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Solving Ramanujan's differential equations for Eisenstein series via a first order Riccati equation

by

JAMES M. HILL (Wollongong), BRUCE C. BERNDT (Urbana, IL) and TIM HUBER (Urbana, IL)

1. Introduction. Ramanujan [16], [18, pp. 136–162] introduced the three Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ defined for $|q| < 1$ by

$$P(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}, \tag{1.1}$$

$$Q(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k}, \tag{1.2}$$

$$R(q) = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1 - q^k}. \tag{1.3}$$

Amongst many results, he established that $P(q)$, $Q(q)$, and $R(q)$ satisfy the differential equations

$$q\,\frac{dP}{dq} = \frac{1}{12}\,(P^2 - Q), \qquad q\,\frac{dQ}{dq} = \frac{1}{3}\,(PQ - R), \qquad q\,\frac{dR}{dq} = \frac{1}{2}\,(PR - Q^2). \tag{1.4}$$

The three series $P(q)$, $Q(q)$, and $R(q)$ in (1.1)–(1.3) and their differential equations (1.4) arise in an enormous variety of contexts in number theory, and, in particular, in Ramanujan's own work. We cite just a small sample of books and papers where (1.1)–(1.3) and (1.4) arise: [1, Chapters 11–16], [3, Chapter 15], [4, pp. 484–486], [5, Chapters 4–6], [6]–[13], [16], [17]. Furthermore, Q and R are the building blocks in the theory of

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modular forms; e.g., see R. A. Rankin's text [20] for a development of the theory of Eisenstein series from the viewpoint of modular forms.

In this paper we show that this system of ordinary differential equations remains invariant under the simple one-parameter stretching group of transformations (3.2). This means that we may choose new variables (3.3) so that (1.4) can be reduced to a first order Riccati differential equation (3.8), which may be solved explicitly in terms of hypergeometric functions, defined for $|z| < 1$ by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, $$

where

$$ (\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1), \quad k \geq 1. $$

Furthermore, an additional transformation may be effected to give a solution in terms of particular associated Legendre functions, but this particular line of investigation is not explored further since these functions are themselves hypergeometric functions. So far as the authors are aware, neither the group invariance of the differential equations nor the reduction to a Riccati equation and its subsequent solution in terms of hypergeometric functions have been previously explicitly stated in the literature. Accordingly, the results of the present paper may provide a possible elementary mechanism for the derivation of results relating to Eisenstein series.

In the following section we cite some of the standard formulas due to Ramanujan relating P, Q, and R, and also formulas relating these series to theta functions, complete elliptic integrals, and their representations in terms of hypergeometric functions. In the section thereafter we show that the differential equations (1.4) remain invariant under the simple one-parameter group of stretching transformations, and therefore we may reduce the system to a single first order Riccati ordinary differential equation. In Section 5, we give the final parametric expressions for the three Eisenstein series. These representations are different from the standard representations (2.4)–(2.6) and are also different from the representations in Ramanujan's cubic theory [4, pp. 105–106]. In some sense, our new representations are variants of each of these sets of representations.

2. Some basic results for $P(q)$, $Q(q)$, and $R(q)$. In this section we present some basic formulas for the three Eisenstein series (1.1)–(1.3). These arise from the fundamental theorem connecting theta functions and Eisenstein series with elliptic integrals of the first kind and hypergeometric series. These formulas are due to Ramanujan [16], [19] and can also be found in [3, pp. 101–102, 120, 126–127].

The complete elliptic integral of the first kind associated with the *modulus* k, $0 < k < 1$, is defined by [22, p. 499]

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2}\, z, \tag{2.1}$$

where

$$z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

and where the last equality in (2.1) can be obtained by expanding the integrand in a binomial series and integrating termwise. Let $q := e^{-y}$, where

$$y = \pi\, \frac{{}_2F_1(1/2, 1/2; 1; 1 - x)}{{}_2F_1(1/2, 1/2; 1; x)}. \tag{2.2}$$

Then the most important formula in Ramanujan's theory of theta functions is given by [19, Chapter 17, Entry 6], [3, pp. 101–102]

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \sqrt{z}. \tag{2.3}$$

This formula then leads to the following representations for P, Q, and R [3, pp. 120, 126–127]:

$$P(q) = z^2(1 - 5x) + 12x(1 - x)z\, \frac{dz}{dx}, \tag{2.4}$$

$$Q(q) = z^4(1 + 14x + x^2), \tag{2.5}$$

$$R(q) = z^6(1 + x)(1 - 34x + x^2). \tag{2.6}$$

These are the focus of this paper; we derive analogues of (2.4)–(2.6).

3. First order Riccati differential equation. On making the substitution $q = e^{-y}$, we find that the three first order differential equations (1.4) become

$$\frac{dP}{dy} = -\frac{1}{12}\,(P^2 - Q), \qquad \frac{dQ}{dy} = -\frac{1}{3}\,(PQ - R), \qquad \frac{dR}{dy} = -\frac{1}{2}\,(PR - Q^2). \tag{3.1}$$

THEOREM 3.1. *The differential equations* (3.1) *are invariant under the simple one-parameter group of stretching transformations*

$$y_1 = e^{\varepsilon} y, \qquad P_1 = e^{-\varepsilon} P, \qquad Q_1 = e^{-2\varepsilon} Q, \qquad R_1 = e^{-3\varepsilon} R. \tag{3.2}$$

Proof. The invariance of each equation follows in a straightforward way from the chain rule. For example, by (3.1),

$$\frac{dP_1}{dy_1} = e^{-2\varepsilon}\, \frac{dP}{dy} = -e^{-2\varepsilon} q\, \frac{dP}{dq} = -\frac{1}{12}\,(P_1^2 - Q_1). \ ∎$$

We are thus motivated to consider the parameters

$$u = \frac{R}{Q^{3/2}}, \qquad v = \frac{Q^{1/2}}{P}, \qquad w = yQ^{1/2}, \tag{3.3}$$

where here and throughout the paper $z^{1/n}$ denotes the principal nth root of z. In terms of these new variables, the three differential equations (3.1) become

$$y\,\frac{du}{dy} = -\frac{w}{2}\,(u^2 - 1), \tag{3.4}$$

$$y\,\frac{dv}{dy} = -\frac{w}{12}\,(v^2 - 2uv + 1), \tag{3.5}$$

$$y\,\frac{dw}{dy} = w\left\{1 + \frac{w}{6}\left(u - \frac{1}{v}\right)\right\}, \tag{3.6}$$

subject to the conditions

$$u, v, \frac{w}{y} \to 1 \qquad \text{as } y \to \infty, \tag{3.7}$$

corresponding to the requirements of (1.1)–(1.3) that $P(0) = Q(0) = R(0) = 1$. Dividing, respectively, (3.5) by (3.4) and (3.4) by (3.6), we deduce that

$$(u^2 - 1)\,\frac{dv}{du} = \frac{1}{6}\,(v^2 - 2uv + 1), \tag{3.8}$$

$$-\frac{2}{u^2 - 1} = \frac{dw}{du} + \frac{w}{3}\,\frac{uv - 1}{v(u^2 - 1)}. \tag{3.9}$$

The form of equation (3.8) is familiar in the theory of ordinary differential equations and is known as a Riccati differential equation. With this in mind, we make the transformation (see Polyanin and Zaitsev [15, p. 2])

$$v(u) = -\frac{6(u^2 - 1)}{X}\,\frac{dX}{du}. \tag{3.10}$$

Differentiating (3.10) gives

$$\frac{dv}{du} = -\frac{12u}{X}\,\frac{dX}{du} + \frac{6(u^2 - 1)}{X^2}\left(\frac{dX}{du}\right)^2 - \frac{6(u^2 - 1)}{X}\,\frac{d^2X}{du^2}. \tag{3.11}$$

Inserting (3.11) and (3.10) into (3.8), and simplifying, we see that

$$(u^2 - 1)^2\,\frac{d^2X}{du^2} + \frac{7u}{3}\,(u^2 - 1)\,\frac{dX}{du} + \frac{X}{36} = 0. \tag{3.12}$$

As we shall show, solutions to (3.12) play an important role in equations expressing v, w, and y in terms of the parameter u. We study solutions to

(3.12) and their relation to hypergeometric functions in the next section. For the present discussion, it suffices to note that if $X(u)$ is a solution to (3.12), then there exists a linearly independent solution $X^*(u)$ [21, pp. 81–82] such that

$$X^*(u) = X(u)\int \frac{1}{X(u)^2} \exp\left(-\int \frac{7u}{3(u^2-1)}\,du\right) du = X(u)\int \frac{du}{(u^2-1)^{7/6}X(u)^2}, \tag{3.13}$$

and where the Wronskian is given by

$$X\,\frac{dX^*}{du} - X^*\,\frac{dX}{du} = \frac{1}{(u^2-1)^{7/6}}. \tag{3.14}$$

The goal of the remainder of this section is to prove the following theorem.

Theorem 3.2. *Suppose that $v(u)$ and $w(u)$ are solutions to* (3.4)–(3.7) *and that $X(u)$ satisfies* (3.10). *Then for some constant A,*

$$w(u) = 36Ay(u^2-1)^{13/6}\left(\frac{dX}{du}\right)^2. \tag{3.15}$$

Furthermore, if X and X^* are linearly independent solutions to (3.12), *then*

$$2\,\frac{dX^*}{dX} = C_0 - Ay \tag{3.16}$$

for some constant C_0.

To make the proof of Theorem 3.2 palatable, we first establish two lemmas which include the basic ingredients for what follows.

Lemma 3.3. *The general solution to* (3.9) *is given by*

$$\frac{w(u)}{(vX)^2(u^2-1)^{1/6}} = -2\int \frac{du}{(vX)^2(u^2-1)^{7/6}} + C_0, \tag{3.17}$$

where C_0 denotes the constant of integration.

Proof. Since (3.9) is a linear first order differential equation, we can solve it by multiplying both sides by an integrating factor $I(u)$. Thus,

$$\frac{d}{du}(wI) = -I\frac{2}{u^2-1}, \tag{3.18}$$

and

$$wI = -\int \frac{2I}{u^2-1}\,du, \tag{3.19}$$

where the integrating factor I can by written via (3.8) and (3.10) as

$$I = \exp\left(\frac{1}{3}\int \frac{uv-1}{v(u^2-1)}\,du\right) \tag{3.20}$$
$$= \exp\left(-\frac{1}{3}\int \frac{(v^2-2uv+1)+uv-v^2}{v(u^2-1)}\,du\right)$$
$$= \exp\left(-\frac{1}{3}\int\left\{6\,\frac{dv}{v}+\frac{u\,du}{u^2-1}+6\,\frac{dX}{X}\right\}\right) = (vX)^{-2}(u^2-1)^{-1/6}.$$

Inserting (3.20) into (3.19), we obtain (3.17). ∎

Our next objective is to eliminate the integral appearing in the solution (3.17) of $w(u)$.

LEMMA 3.4. *If X and X^* are linearly independent solutions of* (3.12), *then for some constant C_0,*

$$w(u) = 36(u^2-1)^{13/6}\,\frac{dX}{du}\left(C_0\,\frac{dX}{du} - 2\,\frac{dX^*}{du}\right). \tag{3.21}$$

Proof. We begin by dividing both sides of (3.12) by $(u^2-1)^{5/6}$, so that we may recast (3.12) in the self-adjoint form

$$\frac{d}{du}\left\{(u^2-1)^{7/6}\,\frac{dX}{du}\right\} + \frac{X}{36(u^2-1)^{5/6}} = 0. \tag{3.22}$$

For brevity, we introduce the parameters

$$\xi = (u^2-1)^{7/6}\,\frac{dX}{du}, \qquad \xi^* = (u^2-1)^{7/6}\,\frac{dX^*}{du}. \tag{3.23}$$

Employing this notation, we find from (3.22) that

$$\frac{d\xi}{du} = -\frac{X}{36(u^2-1)^{5/6}}, \qquad \frac{d\xi^*}{du} = -\frac{X^*}{36(u^2-1)^{5/6}}. \tag{3.24}$$

Using (3.17), (3.10), and the notation of (3.23) and (3.24), we find that

$$w(u) = (vX)^2(u^2-1)^{1/6}\left\{-2\int\frac{du}{(vX)^2(u^2-1)^{7/6}} + C_0\right\} \tag{3.25}$$
$$= (vX)^2(u^2-1)^{1/6}\left\{-2\int\frac{du}{36(u^2-1)^2(dX/du)^2(u^2-1)^{7/6}} + C_0\right\}$$
$$= (vX)^2(u^2-1)^{1/6}\left\{-2\int\frac{X\,du}{36(u^2-1)^{5/6}[(u^2-1)^{7/6}dX/du]^2X} + C_0\right\}$$
$$= (vX)^2(u^2-1)^{1/6}\left\{2\int\frac{d\xi}{X\xi^2} + C_0\right\}.$$

Integrating by parts in (3.25) and using (3.10) and (3.23), we find that

$$
\begin{aligned}
(3.26)\qquad w(u) &= (vX)^2(u^2-1)^{1/6}\Big\{2\int \frac{1}{X}\,\frac{d\xi}{\xi^2}+C_0\Big\}\\
&= (vX)^2(u^2-1)^{1/6}\Big\{-\frac{2}{X\xi}-2\int\frac{dX}{\xi X^2}+C_0\Big\}\\
&= \;-72\Big\{\frac{u^2-1}{X}\,\frac{dX}{du}+(u^2-1)^{13/6}\Big(\frac{dX}{du}\Big)^2\int\frac{du}{(u^2-1)^{7/6}X^2}\Big\}\\
&\quad + 36C_0(u^2-1)^{13/6}\Big(\frac{dX}{du}\Big)^2\\
&= 36(u^2-1)^{13/6}\,\frac{dX}{du}\Big(C_0\,\frac{dX}{du}-2\,\frac{dX^*}{du}\Big),
\end{aligned}
$$

where the relations (3.13) and (3.14) were applied to derive the expression on the last line. ∎

We are finally poised to prove the main result of this section.

Proof of Theorem 3.2. We may write (3.4) as

$$\frac{1}{y}\,\frac{dy}{du}=\frac{-2}{(u^2-1)w(u)}.$$

Using (3.21) and the notation defined in the proof of Lemma 3.4, namely (3.23) and (3.24), we may recast this identity as

$$\frac{1}{y}\,\frac{dy}{d\xi}=\frac{2}{X\xi(C_0\xi-2\xi^*)}. \tag{3.27}$$

From (3.23) and (3.14),

$$X\xi^*-X^*\xi=1, \tag{3.28}$$

so that (3.24) implies that

$$\xi\,\frac{d\xi^*}{du}-\xi^*\,\frac{d\xi}{du}=\frac{1}{36(u^2-1)^{5/6}}. \tag{3.29}$$

Equation (3.29) and the obvious relation

$$\frac{d\xi^*}{du}=\frac{d}{du}(\xi\cdot\xi^*/\xi)=\frac{\xi^*}{\xi}\,\frac{d\xi}{du}+\xi\,\frac{d}{du}(\xi^*/\xi) \tag{3.30}$$

together imply that

$$\frac{d}{du}(\xi^*/\xi)=\frac{1}{36(u^2-1)^{5/6}\xi^2(u)}. \tag{3.31}$$

Integrating both sides of (3.31) with respect to u and employing (3.24), we see that

$$\xi^*(u) = \xi(u) \int \frac{du}{36(u^2-1)^{5/6}\xi^2(u)} + C_1 = -\xi(u) \int \frac{d\xi}{X\xi^2(u)} + C_1, \tag{3.32}$$

where C_1 is the constant of integration. The first representation for $\xi^*(u)$ in (3.32) allows us to write identity (3.29) as

$$\begin{aligned} \xi(u)\left(\frac{d\xi(u)}{du} \cdot \int \frac{du}{36(u^2-1)^{5/6}\xi^2(u)} + \frac{\xi(u)}{36(u^2-1)^{5/6}\xi^2(u)}\right) \\ - \frac{d\xi(u)}{du}\left(\xi(u)\int \frac{du}{36(u^2-1)^{5/6}\xi^2(u)} + C_1\right) = \frac{1}{36(u^2-1)^{5/6}}. \end{aligned} \tag{3.33}$$

Simplifying (3.33) and applying (3.24), we see that

$$0 = C_1 \frac{d\xi}{du} = \frac{C_1 X(u)}{36(u^2-1)^{5/6}}, \tag{3.34}$$

so that $C_1 = 0$, provided $X(u)$ is a nontrivial solution to (3.12). Substituting the expression for ξ^* on the extreme right side of (3.32) into (3.27), we find that

$$\frac{2}{X\xi^2\left(C_0 + 2\int \frac{d\xi}{X\xi^2}\right)} = \frac{1}{y}\frac{dy}{d\xi}. \tag{3.35}$$

Integrating both sides of this equation with respect to ξ and exponentiating the result, we deduce that

$$C_0 + 2\int \frac{d\xi}{X\xi^2} = Ay, \tag{3.36}$$

where A is the constant of integration. Rewrite (3.25) in the form

$$\frac{w(u)}{(vX)^2(u^2-1)^{1/6}} = C_0 + 2\int \frac{d\xi}{X\xi^2}. \tag{3.37}$$

We complete the proof of (3.15) by substituting (3.37) into (3.36) and using the expansion for $v(u)$ given in (3.10).

To prove (3.16) we refer to the evaluation of the integral $\int \frac{d\xi}{X\xi^2}$ appearing within the parentheses of equation (3.26). Using (3.23) and inserting this calculation on the left side of (3.36), we find that

$$\begin{aligned} Ay &= C_0 + 2\int \frac{d\xi}{X\xi^2} \\ &= C_0 - \frac{2}{X(dX/du)}\left\{\frac{1}{(u^2-1)^{7/6}} + \frac{dX}{du} \cdot X \int \frac{du}{X^2(u^2-1)^{7/6}}\right\}. \end{aligned} \tag{3.38}$$

The identity (3.38) simplifies via (3.13) and (3.14) to

$$C_0 - 2\,\frac{dX^*/du}{dX/du} = Ay, \tag{3.39}$$

or alternatively

$$2\,\frac{dX^*}{dX} = C_0 - Ay. \tag{3.40} \blacksquare$$

In summary, (3.10), (3.15), and (3.16) constitute the three basic equations for v, w, and y in terms of the parameter u and the two linearly independent solutions of (3.12), namely $X(u)$ and $X^*(u)$.

Thus, the problem hinges on solving the linear second order homogeneous differential equation (3.12), and in the following section we present these solutions $X(u)$ in terms of the hypergeometric function. The function $X(u)$ can also be represented in terms of certain associated Legendre functions, but since they are also hypergeometric functions, we do not further pursue this particular line of investigation.

4. Hypergeometric solution of the Riccati equation. We devote this section to proving the following result.

THEOREM 4.1. *Suppose that $v(u)$ and $w(u)$ are solutions to* (3.4)–(3.7) *and that $X(u)$ satisfies* (3.12). *If $\lambda(u) = 1 - \{u - (u^2-1)^{1/2}\}^2$, then for $|\lambda| < 1$,*

$$v(\lambda) = -6\,\frac{(1-\lambda)^{1/2}\lambda}{X}\,\frac{dX}{d\lambda}, \tag{4.1}$$

$$w(\lambda) = 36Ay(1-\lambda)^{5/6}\,\frac{\lambda^{7/3}}{2^{1/3}}\left(\frac{dX}{d\lambda}\right)^2, \tag{4.2}$$

$$X(\lambda) = \frac{C}{\lambda^{1/6}}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{2};1;\lambda\right), \tag{4.3}$$

for some constants A and C such that $AC^2 = 2^{1/3}$.

Proof. On making the substitution $\omega = u/(u^2-1)^{1/2}$, we can write (3.12) in the form

$$(\omega^2-1)\,\frac{d^2X}{d\omega^2} + \frac{2}{3}\,\omega\,\frac{dX}{d\omega} + \frac{X}{36} = 0. \tag{4.4}$$

The linear change of variable $\gamma = (\omega+1)/2$ translates (4.4) into the hypergeometric equation

$$\gamma(1-\gamma)\,\frac{d^2X}{d\gamma^2} + \left(\frac{1}{3} - \frac{2}{3}\,\gamma\right)\frac{dX}{d\gamma} - \frac{X}{36} = 0. \tag{4.5}$$

These transform (3.10) and (3.15), respectively, into

$$v(\gamma) = 6\,\frac{\{\gamma(\gamma-1)\}^{1/2}}{X}\,\frac{dX}{d\gamma}, \tag{4.6}$$

$$w(\gamma) = 36Ay\,\frac{\{\gamma(\gamma-1)\}^{5/6}}{2^{1/3}}\left(\frac{dX}{d\gamma}\right)^2. \tag{4.7}$$

We next make the substitutions $\gamma = 1/\lambda$ and $X(\gamma) = \lambda^{-1/6}Y(\lambda)$ in (4.5). Let us denote by $Y'(\lambda)$ differentiation with respect to λ, so that

$$\frac{dX}{d\gamma} = \frac{d\lambda}{d\gamma}\frac{dX}{d\lambda} = -\lambda^2 \frac{d}{d\lambda}(\lambda^{-1/6}Y) = \frac{1}{6}\lambda^{5/6}(Y(\lambda) - 6\lambda Y'(\lambda)),$$

and also

$$\begin{aligned}\frac{d^2X}{d\gamma^2} &= \frac{d\lambda}{d\gamma}\frac{d}{d\lambda}\left(\frac{d\lambda}{d\gamma}\frac{dX}{d\lambda}\right) = -\lambda^2\frac{d}{d\lambda}\left(\frac{1}{6}\lambda^{5/6}(Y - 6\lambda Y')\right)\\ &= \frac{1}{36}\lambda^{11/6}(-5Y + 12\lambda(5Y' + 3Y'')).\end{aligned}$$

Inserting these calculations into (4.5) and multiplying both sides by $-\lambda^{-5/6}$, we obtain the equation

$$\lambda(1-\lambda)\frac{d^2Y}{d\lambda^2} + \left\{1 - \left(-\frac{1}{6} + \frac{1}{2} + 1\right)\lambda\right\}\frac{dY}{d\lambda} + \frac{1}{12}Y = 0. \tag{4.8}$$

With these changes of variables, relations (4.6) and (4.7) transform into (4.1) and (4.2), respectively. Since the general solution of (4.8) that is analytic at the origin is given by [2, p. 1]

$$Y(\lambda) = C\cdot {}_2F_1\left(-\frac{1}{6}, \frac{1}{2}; 1; \lambda\right), \qquad |\lambda| < 1, \tag{4.9}$$

we see that (4.3) holds for $0 < |\lambda| < 1$ and for some constant C. By (3.7) and (4.2),

$$1 = \lim_{\lambda\to 0}\frac{w(\lambda)}{y} = \lim_{\lambda\to 0} 36A(1-\lambda)^{5/6}\frac{\lambda^{7/3}}{2^{1/3}}\left(\frac{dX}{d\lambda}\right)^2 = \frac{AC^2}{2^{1/3}}. \tag{4.10}$$

In fact, we see that each of the initial conditions in (3.7) is satisfied as long as A and C satisfy (4.10). (If we had chosen a linear combination of (4.9) and a second linearly independent solution of (4.8), then (4.10) would not be satisfied.) ■

If instead of proceeding from (4.4) to (4.5), we make the transformation

$$X(s) = (1-s^2)^{1/3}Z(s), \tag{4.11}$$

then from (4.4) we may show that $Z(s)$ satisfies

$$(1-s^2)\frac{d^2Z}{ds^2} - 2s\frac{dZ}{ds} - \left\{\frac{1}{4} + \frac{4}{9}\frac{1}{1-s^2}\right\}Z = 0. \tag{4.12}$$

Two linearly dependent solutions for this associated Legendre equation are P_ν^μ and Q_ν^μ with $\nu = -1/2$ and $\mu = 2/3$. Since these functions can also be expressed in terms of hypergeometric functions (see, for example, [14, p. 999]), we do not further pursue this line of investigation.

5. Final parametric expressions for P, Q and R. Recalling the relations between the parameters (2.2) and the substitutions from the previous section

$$\omega = \frac{u}{(u^2-1)^{1/2}}, \qquad \gamma = \frac{\omega+1}{2}, \qquad \gamma = \frac{1}{\lambda}, \tag{5.1}$$

we may show that $u = (1-\lambda/2)/(1-\lambda)^{1/2}$. Employing (4.1) and (4.2) and noting that, by (3.3),

$$P(e^{-y}) = \frac{w}{vy}, \qquad Q(e^{-y}) = \frac{w^2}{y^2}, \qquad R(e^{-y}) = \frac{w^3 u}{y^3},$$

we deduce from Theorem 4.1 the following parametric representations for Eisenstein series.

THEOREM 5.1. *For* $|\lambda| < 1$,

$$P(q) = P(e^{-y}) = -6(1-\lambda)^{1/3}\lambda^{4/3} X \frac{dX}{d\lambda}, \tag{5.2}$$

$$Q(q) = Q(e^{-y}) = 6^4(1-\lambda)^{5/3}\lambda^{14/3}\left(\frac{dX}{d\lambda}\right)^4, \tag{5.3}$$

$$R(q) = R(e^{-y}) = 6^6\left(1-\frac{\lambda}{2}\right)(1-\lambda)^2\lambda^7\left(\frac{dX}{d\lambda}\right)^6, \tag{5.4}$$

where $X(\lambda)$ *is defined by the equation*

$$X(\lambda) = \lambda^{-1/6}\,{}_2F_1\left(-\frac{1}{6}, \frac{1}{2}; 1; \lambda\right), \tag{5.5}$$

and where y *is defined by* (2.2).

Here we have taken $C = 1$ and $A = 2^{1/3}$ in accordance with (4.10).

A few remarks should be made on the region of validity of Theorem 5.1. First note that each variable appearing in the theorem is a function of x, the square of the elliptic modulus, appearing in (2.2) and (2.1). Using (2.5), (2.6), and (3.3), we find that

$$\lambda(u) = 1 - [u - (u^2-1)^{1/2}]^2, \qquad u(x) = \frac{(1+x)(1-34x+x^2)}{(1+14x+x^2)^{3/2}}, \tag{5.6}$$

and so, for $0 < x < 1$,

$$\begin{aligned}(\lambda \circ u)(x) &\\ &= 1 - \left(-6\sqrt{3}\sqrt{-\frac{x(x-1)^4}{x^2+14x+1}} + \frac{(1+x)(x^2-34x+1)}{(x^2+14x+1)^{3/2}}\right)^2 \\ &= \frac{216(x-1)^4 x}{(x^2+14x+1)^3} + \frac{12\sqrt{3x}\,(x-1)^2(1+x)(x^2-34x+1)}{(x^2+14x+1)^3}\,i.\end{aligned} \tag{5.7}$$

It follows that

$$(5.8)\qquad |(\lambda\circ u)(x)-1| = \left(\frac{(x^6-174x^5+1455x^4+1532x^3+1455x^2-174x-1)^2}{(x^2+14x+1)^6} + \frac{432x(x-1)^4(x+1)^2(1-34x+x^2)^2}{(x^2+14x+1)^6}\right)^{1/2} = 1.$$

After a tedious but elementary calculation, we find that

$$(5.9)\qquad \frac{d}{dx}\operatorname{Re}(\lambda\circ u(x)) = \frac{216(1-x)^3(1+x)(1-34x+x^2)}{(1+14x+x^2)^4}$$

and

$$(5.10)\qquad \frac{d}{dx}\operatorname{Im}(\lambda\circ u(x)) = \frac{6\sqrt{3}(1-x)(1-10x+x^2)(1-164x-186x^2-164x^3+x^4)}{x^{1/2}(1+14x+x^2)^4}.$$

The polynomial $1-164x-186x^2-164x^3+x^4$ has the zero set

$$\left\{41+24\sqrt{3}\pm 4\sqrt{3(71+41\sqrt{3})},\ 41+24\sqrt{3}\pm 4i\sqrt{3(71+41\sqrt{3})}\right\}.$$

Likewise, the zeros of $1-10x+x^2$ and $1-34x+x^2$ are, respectively,

$$\{5\pm 2\sqrt{6}\}\quad\text{and}\quad\{(17+12\sqrt{2})^{\pm 1}\}.$$

Noting the sign of each polynomial appearing in (5.9) on the intervals corresponding to zeros lying in $(0,1)$, we see that $\operatorname{Re}(\lambda\circ u)$ increases for

$$0<x<(17+12\sqrt{2})^{-1}$$

and decreases for

$$(17+12\sqrt{2})^{-1}<x<1.$$

A similar analysis shows that $\operatorname{Im}(\lambda\circ u)$ is increasing for

$$0<x<41+24\sqrt{3}-4\sqrt{3(71+41\sqrt{3})}\quad\text{and}\quad 5-2\sqrt{6}<x<1,$$

and decreasing for

$$41+24\sqrt{3}-4\sqrt{3(71+41\sqrt{3})}<x<5-2\sqrt{6}.$$

In addition, we list the values taken on by the real and imaginary parts of $\lambda\circ u$ at these extrema.

x	$0,1$	$41+24\sqrt{3}-4\sqrt{3(71+41\sqrt{3})}$	$\frac{1}{17+12\sqrt{2}}$	$5-2\sqrt{6}$
$\operatorname{Re}(\lambda\circ u)$	0	1	2	1
$\operatorname{Im}(\lambda\circ u)$	0	1	0	-1

Thus we see that as x ranges from 0 to 1, $(\lambda\circ u)(x)$ gives a parametrization of the circle $\{1+e^{i\theta} \mid 0<\theta<2\pi\}$. These observations imply that the solution $X(\lambda)$ to (4.5) and the subsequent parametrizations for the Eisenstein series in Theorem 5.1 are valid for values of x such that $0<x<\zeta$ or $\alpha<x<1$, where $\alpha,\zeta\in(0,1)$ are the unique constants satisfying, respectively,

$$(\lambda\circ u)(\zeta)=e^{\pi i/3} \quad \text{and} \quad (\lambda\circ u)(\alpha)=e^{-\pi i/3}. \tag{5.11}$$

6. Conclusions. We have examined Ramanujan's differential relations (1.4) for the three Eisenstein series $P(q)$, $Q(q)$, and $R(q)$. The differential relations (1.4) remain invariant under the stretching group of transformations (3.2), giving rise to the invariants $u=R/Q^{3/2}$, $v=Q^{1/2}/P$, and $w=yQ^{1/2}$, where $y=-\log q$. By solving a first order Riccati equation, we have shown that the three Eisenstein series are given parametrically by (5.2)–(5.5) in terms of the parameter λ defined by $\lambda=1-\{u-(u^2-1)^{1/2}\}^2$. Using these formulas and (5.6), we have computed power series representations for the series P, Q, and R in the variable x, the square of the elliptic modulus. We have verified that, within the specified domains, the parametrizations presented in this paper agree with the classical representations for these series in terms of the elliptic parameters [3, pp. 126–129].

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School of Mathematics
and Applied Statistics
University of Wollongong
Wollongong, N.S.W. 2522, Australia
E-mail: jhill@uow.edu.au

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green St.
Urbana, IL 61801, U.S.A.
E-mail: berndt@math.uiuc.edu
tjhuber@math.uiuc.edu

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