\rho}_0, s) = \det\left(I \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ell^{-s}\right)^{-1} = (1 - \ell^{-2s})^{-1}.$$

If $\ell \equiv 1 \bmod 8$, then $\mathrm{Fr}_\ell = 1$ and thus $\widetilde{\mathrm{Fr}}_\ell = 1$ or ε. In this case we must determine $\widetilde{\mathrm{Fr}}_\ell$ completely. Since $\widetilde{\mathrm{Fr}}_\ell(\sqrt[4]{2}) \equiv (\sqrt[4]{2})^\ell \bmod \wp$ for the prime ideal $\wp$ of $\widetilde{K}$ over ℓ associated to $\widetilde{\mathrm{Fr}}_\ell$, we have $\widetilde{\mathrm{Fr}}_\ell = 1$ if $2^{(\ell-1)/4} \equiv 1 \bmod \ell$, and $\widetilde{\mathrm{Fr}}_\ell = \varepsilon$ if $2^{(\ell-1)/4} \equiv -1 \bmod \ell$. As in the previous section, we find that for $\ell \equiv 1 \bmod 8$, $2^{(\ell-1)/4} \equiv 1 \bmod \ell$ if and only if $\ell \in P_0$. So we have

$$L_\ell(\widetilde{\rho}_0, s) = \begin{cases} (1 - \ell^{-s})^{-2} & \text{if } \ell \in P_0, \\ (1 + \ell^{-s})^{-2} & \text{otherwise.} \end{cases}$$

We get the Artin L-function in the case $(p, q) = (-1, 2)$ as follows:

$$L(\widetilde{\rho}_0, s) = \prod_{\ell \equiv 3 \text{ or } 7 \bmod 8} (1 - \ell^{-2s})^{-1} \cdot \prod_{\ell \equiv 5 \bmod 8} (1 + \ell^{-2s})^{-1} \times \prod_{\ell \in P_0} (1 - \ell^{-s})^{-2} \cdot \prod_{\ell \equiv 1 \bmod 8,\, \ell \notin P_0} (1 + \ell^{-s})^{-2}. \tag{6.1}$$

6.2. Case of q odd. In this case, all 2-dimensional irreducible representations of $\widetilde{G}$ are $\widetilde{\rho}_j$ with $0 \le j < q - 1, 2 \mid j$ if $q \equiv 1 \bmod 4$, and $0 \le j < q - 1, 2 \nmid j$ if $q \equiv 3 \bmod 4$, where $\widetilde{\rho}_j$ is defined by

$$\widetilde{\rho}_j(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}_j(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix}, \quad \widetilde{\rho}_j(\varepsilon) = -I.$$

We first determine the local factors $L_\ell(\widetilde{\rho}_j, s)$ for $\ell \ne 2, q$. For such ℓ we have $V^{\widetilde{I}_\ell} = V$. Let $\mathrm{Fr}_\ell = \sigma_{-1}^{a_\ell}\sigma_q^{b_\ell}$, which is equivalent to $\ell \equiv (-1)^{a_\ell} \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$, where g is the primitive root mod q associated to σ_q. It is easy to compute that

$$\widetilde{\rho}_j(\widetilde{\sigma}_q^{b_\ell}) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix}^{b_\ell} = \begin{cases} \zeta_{2(q-1)}^{jb_\ell} I & \text{if } 2 \mid b_\ell, \\ \begin{pmatrix} 0 & \zeta_{2(q-1)}^{j(b_\ell+1)} \\ \zeta_{2(q-1)}^{j(b_\ell-1)} & 0 \end{pmatrix} & \text{if } 2 \nmid b_\ell. \end{cases}$$

Furthermore,

$$\det(I-\widetilde{\rho}_j(\widetilde{\sigma}_{-1}^{a_\ell}\widetilde{\sigma}_q^{b_\ell})\ell^{-s}) = \begin{cases} (1-\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^2 & \text{if } a_\ell=0,\ 2\,|\,b_\ell, \\ 1-\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell=0,\ 2\nmid b_\ell, \\ & \text{or } a_\ell=1,\ 2\,|\,b_\ell, \\ 1+\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell=1,\ 2\nmid b_\ell \end{cases}$$

and

$$\det(I+\widetilde{\rho}_j(\widetilde{\sigma}_{-1}^{a_\ell}\widetilde{\sigma}_q^{b_\ell})\ell^{-s}) = \begin{cases} (1+\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^2 & \text{if } a_\ell=0,\ 2\,|\,b_\ell, \\ 1-\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell=0,\ 2\nmid b_\ell, \\ & \text{or } a_\ell=1,\ 2\,|\,b_\ell, \\ 1+\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell=1,\ 2\nmid b_\ell. \end{cases}$$

So we get

$$L_\ell(\widetilde{\rho}_j,s) = (1-\zeta_{q-1}^{jb_\ell}\ell^{-2s})^{-1}$$

if $\ell\equiv 1 \bmod 4$ and $\ell\equiv g^{b_\ell} \bmod q$ with $2\nmid b_\ell$, and also if $\ell\equiv 3 \bmod 4$ and $\ell\equiv g^{b_\ell} \bmod q$ with $2\,|\,b_\ell$, while

$$L_\ell(\widetilde{\rho}_j,s) = (1+\zeta_{q-1}^{jb_\ell}\ell^{-2s})^{-1}$$

if $\ell\equiv 3 \bmod 4$ and $\ell\equiv g^{b_\ell} \bmod q$ with $2\nmid b_\ell$.

To compute the local factors when $\ell\equiv 1 \bmod 4$ and $\ell\equiv g^{b_\ell} \bmod q$ with $2\,|\,b_\ell$ we must determine $\widetilde{\mathrm{Fr}}_\ell$ completely. Since $\left(\frac{\ell}{q}\right)=1$, we have $\left(\frac{q}{\ell}\right)=1$ and $\left(\frac{q^*}{\ell}\right)=1$. Let $\alpha_\ell\in\mathbb{Z}$ be such that $\alpha_\ell^2\equiv q^* \bmod \ell$. From $\widetilde{\sigma}_q^{b_\ell}(\sqrt[4]{q^*}) = (-1)^{b_\ell/2}\sqrt[4]{q^*}$, we see that $\widetilde{\mathrm{Fr}}_\ell=\widetilde{\sigma}_q^{b_\ell}$ if $\left(\frac{\alpha_\ell}{\ell}\right)=(-1)^{b_\ell/2}$, and $\widetilde{\mathrm{Fr}}_\ell=\widetilde{\sigma}_q^{b_\ell}\varepsilon$ if $\left(\frac{\alpha_\ell}{\ell}\right)=(-1)^{b_\ell/2+1}$. So when $\ell\equiv 1 \bmod 4$ and $\ell\equiv g^{b_\ell} \bmod q$ with $2\,|\,b_\ell$, we have

$$L_\ell(\widetilde{\rho}_j,s) = \begin{cases} (1-\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^{-2} & \text{if } \left(\frac{\alpha_\ell}{\ell}\right)=(-1)^{b_\ell/2}, \\ (1+\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^{-2} & \text{if } \left(\frac{\alpha_\ell}{\ell}\right)=(-1)^{b_\ell/2+1}. \end{cases}$$

Next we compute the local factors $L_2(\widetilde{\rho}_j,s)$ and $L_q(\widetilde{\rho}_j,s)$. When $\left(\frac{2}{q}\right)=-1$, we know from the previous section that 2 is ramified in $\widetilde{K}/K$. So $L_2(\widetilde{\rho}_j,s)=1$ in this case. Now we assume $\left(\frac{2}{q}\right)=1$. Since $I_2=\langle\sigma_{-1}\rangle$ and 2 is unramified in $\widetilde{K}/K$, we have $\widetilde{I}_2=\langle\widetilde{\sigma}_{-1}\rangle$ or $\widetilde{I}_2=\langle\widetilde{\sigma}_{-1}\varepsilon\rangle$. The matrices $I+\widetilde{\rho}_j(\widetilde{\sigma}_{-1})$ and $I+\widetilde{\rho}_j(\widetilde{\sigma}_{-1}\varepsilon)$ have rank 1, thus $V^{\widetilde{I}_2}$ has dimension 1. Write $\mathrm{Fr}_2=\sigma_2^{b_2}$ with $2\,|\,b_2$. As in the previous section, we always assume $b_2\equiv 2 \bmod 4$ if $q\equiv 3 \bmod 4$. Recall that P_0 is the set of all prime numbers of the form A^2+64B^2 with $A,B\in\mathbb{Z}$. Since $\widetilde{\rho}_j(\widetilde{\sigma}_2^{b_2})=\zeta_{2(q-1)}^{jb_2}I$, by Lemma 4.3

we have

$$L_2(\widetilde{\rho}_j, s) = \begin{cases} 1 - \zeta_{2(q-1)}^{jb_2} 2^{-s} & \text{if } q \notin P_0,\ 16 \nmid q^* - 1, \text{ or } q \in P_0,\ 16 \,|\, q^* - 1, \\ 1 + \zeta_{2(q-1)}^{jb_2} 2^{-s} & \text{if } q \in P_0,\ 16 \,|\, q^* - 1, \text{ or } q \notin P_0,\ 16 \mid q^* - 1. \end{cases}$$

When $q \equiv 3 \bmod 4$, we know that q is ramified in $\widetilde{K}/K$. So $L_q(\widetilde{\rho}_j, s) = 1$ for odd j in this case. Assume $q \equiv 1 \bmod 4$. Since $I_q = \langle \sigma_q \rangle$ and q is unramified in $\widetilde{K}/K$, we have $\widetilde{I}_q = \langle \widetilde{\sigma}_q \rangle$ or $\widetilde{I}_2 = \langle \widetilde{\sigma}_q \varepsilon \rangle$. Thus $V^{\widetilde{I}_q} = 0$ if $j \neq 0$, and $V^{\widetilde{I}_q}$ has dimension 1 if $j = 0$.

The Frobenius map Fr_q of q in G modulo I_q is the identity map. So $\widetilde{\mathrm{Fr}}_q = 1$ or ε. In [7, Sect. 5] we have shown that q splits in $\widetilde{K}/K$ if $q \equiv 1 \bmod 8$, and is inert if $q \equiv 5 \bmod 8$. So $\widetilde{\mathrm{Fr}}_2 = 1$ if $q \equiv 1 \bmod 8$, and $\widetilde{\mathrm{Fr}}_2 = \varepsilon$ if $q \equiv 5 \bmod 8$. Thus we get

$$L_q(\widetilde{\rho}_j, s) = \begin{cases} 1 & \text{if } j \neq 0, \\ 1 - q^{-s} & \text{if } j = 0,\ q \equiv 1 \bmod 8, \\ 1 + q^{-s} & \text{if } j = 0,\ q \equiv 5 \bmod 8. \end{cases}$$

We have computed all the local factors, obtaining

$$\begin{aligned} (6.2) \quad L(\widetilde{\rho}_j, s) &= (1 - u_q \zeta_{2(q-1)}^{jb_2} 2^{-s})^{-1} (1 - (-1)^{(q-1)/4} q^{-s})^{-\delta_{0j}} \\ &\quad \times \prod_{\ell \equiv 1,\, 2 \nmid b_\ell \text{ or } \ell \equiv 3,\, 2 \,|\, b_\ell} (1 - \zeta_{q-1}^{jb_\ell} \ell^{-2s})^{-1} \\ &\quad \times \prod_{\ell \equiv 3,\, 2 | b_\ell} (1 + \zeta_{q-1}^{jb_\ell} \ell^{-2s})^{-1} \prod_{\ell \equiv 1,\, 2 | b_\ell} (1 - u_\ell \zeta_{2(q-1)}^{jb_\ell} \ell^{-s})^{-2}, \end{aligned}$$

where $u_q = 1$ if $q \notin P_0$, $16 \nmid q^* - 1$ or $q \in P_0$, $16 \,|\, q^* - 1$, and $u_q = -1$ otherwise; $\delta_{0j} = 0$ if $j \neq 0$ and $\delta_{00} = 1$; and $u_\ell = \left(\frac{\alpha_\ell}{\ell}\right)(-1)^{b_\ell/2}$. In the above products, "≡" denotes congruence modulo 4.

THEOREM 6.1. *Except for the Dirichlet L-functions, all Artin L-functions of the Galois extension* $\widetilde{K}/\mathbb{Q}$ *are explicitly given by* (6.1) *in the case* $q = 2$ *and by* (6.2) *in the case of* q *odd, where in* (6.2) $0 \le j < q - 1$, $2 \,|\, j$ *if* $q \equiv 1 \bmod 4$ *and* $0 \le j < q - 1$, $2 \nmid j$ *if* $q \equiv 3 \bmod 4$.

6.3. A formula. Let $\zeta_{\widetilde{K}}(s)$ and $\zeta_K(s)$ be the Dedekind zeta functions of $\widetilde{K}$ and K respectively. By Artin's formula for the decomposition of Dedekind zeta functions, we have

$$\frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} = \prod_{\widetilde{\rho}_j} \prod_{\ell \text{ prime}} L_\ell(\widetilde{\rho}_j, s)^2,$$

where $\widetilde{\rho}_j$ runs over all 2-dimensional irreducible representations of $\widetilde{G}$. When $q = 2$, there is only one 2-dimensional irreducible representation of $\widetilde{G}$. So the square of (6.1) gives the formula. When q is odd, by computing $\prod_{\widetilde{\rho}_j} L_\ell(\widetilde{\rho}_j, s)$, we get the following

COROLLARY 6.2. *For a prime number* $\ell \neq q$, *let*

$$f_\ell = \frac{q-1}{\gcd(b_\ell, q-1)}$$

be the order of $\ell \bmod q$ *and let*

$$g_\ell = \gcd(b_\ell, q-1) = \frac{q-1}{f_\ell}.$$

If $q \equiv 1 \bmod 4$, *then*

$$\frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} = (1 - u_q^{f_2} 2^{-f_2 s})^{-g_2} (1 - (-1)^{(q-1)/4} q^{-s})^{-2} \prod_{\ell \equiv 1,\, 2 \nmid b_\ell \text{ or } \ell \equiv 3} (1 - \ell^{-f_\ell s})^{-2g_\ell}$$
$$\times \prod_{\ell \equiv 1,\, 2 | b_\ell} (1 - u_\ell^{f_\ell} \ell^{-f_\ell s})^{-2g_\ell},$$

and if $q \equiv 3 \bmod 4$, *then*

$$\frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} = (1 + u_q^{f_2} 2^{-f_2 s})^{-g_2} \prod_{\ell \equiv 1,\, 2 \nmid b_\ell} (1 + \ell^{-f_\ell s})^{-2g_\ell} \prod_{\ell \equiv 3} (1 - \ell^{-2 f_\ell s})^{-g_\ell}$$
$$\times \prod_{\ell \equiv 1,\, 2 | b_\ell} (1 + u_\ell^{f_\ell} \ell^{-f_\ell s})^{-2g_\ell},$$

where u_q *and* u_ℓ *are as above.*

6.4. The corresponding modular forms. All the 2-dimensional irreducible representations of $\widetilde{G}$ in the case $p = -1$ are monomial. It is easy to see that they are odd. By Deligne–Serre's theorem [6, Th. 2], these Artin L-functions above are equal to the L-functions of some normalized newforms of weight one, which allows one to determine a newform of weight one from a 2-dimensional irreducible odd representation of $\widetilde{G}$. More precisely, the irreducible representation $\widetilde{\rho}_j$ of conductor N corresponds to a normalized newform $f_j(z)$ of weight one on $\Gamma_0(N)$ with nebentype $\phi_j = \det(\widetilde{\rho}_j)$, which has a Fourier expansion at infinity

$$f_j(z) = \sum_{n=1}^{\infty} a_n^{(j)} q^n, \qquad q = e^{2\pi i z},$$

where $a_1^{(j)} = 1$ and the other coefficients a_n are equal to those of the L-function $L(\phi_j, s) = \sum_{n=1}^{\infty} a_n n^{-s}$. In this subsection we describe these modular forms explicitly. Since these newforms are eigenfunctions of Hecke operators, to determine all $a_n^{(j)}$ it is enough to determine $a_\ell^{(j)}$ for all primes ℓ.

When $q = 2$, we get one normalized newform $f_0(z)$ of weight 1 on $\Gamma_0(2^7)$ with nebentype $\phi_0 : (\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{C}^*$, where $\phi_0(\sigma_{-1}) = -1$ and $\phi_0(\sigma_2) = 1$.

By (6.1), we directly see that for primes ℓ the coefficients $a_\ell^{(0)}$ of the newform are given by

$$a_\ell^{(0)} = \begin{cases} 0 & \text{if } \ell = 2 \text{ or } \ell \equiv 3,5,7 \bmod 8, \\ 2 & \text{if } \ell \in P_0, \\ -2 & \text{if } \ell \equiv 1 \bmod 8 \text{ but } \ell \notin P_0. \end{cases}$$

When q is odd, we get $(q-1)/2$ normalized newforms $f_j(z)$ of weight 1 on $\Gamma_0(4^{1+\log_{-1}(\frac{2}{q})}q^2)$ with nebentype $\phi_j : (\mathbb{Z}/4q\mathbb{Z})^* \to \mathbb{C}^*$, where $\phi_j(\sigma_{-1}) = -1$ and $\phi_j(\sigma_q) = -\zeta_{q-1}^j$. By (6.2) we directly see that for primes $\ell \neq q$ the coefficients of the newforms are given by

$$a_\ell^{(j)} = \begin{cases} u_q \zeta_{2(q-1)}^{jb_2} & \text{if } \ell = 2, \\ 2u_\ell \zeta_{2(q-1)}^{jb_\ell} & \text{if } \ell \equiv 1 \bmod 4 \text{ and } 2 \mid b_\ell, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a_q^{(j)} = \begin{cases} 0 & \text{if } j \neq 0, \\ (-1)^{(q-1)/4} & \text{if } j = 0, \end{cases}$$

where $0 \le j < q-1$, $2 \mid j$ if $q \equiv 1 \bmod 4$, and $0 \le j < q-1$, $2 \nmid j$ if $q \equiv 3 \bmod 4$; b_ℓ is defined by $\ell \equiv g^{b_\ell} \bmod q$ for a primitive root g modulo q; $u_\ell = \left(\frac{\alpha_\ell}{\ell}\right)(-1)^{b_\ell/2}$ for an integer α_ℓ such that $\alpha_\ell^2 \equiv q^* \bmod \ell$; and

$$u_q = \begin{cases} 1 & \text{if } q \notin P_0,\ 16 \nmid q^*-1 \text{ or } q \in P_0,\ 16 \mid q^*-1, \\ -1 & \text{if } q \notin P_0,\ 16 \mid q^*-1 \text{ or } q \in P_0,\ 16 \nmid q^*-1. \end{cases}$$

Here P_0 is the set of all primes of the form $A^2 + 64B^2$ with $A, B \in \mathbb{Z}$.

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Sunghan Bae
Department of Mathematics
KAIST
Daejeon, Korea

Yong Hu, Linsheng Yin (corresponding author)
Department of Mathematical Sciences
Tsinghua University
Beijing 100084, China
E-mail: lsyin@math.tsinghua.edu.cn

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