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On the Stark–Shintani units and the ideal class groups in the cyclotomic $\mathbb{Z}_p$-extensions of class fields over real quadratic fields

by

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1. Introduction. First, we briefly explain the Stark–Shintani conjecture. Let F be a real quadratic field and $H(\mathfrak{f})$ (resp. $K(\mathfrak{f})$) the narrow ray class group (resp. the narrow ray class field) of F modulo $\mathfrak{f}$, where $\mathfrak{f}$ is an integral ideal of F. Let M be an abelian extension of F such that exactly one infinite prime of F (corresponding to the prescribed embedding into the real number field $\mathbb{R}$) splits in M. Let $\mathfrak{f}$ be the conductor of M over F and ν a totally positive integer of F with the property that $\nu+1 \in \mathfrak{f}$ and denote by the same letter ν the narrow ray class modulo $\mathfrak{f}$ represented by (ν). We know that M is a quadratic extension of the maximal totally real subfield M^+ of M and that the Galois group $\mathrm{Gal}(M/M^+)$ is generated by $\sigma(\nu)$, where $\sigma : H(\mathfrak{f}) \to \mathrm{Gal}(K(\mathfrak{f})/F)$ denotes the Artin map.

We define the Stark–Shintani ray class invariants by

$$X_{\mathfrak{f}}(c) = \exp(\zeta_F'(0,c) - \zeta_F'(0,c\nu))$$

for $c \in H(\mathfrak{f})$, where

$$\zeta_F(s,c) = \sum_{\mathfrak{a}\in c} \frac{1}{N(\mathfrak{a})^s},$$

and $\zeta_F'(s,c)$ denotes its derivative. For the subgroup G of $H(\mathfrak{f})$ corresponding to the extension M/F we put

$$X_{\mathfrak{f}}(c,G) = \prod_{g\in G} X_{\mathfrak{f}}(cg)$$

for $c \in H(\mathfrak{f})$.

In this situation, the Stark–Shintani conjecture is formulated as follows:

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CONJECTURE. *There exists a positive integer* m *such that*:

(i) $X_{\mathfrak{f}}(c,G)^m$ *is a unit of* M *for each* $c \in H(\mathfrak{f})$,

(ii) $\{X_{\mathfrak{f}}(c,G)^m\}^{\sigma(c')} = X_{\mathfrak{f}}(cc',G)^m$ *for any* $c, c' \in H(\mathfrak{f})$.

Shintani proved that the conjecture is true when M^+ is an abelian extension over $\mathbb{Q}$ (cf. [7, Theorem 2]).

Let p be an odd prime, M_∞ the cyclotomic $\mathbb{Z}_p$-extension of M, and M_n the n-th layer of M_∞/M, that is, M_n is the unique cyclic extension field over M of degree p^n which is contained in M_∞. Arakawa pointed out that if M^+ is an abelian extension over $\mathbb{Q}$, the conjecture is true for all M_n, and he also gave the class number formula in terms of these units (cf. [1]).

In this paper, we assume that M and p satisfy the following two conditions (P) and (D):

(P) The Stark–Shintani conjecture holds for all M_n. Namely, for each n there exists an integer $t(n)$ such that $X_{\mathfrak{f}_n}(c,G_n)^{t(n)}$ is a unit in M_n which satisfies $\{X_{\mathfrak{f}_n}(c,G_n)^{t(n)}\}^{\sigma(c')} = X_{\mathfrak{f}_n}(cc',G_n)^{t(n)}$ for all $c, c' \in H(\mathfrak{f}_n)$, where $\mathfrak{f}_n$ is the conductor of M_n and G_n is the subgroup of $H(\mathfrak{f}_n)$ which corresponds to M_n. Moreover, $t(n)$ is prime to p for each n.

(D) The prime p does not divide $[M:F]$ and for any subfield M' of M over F with $M' \not\subset M^+$, any prime divisor $\mathfrak{p}$ of $\mathfrak{f}$ is a divisor of the conductor of M'/F or a divisor of p. Moreover, if $\mathfrak{p}$ is a prime divisor of p which does not divide the conductor of M', then the decomposition field of $\mathfrak{p}$ in M'/F is $(M')^+$.

REMARK. If M is a quadratic extension of F and no prime above p splits in M/F, then conditions (P) and (D) are satisfied (cf. [4, Theorem 1]).

Under conditions (P) and (D), we let E_n be the full unit group of M_n and C_n the subgroup of E_n generated by $X_{\mathfrak{f}_n}(c,G_n)$ with $c \in H(\mathfrak{f}_n)/G_n$. We also put

$$E_n^- = \{u \in E_n \mid N_{M_n/M_n^+}(u) = 1\}.$$

Because $X_{\mathfrak{f}_n}(c,G_n)^{\sigma(\nu_n)} = X_{\mathfrak{f}_n}(c,G_n)^{-1}$ where ν_n is an element of $H(\mathfrak{f}_n)/G_n$ which corresponds to the generator of $\mathrm{Gal}(M_n/M_n^+)$ by class field theory, we see that $C_n \subset E_n^-$. Then by Arakawa's class number formula which we recall later, we can see that C_n has a finite index in E_n^-.

Our main theorem in this paper is the following:

MAIN THEOREM. *Let* F *be a real quadratic field and* M *an abelian extension over* F *in which exactly one infinite prime of* F *corresponding to the prescribed embedding of* F *into* $\mathbb{R}$ *splits. Suppose that* M *and* p *satisfy conditions* (P) *and* (D). *Let* A_n *be a Sylow* p*-subgroup of the ideal class group of* M_n *and* $A_n^- = \{c \in A_n \mid c^{\sigma(\nu_n)} = c^{-1}\}$, *where* M_n^+ *is the maximal totally real subfield of* M_n, *and let* B_n *be a Sylow* p*-subgroup of* E_n^-/C_n.

Moreover, we assume that $|A_n^-|$ *is bounded with respect to* n *and that all primes of* F *above* p *do not split in* M/M^+. *Then for any sufficiently large* n, *there exists an isomorphism*

$$A_n^- \cong B_n$$

as Galois modules.

Remark. This theorem is an analogue to Ozaki's result for the cyclotomic $\mathbb{Z}_p$-extension and the cyclotomic units of real abelian fields (cf. [6]).

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2. Properties of Stark–Shintani units. Nakagawa dealt in [4], [5] with the Stark–Shintani units in the cyclotomic $\mathbb{Z}_p$-extension of certain abelian extensions of real quadratic fields. He rewrote Arakawa's class number formula in terms of $X_{\mathfrak{f}}(c,G)$ and showed that each Tate cohomology group $\widehat{H}^i(\Gamma_{m,n}, C_m) = 0$ always vanishes for $i = 0,1$.

Assume that M and p satisfy conditions (P) and (D). Let $\mathfrak{f}_n, c_n, G_n$ and ν_n be as in the previous section.

For any character χ of $H(\mathfrak{f}_n)/G_n$ with $\chi(\nu_n) = -1$, we know that

$$L_F'(0,\chi) = \sum_{c\in H(\mathfrak{f}_n)/\langle G_n,\nu_n\rangle} \chi(c)\log X_{\mathfrak{f}_n}(c,G_n),$$

where $L_F(s,\chi)$ is the Hecke L-function of F associated to χ. Let $\mathfrak{f}_\chi$ be the conductor of χ and $\widetilde{\chi}$ the primitive character associated to χ. Then we have the equality

$$L_F'(0,\chi) = L_F'(0,\widetilde{\chi}) \prod_{\mathfrak{p}\mid\mathfrak{f}_n,\, \mathfrak{p}\nmid\mathfrak{f}_\chi} (1-\widetilde{\chi}(\mathfrak{p})).$$

Under assumption (D), we know that $\widetilde{\chi}(\mathfrak{p}) = -1$ for every $\mathfrak{p}$ such that $\mathfrak{p}\mid\mathfrak{f}_n$ and $\mathfrak{p}\nmid\mathfrak{f}_\chi$.

Using the same method as in [1] or [5], we have another version of Arakawa's class number formula.

Theorem 1 (cf. [5, Theorem 1]). *Assume that* M *and* p *satisfy conditions* (P) *and* (D). *Let* $h(M_n)$ (*resp.* $h(M_n^+)$) *be the class number of* M_n (*resp.* M_n^+). *Then*

$$h(M_n)/h(M_n^+) = (\textit{power of } 2)\cdot t(n)^{-[M_n^+:F]}\cdot [E_n^- : C_n].$$

Moreover, Nakagawa showed the following:

Theorem 2 (cf. [5, Proposition 2]). *For any integers* $m \ge n > 0$,

(i) $\widehat{H}^i(\Gamma_{m,n}, C_m) = 0$ *for all* i,
(ii) *the natural map* $E_n^-/C_n \to E_m^-/C_m$ *is injective.*

3. Proof of the main theorem. We fix an odd prime p. Let I_n (resp. P_n) be the ideal group (resp. the principal ideal group) of M_n and A_n a Sylow p-subgroup of the ideal class group of M_n. We identify all $\sigma(\nu_n)$ and write this as τ. For any $\mathrm{Gal}(M_n/M_n^+)$-module N, let the relative part of N be $N^- = \{c \in N \mid c^\tau = c^{-1}\}$. Let $I_n^{(p)}$ be the subgroup of I_n such that $I_n^{(p)}/P_n \cong A_n$. Since p is odd, we find that

$$0 \to P_n^- \to (I_n^{(p)})^- \to A_n^- \to 0$$

is an exact sequence.

Moreover, we put

$$P_n' = \{(\alpha) \in P_n \mid \alpha\alpha^\tau = \varepsilon\varepsilon^\tau \text{ with some } \varepsilon \in E_n\},$$

which is a subgroup of P_n^-.

Lemma 1. *For any non-negative integer n,*

(i) P_n^-/P_n' *has exponent* 2,
(ii) *the following sequence is exact*:

$$0 \to E_n^- \to (M_n^\times)^- \to P_n' \to 0.$$

Proof. For any $(\alpha) \in P_n^-$, we see that $\alpha\alpha^\tau = \varepsilon$ with some $\varepsilon \in E_n^+$, where E_n^+ is the unit group of M_n^+. Then (i) follows.

(ii) We only show the surjectivity of the mapping $(M_n^\times)^- \to P_n'$, since the remainder is clear. For any $(\alpha) \in P_n'$, we have $\alpha\alpha^\tau = \varepsilon\varepsilon^\tau$ with some $\varepsilon \in E_n$. Then the fact that $(\alpha) = (\varepsilon^{-1}\alpha)$ and $\varepsilon^{-1}\alpha \in M_n^-$ gives the surjectivity. ■

Lemma 2. *For any $m \geq n \geq 0$,*

(i) $H^1(\Gamma_{m,n}, (M_m^\times)^-) = 0$,
(ii) $H^1(\Gamma_{m,n}, E_m^-) \cong (P_m')^{\Gamma_{m,n}}/P_n'$.

Proof. (i) We denote by $N_{m,n}$ the norm from M_m to M_n and fix a generator γ of $\mathrm{Gal}(M_m/M_n)$. For any element α of $(M_m^\times)^-$ which satisfies $N_{m,n}(\alpha) = 1$, there exists an element β of $M_m^\times$ such that $\alpha = \beta/\beta^\gamma$ by Hilbert's Theorem 90. It is sufficient to show that β can be taken from $(M_m^\times)^-$.

Because $\alpha \in (M_m^\times)^-$, we see that $x := \beta\beta^\tau$ is in $(N_{M_m/M_m^+}(M_m^\times)) \cap (M_n^+)^\times$. Therefore x^2 is in $N_{M_n/M_n^+}(M_n^\times)$. On the other hand, $x^{p^{m-n}} = N_{m,n}(\beta)N_{m,n}(\beta)^\tau$ is also in $N_{M_n/M_n^+}(M_n^\times)$. Since p is prime to 2, we have $x \in N_{M_n/M_n^+}(M_n^\times)$. Let $x = yy^\tau$ with some $y \in M_n^\times$. Then

$$\alpha = \beta/\beta^\gamma = \beta y^{-1}/(\beta y^{-1})^\gamma$$

with $\beta y^{-1} \in (M_m^\times)^-$. This completes the proof of (i).

(ii) Taking the cohomology groups of the exact sequence of Lemma 1(ii), we have the exact sequence

$$0 \to E_n^- \to (M_n^\times)^- \to (P'_m)^{\Gamma_{m,n}} \to H^1(\Gamma_{m,n}, E_m^-) \to H^1(\Gamma_{m,n}, (M_m^\times)^-) = 0.$$

From this (ii) follows. ∎

Taking the p-part of the cohomology of the exact sequence

$$0 \to C_m \to E_m^- \to E_m^-/C_m \to 0,$$

by Lemma 2(ii) we obtain

$$(1) \qquad (P'_m)^{\Gamma_{m,n}}/P'_n \cong \widehat{H}^{-1}(\Gamma_{m,n}, E_m^-) \cong \widehat{H}^{-1}(\Gamma_{m,n}, B_m).$$

PROPOSITION 1. *Assume that no prime of M^+ above p splits in M. Then*

$$(P_m^-)^{\Gamma_{m,n}}/P_n^- \cong \ker(j_{m,n})$$

for all $m > n \geq 0$. Here $j_{m,n} : A_n^- \to A_m^-$ is the lift map of the ideal class groups.

Proof. For any principal ideal (α) in $(P_m^-)^{\Gamma_{m,n}}$, we have $(\alpha) = \mathfrak{A} \in I_n^-$ by assumption because all ramified primes in M_m/M_n are above p. Then

$$(P_m^-)^{\Gamma_{m,n}}/P_n^- \cong (I_n^- \cap P_m^-)/P_n^- \cong \ker(j_{m,n}). \ ∎$$

We now finish the proof of our main theorem. Since $|A_n^-|$ is bounded with respect to n, $|B_n|$ is also bounded by Theorem 1. Therefore there exists a positive integer n_0 such that $A_m^- \cong A_n^-$ (by the norm map) and $B_n \cong B_m$ (by natural injection) for all $m > n \geq n_0$. Then for any $n \geq n_0$, taking a sufficiently large m, we have

$$\widehat{H}^{-1}(\Gamma_{m,n}, B_m) \cong B_m \cong B_n \quad \text{and} \quad \ker(j_{m,n}) = A_n^-.$$

Then by (1) and Proposition 1, it is sufficient to show that $(P'_m)^{\Gamma_{m,n}}/P'_n$ is isomorphic to $(P_m^-)^{\Gamma_{m,n}}/P_n^-$. We note that both groups are p-groups with the same order. Therefore the isomorphism easily follows by Lemma 1(i). This completes the proof of the main theorem. ∎

REMARK. We can show the main theorem replacing $X_\mathfrak{f}(c, G)$ by another invariant $Y_\mathfrak{f}(c, G)$ introduced in [7], by using Arakawa's original class number formula. In this case, assumption (D) is not necessary.

4. Examples. Since the field M considered in the preceding sections is not totally real, the assumption that $|A_n^-|$ is bounded with respect to n may seem strong. In fact, if p splits completely in M, then $|A_n|$ is always unbounded and in the case of the CM-field, if p divides the "minus part" of the ideal class group, then the "minus part" in the cyclotomic $\mathbb{Z}_p$-extension is unbounded (cf. [3]). Thus there is the problem to find "non-trivial" examples which satisfy the assumptions of the main theorem. The author has found some such examples. Let $F = \mathbb{Q}(\sqrt{6})$, $M = F(\sqrt{6+20\sqrt{6}})$ and $p = 3$. Then p does not split in M. Since M is a quadratic extension over F, assumptions (P) and (D) are satisfied. By using KASH, we see that the class number of M is 6 and that of M_1 is 24. Then by Theorem 1 of [2], $|A_n| = |A_n^-|$ is bounded. By the same method, we find that the case of $F = \mathbb{Q}(\sqrt{2})$ and $M = F(\sqrt{5+13\sqrt{2}})$ also satisfies the assumptions of the main theorem for $p = 3$.

References

[1] T. Arakawa, *On the Stark–Shintani conjecture and certain relative class numbers*, Adv. Stud. Pure Math. 7 (1985), 1–6.

[2] T. Fukuda, *Remarks on $\mathbb{Z}_p$-extensions on number fields*, Proc. Japan Acad. Ser. A 70 (1994), 264–266.

[3] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. 93 (1976), 263–284.

[4] J. Nakagawa, *On the Stark–Shintani conjecture and cyclotomic $\mathbb{Z}_p$-extensions of class fields over real quadratic fields*, J. Math. Soc. Japan 36 (1984), 577–588.

[5] —, *On the Stark–Shintani conjecture and cyclotomic $\mathbb{Z}_p$-extensions of class fields over real quadratic fields II*, Tôhoku Math. J. 36 (1984), 439–452.

[6] M. Ozaki, *On the cyclotomic unit group and the ideal class group of a real abelian number field*, J. Number Theory 64 (1997), 211–222.

[7] T. Shintani, *On certain ray class invariants of real quadratic fields*, J. Math. Soc. Japan 30 (1978), 139–167.

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