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Densities of 4-ranks of $K_2(\mathcal{O})$

by

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1. Introduction. Since the 1960's, relationships between algebraic K-theory and number theory have been intensely studied. For number fields F and their rings of integers $\mathcal{O}_F$, the K-groups $K_0(\mathcal{O}_F), K_1(\mathcal{O}_F), K_2(\mathcal{O}_F), \ldots$ were a main focus of attention. From [7] we have

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \times C(F)$$

where $C(F)$ is the ideal class group of F, and

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^*,$$

the group of units of $\mathcal{O}_F$.

What can we say in general about $K_2(\mathcal{O}_F)$? Garland and Quillen in [3] and [10] showed that $K_2(\mathcal{O}_F)$ is finite. A conjecture of Birch and Tate connects the order of $K_2(\mathcal{O}_F)$ and the value of the zeta function of F at -1 when F is a totally real field. For abelian number fields, this conjecture has been confirmed. For totally real fields, it has been confirmed up to powers of 2 (see [13]). In [11] a 2-rank formula for $K_2(\mathcal{O}_F)$ was given by Tate. Some results on the 4-rank of $K_2(\mathcal{O}_F)$ were given in [8], [9], and [12]. To gain further insight into the 4-rank of $K_2(\mathcal{O}_F)$, we consider the following specific families of fields.

In this paper we deal with the 4-rank of the Milnor K-group $K_2(\mathcal{O})$ for the quadratic number fields $\mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$, $\mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$ for primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = 1$. In [1], the authors show that for the fields $E = \mathbb{Q}(\sqrt{pl})$, $\mathbb{Q}(\sqrt{2pl})$ and $F = \mathbb{Q}(\sqrt{-pl})$, $\mathbb{Q}(\sqrt{-2pl})$,

$$\text{4-rank } K_2(\mathcal{O}_E) = 1 \text{ or } 2,$$
$$\text{4-rank } K_2(\mathcal{O}_F) = 0 \text{ or } 1.$$

Each of the possible values of 4-ranks is then characterized by checking which ones of the quadratic forms $X^2 + 32Y^2$, $X^2 + 2pY^2$, $2X^2 + pY^2$

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represent a certain power of l over $\mathbb{Z}$. This approach makes numerical computations accessible. We should note that this approach involves quadratic symbols and determining the matrix rank over $\mathbb{F}_2$ of 3×3 matrices with Hilbert symbols as entries (see [4]). Fix a prime $p \equiv 7 \bmod 8$ and consider the set

$$\Omega = \left\{ l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1 \right\}.$$

Let

$$\begin{aligned} \upsilon &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}), \\ \mu &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{2pl})}), \\ \sigma &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}), \\ \tau &= 4\text{-rank } K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2pl})}), \end{aligned}$$

and also consider the sets

$$\begin{aligned} \Omega_1 &= \{l \in \Omega : \upsilon = 1\}, \\ \Omega_2 &= \{l \in \Omega : \upsilon = 2\}, \\ \Omega_3 &= \{l \in \Omega : \mu = 1\}, \\ \Omega_4 &= \{l \in \Omega : \mu = 2\}, \\ \Lambda_1 &= \{l \in \Omega : \sigma = 0\}, \\ \Lambda_2 &= \{l \in \Omega : \sigma = 1\}, \\ \Lambda_3 &= \{l \in \Omega : \tau = 0\}, \\ \Lambda_4 &= \{l \in \Omega : \tau = 1\}. \end{aligned}$$

We have computed the following (see Table 1 in Appendix): For $p = 7$, there are 9730 primes l in Ω with $l \leq 10^6$. Among them, there are 4866 primes (50.01%) in Ω_1 and Ω_3 and 4864 primes (49.99%) in Ω_2 and Ω_4. Also, there are 4878 primes (50.13%) in Λ_1 and Λ_3 and 4852 primes in Λ_2 and Λ_4. The goal of this paper is to prove the following theorem.

THEOREM 1.1. *For the fields* $\mathbb{Q}(\sqrt{pl})$ *and* $\mathbb{Q}(\sqrt{2pl})$, 4*-rank* 1 *and* 2 *each appear with natural density* $1/2$ *in* Ω. *For the fields* $\mathbb{Q}(\sqrt{-pl})$ *and* $\mathbb{Q}(\sqrt{-2pl})$, 4*-rank* 0 *and* 1 *each appear with natural density* $1/2$ *in* Ω.

Now consider the tuple of 4-ranks $(\upsilon, \mu, \sigma, \tau)$. By Corollary 5.6 in [1], there are eight possible tuples of 4-ranks. For $p = 7$, among the 9730 primes $l \in \Omega$ with $l \leq 10^6$, the eight possible tuples are realized by 1215, 1213, 1228, 1210, 1210, 1228, 1225, 1201 primes l respectively (see Table 2 in Appendix). And, in fact:

THEOREM 1.2. *Each of the eight possible tuples of* 4*-ranks appear with natural density* $1/8$ *in* Ω.

2. Preliminaries. Let $\mathcal{D}$ be a Galois extension of $\mathbb{Q}$, and $G=\mathrm{Gal}(\mathcal{D}/\mathbb{Q})$. Let $Z(G)$ be the center of G and $\mathcal{D}^{Z(G)}$ the fixed field of $Z(G)$. Let p be a rational prime which is unramified in $\mathcal{D}$ and β a prime of $\mathcal{D}$ containing p. Let $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right)$ denote the Artin symbol of p and $\{g\}$ the conjugacy class containing one element $g \in G$.

LEMMA 2.1. *$\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ for some $g \in Z(G)$ if and only if p splits completely in $\mathcal{D}^{Z(G)}$.*

Proof. $\left(\frac{\mathcal{D}/\mathbb{Q}}{p}\right) = \{g\}$ for some $g \in Z(G)$ if and only if $\left(\frac{\mathcal{D}/\mathbb{Q}}{\beta}\right) = g$ if and only if $\left(\frac{\mathcal{D}^{Z(G)}/\mathbb{Q}}{\beta}\right) = \left(\frac{\mathcal{D}/\mathbb{Q}}{\beta}\right)\big|_{\mathcal{D}^{Z(G)}} = g|_{\mathcal{D}^{Z(G)}} = \mathrm{Id}_{\mathrm{Gal}(\mathcal{D}^{Z(G)}/\mathbb{Q})}$ if and only if p splits completely in $\mathcal{D}^{Z(G)}$. ∎

Thus if we can show that rational primes split completely in the fixed field of the center of a certain Galois group G, then we know the associated Artin symbol is a conjugacy class containing one element. Hence we may identify the Artin symbol with this element and consider the symbol to be an automorphism which lies in $Z(G)$. Thus determining the order of $Z(G)$ gives us the number of possible choices for the Artin symbol.

Let G_1 and G_2 be finite groups and A a finite abelian group. Suppose $r_1 : G_1 \to A$ and $r_2 : G_2 \to A$ are two epimorphisms and $\mathcal{G} \subset G_1 \times G_2$ is the set $\{(g_1, g_2) \in G_1 \times G_2 : r_1(g_1) = r_2(g_2)\}$. Since A is abelian, there is an epimorphism $r : G_1 \times G_2 \to A$ given by $r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$. Thus $\mathcal{G} = \ker(r) \subset G_1 \times G_2$. One can check that $Z(\mathcal{G}) = \mathcal{G} \cap Z(G_1 \times G_2)$.

LEMMA 2.2. (i) *If $r_2|_{Z(G_2)}$ is trivial, then $Z(\mathcal{G}) = \ker(r_1|_{Z(G_1)}) \times Z(G_2)$.*
(ii) *$Z(\mathcal{G}) = Z(G_1) \times Z(G_2) \Leftrightarrow r_1|_{Z(G_1)}$ and $r_2|_{Z(G_2)}$ are both trivial.*

Proof. (i) Suppose $(g_1, g_2) \in Z(\mathcal{G}) \subset \ker(r)$ where $g_1 \in Z(G_1)$, $g_2 \in Z(G_2)$. Thus $1 = r(g_1, g_2) = r_1(g_1)r_2(g_2)^{-1}$ and so $r_1(g_1) = r_2(g_2)$. But $r_2(g_2) = 1$, which yields $r_1(g_1) = 1$. Thus $g_1 \in \ker(r_1|_{Z(G_1)})$. The other inclusion is clear.

(ii) Take $(g_1, 1), (1, g_2) \in Z(G_1) \times Z(G_2) = Z(\mathcal{G}) \subset \ker(r)$ to obtain that $r_1|_{Z(G_1)}$ and $r_2|_{Z(G_2)}$ are both trivial. The converse follows from part (i). ∎

We will use the following definition throughout this paper.

DEFINITION 2.3. For primes $p \equiv 7 \bmod 8$, $l \equiv 1 \bmod 8$ with $\left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1$, $\mathcal{K} = \mathbb{Q}(\sqrt{-2p})$, and $h(\mathcal{K})$ the class number of $\mathcal{K}$, we say:

- *l satisfies* $\langle 1, 32\rangle$ if $l = x^2 + 32y^2$ for some $x, y \in \mathbb{Z}$,
- *l satisfies* $\langle 2, p\rangle$ if $l^{h(\mathcal{K})/4} = 2n^2 + pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$,
- *l satisfies* $\langle 1, 2p\rangle$ if $l^{h(\mathcal{K})/4} = n^2 + 2pm^2$ for some $n, m \in \mathbb{Z}$ with $m \not\equiv 0 \bmod l$.

3. Three extensions. In this section, we consider three degree eight field extensions of $\mathbb{Q}$. The idea will be to study composites of these fields and relate Artin symbols to 4-ranks. Rational primes which split completely in a degree 64 extension of $\mathbb{Q}$ will relate to Artin symbols and thus 4-ranks. Therefore calculating the density of these primes will answer density questions involving 4-ranks.

3.1. *First extension.* Consider $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Let $\varepsilon = 1+\sqrt{2} \in (\mathbb{Z}[\sqrt{2}])^*$. Then ε is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ which has norm -1. The degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1})$. Set

$$N_1 = \mathbb{Q}(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{-1}).$$

Note that N_1 is the splitting field of the polynomial $x^4 - 2x^2 - 1$ and so has degree 8 over $\mathbb{Q}$. Therefore $\mathrm{Gal}(N_1/\mathbb{Q})$ is the dihedral group of order 8. Note that the automorphism induced by sending $\sqrt{\varepsilon}$ to $-\sqrt{\varepsilon}$ commutes with every element of $\mathrm{Gal}(N_1/\mathbb{Q})$. Thus $Z(\mathrm{Gal}(N_1/\mathbb{Q})) = \mathrm{Gal}(N_1/\mathbb{Q}(\sqrt{2}, \sqrt{-1}))$.

Observe that only the prime 2 ramifies in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\varepsilon})$, and so only the prime 2 ramifies in the compositum N_1 over $\mathbb{Q}$. Now as $l \in \Omega$ is unramified in N_1 over $\mathbb{Q}$, the Artin symbol $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ is defined for primes β of $\mathcal{O}_{N_1}$ containing l. Let $\left(\frac{N_1/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_1/\mathbb{Q}}{\beta}\right)$ in $\mathrm{Gal}(N_1/\mathbb{Q})$. The primes $l \in \Omega$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-1})$ and $N_1^{Z(\mathrm{Gal}(N_1/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$. Thus by Lemma 2.1, we have $\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{g\} \subset Z(\mathrm{Gal}(N_1/\mathbb{Q}))$. As $Z(\mathrm{Gal}(N_1/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_1/\mathbb{Q}}{l}\right)$. Combining this statement with Addendum 3.7 from [1], we have

REMARK 3.1.

$$\begin{aligned}\left(\frac{N_1/\mathbb{Q}}{l}\right) = \{\mathrm{id}\} &\Leftrightarrow l \text{ splits completely in } N_1\\ &\Leftrightarrow l \text{ satisfies } \langle 1, 32\rangle.\end{aligned}$$

3.2. *Second and third extension.* Consider the fixed prime $p \equiv 7 \bmod 8$. Note p splits completely in $\mathcal{L} = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ and so

$$p\mathcal{O}_{\mathcal{L}} = \mathfrak{B}\mathfrak{B}'$$

for some primes $\mathfrak{B} \neq \mathfrak{B}'$ in $\mathcal{L}$. The field $\mathcal{L}$ has narrow class number $h^+(\mathcal{L}) = 1$ as $h(\mathcal{L}) = 1$ and $N_{\mathcal{L}/\mathbb{Q}}(\varepsilon) = -1$ where $\varepsilon = 1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$ (see [5]). From [1], we have

LEMMA 3.2. *The prime $\mathfrak{B}$ which occurs in the decomposition of $p\mathcal{O}_{\mathcal{L}}$ has a generator $\pi = a + b\sqrt{2} \in \mathcal{O}_{\mathcal{L}}$, unique up to a sign and to multiplication by the square of a unit in $\mathcal{O}_{\mathcal{L}}^*$ for which $N_{\mathcal{L}/\mathbb{Q}}(\pi) = a^2 - 2b^2 = -p$.*

Since $N_{\mathcal{L}/\mathbb{Q}}(\pi) = -p$, the degree 4 extension $\mathbb{Q}(\sqrt{2}, \sqrt{\pi})$ over $\mathbb{Q}$ has normal closure $\mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p})$. Set

$$N_2 = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{-p}).$$

Then N_2 is Galois over $\mathbb{Q}$ and $[N_2 : \mathbb{Q}] = 8$. Such an extension N_2 exists since the 2-Sylow subgroup of the ideal class group of $\mathbb{Q}(\sqrt{-2p})$ is cyclic of order divisible by 4 (see [2]). Thus the Hilbert class field of $\mathbb{Q}(\sqrt{-2p})$ contains a unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{-2p})$. By Lemma 2.3 in [1], N_2 is the unique unramified cyclic degree 4 extension over $\mathbb{Q}(\sqrt{-2p})$. Also compare [6]. Similar to arguments in Section 3.1, $\mathrm{Gal}(N_2/\mathbb{Q})$ is the dihedral group of order 8. Note that the automorphism induced by sending $\sqrt{\pi}$ to $-\sqrt{\pi}$ commutes with every element of $\mathrm{Gal}(N_2/\mathbb{Q})$. Thus $Z(\mathrm{Gal}(N_2/\mathbb{Q})) = \mathrm{Gal}(N_2/\mathbb{Q}(\sqrt{2}, \sqrt{-p}))$.

PROPOSITION 3.3. *If $l \in \Omega$, then l is unramified in N_2 over $\mathbb{Q}$.*

Proof. Since $p \equiv 7 \bmod 8$, the discriminant of $\mathbb{Q}(\sqrt{-2p})$ is $-8p$. For $l \in \Omega$, we have $\left(\frac{-2p}{l}\right) = 1$ and so l is unramified in $\mathbb{Q}(\sqrt{-2p})$. By Lemma 2.3 in [1], we conclude that l is unramified in N_2 over $\mathbb{Q}$. ■

As $l \in \Omega$ is unramified in N_2 over $\mathbb{Q}$, the Artin symbol $\left(\frac{N_2/\mathbb{Q}}{\beta}\right)$ is defined for primes β of $\mathcal{O}_{N_2}$ containing l. Let $\left(\frac{N_2/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{N_2/\mathbb{Q}}{\beta}\right)$ in $\mathrm{Gal}(N_2/\mathbb{Q})$. The primes $l \in \Omega$ split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{-p})$ and $N_2^{Z(\mathrm{Gal}(N_2/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-p})$. By Lemma 2.1, we see that $\left(\frac{N_2/\mathbb{Q}}{l}\right) = \{h\} \subset Z(\mathrm{Gal}(N_2/\mathbb{Q}))$ for some $h \in Z(\mathrm{Gal}(N_2/\mathbb{Q}))$. As $Z(\mathrm{Gal}(N_2/\mathbb{Q}))$ has order 2, there are two possible choices for $\left(\frac{N_2/\mathbb{Q}}{l}\right)$. Combining this statement and Lemmas 3.3 and 3.4 from [1], we have

REMARK 3.4.

$$\begin{aligned}\left(\frac{N_2/\mathbb{Q}}{l}\right) = \{\mathrm{id}\} &\Leftrightarrow l \text{ splits completely in } N_2\\ &\Leftrightarrow l \text{ satisfies } \langle 1, 2p\rangle.\\ \left(\frac{N_2/\mathbb{Q}}{l}\right) \neq \{\mathrm{id}\} &\Leftrightarrow l \text{ does not split completely in } N_2\\ &\Leftrightarrow l \text{ satisfies } \langle 2, p\rangle.\end{aligned}$$

Finally, for $l \in \Omega$, l splits completely in $\mathbb{Q}(\zeta_{16}) \Leftrightarrow l \equiv 1 \bmod 16$. This yields

REMARK 3.5.

$$\begin{aligned}\left(\frac{\mathbb{Q}(\zeta_{16})/\mathbb{Q}}{l}\right) = \{\mathrm{id}\} &\Leftrightarrow l \text{ splits completely in } \mathbb{Q}(\zeta_{16})\\ &\Leftrightarrow l \equiv 1 \bmod 16.\end{aligned}$$

4. The composite and two theorems. In this section we consider the composite field $N_1 N_2 \mathbb{Q}(\zeta_{16})$. Set

$$L = N_1 N_2 \mathbb{Q}(\zeta_{16}).$$

Note that $[L : \mathbb{Q}] = 64$. As N_1, N_2, and $\mathbb{Q}(\zeta_{16})$ are normal extensions of $\mathbb{Q}$, L is a normal extension of $\mathbb{Q}$.

For $l \in \Omega$, l is unramified in L as it is unramified in N_1, N_2, and $\mathbb{Q}(\zeta_{16})$. The Artin symbol $\left(\frac{L/\mathbb{Q}}{\beta}\right)$ is now defined for some prime β of $\mathcal{O}_L$ containing l. Let $\left(\frac{L/\mathbb{Q}}{l}\right)$ denote the conjugacy class of $\left(\frac{L/\mathbb{Q}}{\beta}\right)$ in $\mathrm{Gal}(L/\mathbb{Q})$. Letting $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p}) \subset L$, we prove

Lemma 4.1. *$Z(\mathrm{Gal}(L/\mathbb{Q})) = \mathrm{Gal}(L/M)$ is elementary abelian of order* 8.

Proof. For $\sigma \in \mathrm{Gal}(L/M)$, σ can only change the sign of $\sqrt{\varepsilon}$, $\sqrt{\pi}$, and $\sqrt{\zeta_8}$ as $\varepsilon \in M$. Since $L = M(\sqrt{\varepsilon}, \sqrt{\pi}, \sqrt{\zeta_8})$, $\mathrm{Gal}(L/M)$ is elementary abelian of order 8. Now consider the restrictions $r_1 : G_1 \to \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and $r_2 : G_2 \to \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ where $G_1 = \mathrm{Gal}(N_1/\mathbb{Q})$ and $G_2 = \mathrm{Gal}(N_2/\mathbb{Q})$. Clearly $r_1|_{Z(G_1)}$ and $r_1|_{Z(G_2)}$ are both trivial. Then by Lemma 2.2(ii), $Z(\mathcal{G})$ is elementary abelian of order 4 where $\mathcal{G} = \mathrm{Gal}(N_1 N_2/\mathbb{Q})$. Now consider the restrictions $R_1 : \mathrm{Gal}(\mathbb{Q}(\zeta_{16})/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ and $R_2 : \mathcal{G} \to \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$. Note that $\ker(R_1)$ is cyclic of order 2 and $Z(\mathcal{G}) = \mathrm{Gal}(M/\mathbb{Q})$. Thus $R_2|_{Z(\mathcal{G})}$ is trivial and so by the above and Lemma 2.2(i), $Z(\mathrm{Gal}(L/\mathbb{Q})) \cong \mathbb{Z}/2\mathbb{Z} \times Z(\mathcal{G}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $Z(\mathrm{Gal}(L/\mathbb{Q})) = \mathrm{Gal}(L/M)$. ■

Now for $l \in \Omega$, l splits completely in $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{-p})$ and so splits completely in the composite field $M = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. From Lemma 4.1, $L^{Z(\mathrm{Gal}(L/\mathbb{Q}))} = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-p})$. So by Lemma 2.1, we have

$$\left(\frac{L/\mathbb{Q}}{l}\right) = \{k\} \subset Z(\mathrm{Gal}(L/\mathbb{Q})) \quad \text{for some } k \in \mathrm{Gal}(L/\mathbb{Q}).$$

As $Z(\mathrm{Gal}(L/\mathbb{Q}))$ has order 8, there are eight possible choices for $\left(\frac{L/\mathbb{Q}}{l}\right)$. Using Remarks 3.1, 3.4, and 3.5, we now make the following one-to-one correspondences.

Remark 4.2. (i) $\left(\frac{L/\mathbb{Q}}{l}\right) = \{\mathrm{id}\} \Leftrightarrow l$ splits completely in $L \Leftrightarrow$

$$\left\{\begin{array}{l} l \text{ splits completely in } N_1, \\ N_2, \text{ and } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\}.$$

(ii) $\left(\frac{L/\mathbb{Q}}{l}\right) \neq \{\mathrm{id}\} \Leftrightarrow l$ does not split completely in L. Now there are seven cases:

$$(1)\qquad \left\{\begin{array}{l} l \text{ splits completely in } N_1, \\ \text{but does not in } N_2 \text{ or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\},$$

$$(2)\qquad \left\{\begin{array}{l} l \text{ splits completely in } N_1 \\ \text{and } N_2, \text{ but does not in } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\},$$

$$(3)\qquad \left\{\begin{array}{l} l \text{ splits completely in} \\ N_2, \text{ but does not in } N_1 \\ \text{or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\},$$

$$(4)\qquad \left\{\begin{array}{l} l \text{ splits completely in} \\ N_2 \text{ and } \mathbb{Q}(\zeta_{16}), \\ \text{but does not in } N_1 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 1, 2p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\},$$

$$(5)\qquad \left\{\begin{array}{l} l \text{ splits completely in } N_1 \\ \text{and } \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ satisfies } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\},$$

$$(6)\qquad \left\{\begin{array}{l} l \text{ splits completely in} \\ \mathbb{Q}(\zeta_{16}), \text{ but does not in } N_1 \\ \text{or } N_2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 1 \bmod 16 \end{array}\right\},$$

$$(7)\qquad \left\{\begin{array}{l} l \text{ does not split completely} \\ \text{in } N_1,\ N_2, \text{ or } \mathbb{Q}(\zeta_{16}) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} l \text{ does not satisfy } \langle 1, 32\rangle \\ l \text{ satisfies } \langle 2, p\rangle \\ l \equiv 9 \bmod 16 \end{array}\right\}.$$

Now using Theorems 5.2–5.5 from [1], we relate each Artin symbol $\left(\frac{L/\mathbb{Q}}{l}\right)$ to each of the eight possible tuples of 4-ranks.

Remark 4.3. From Remark 4.2, case (i) occurs if and only if we have $(2, 2, 1, 1)$. For case (ii),

- (1) occurs if and only if we have $(1, 2, 0, 1)$,
- (2) occurs if and only if we have $(2, 1, 1, 0)$,
- (3) occurs if and only if we have $(2, 1, 0, 1)$,
- (4) occurs if and only if we have $(2, 2, 0, 0)$,
- (5) occurs if and only if we have $(1, 1, 0, 0)$,
- (6) occurs if and only if we have $(1, 1, 1, 1)$,
- (7) occurs if and only if we have $(1, 2, 1, 0)$.

We can now prove Theorem 1.2.

Proof. Consider the set $X = \big\{l$ prime : l is unramified in L and $\big(\frac{L/\mathbb{Q}}{l}\big) = \{k\} \subset Z(\mathrm{Gal}(L/\mathbb{Q}))\big\}$ for some $k \in \mathrm{Gal}(L/\mathbb{Q})$. By the Chebotarev Density Theorem, the set X has natural density $1/64$ in the set of all primes. Recall

$$\Omega = \left\{ l \text{ rational prime} : l \equiv 1 \bmod 8 \text{ and } \left(\frac{l}{p}\right) = \left(\frac{p}{l}\right) = 1 \right\}$$

for some fixed prime $p \equiv 7 \bmod 8$. By Dirichlet's Theorem on primes in arithmetic progressions, Ω has natural density $1/8$ in the set of all primes. Thus X has natural density $1/8$ in Ω. By Remarks 4.2 and 4.3, each of the eight choices for $\big(\frac{L/\mathbb{Q}}{l}\big)$ is in one-to-one correspondence with each of the possible tuples of 4-ranks. Thus each of the eight possible tuples of 4-ranks appear with natural density $1/8$ in Ω. ∎

Now we can prove Theorem 1.1.

Proof. We see from Remark 4.3 that:

- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{pl})}) = 1$ in cases (ii)(1), (5), (6), (7),
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{2pl})}) = 2$ in cases (i) and (ii)(1), (4), (7),
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-pl})}) = 0$ in cases (ii)(1), (3), (4), (5),
- 4-rank $K_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-2pl})}) = 1$ in cases (i) and (ii)(1), (3), (6).

As each of the 4-rank tuples occur with natural density $1/8$, for the fields $\mathbb{Q}(\sqrt{pl})$ and $\mathbb{Q}(\sqrt{2pl})$, we have 4-rank 1 and 2 each appear with natural density $4 \cdot \frac{1}{8} = \frac{1}{2}$ in Ω. For the fields $\mathbb{Q}(\sqrt{-pl})$ and $\mathbb{Q}(\sqrt{-2pl})$, 4-rank 0 and 1 each appear with natural density $4 \cdot \frac{1}{8} = \frac{1}{2}$ in Ω. ∎

Appendix. The following tables motivated possible density results of 4-ranks of tame kernels. We consider primes $l \in \Omega$ with $l \leq N$ for a fixed prime $p \equiv 7 \bmod 8$ and positive integer N. For Table 1, we consider the sets $\Omega_1, \ldots, \Omega_4$ and $\Lambda_1, \ldots, \Lambda_4$ as in the introduction. For Table 2, we consider the sets

$$\begin{aligned}
I_1 &= \{l \in \Omega : \text{4-rank tuple is } (1,1,0,0)\},\\
I_2 &= \{l \in \Omega : \text{4-rank tuple is } (1,1,1,1)\},\\
I_3 &= \{l \in \Omega : \text{4-rank tuple is } (2,1,1,0)\},\\
I_4 &= \{l \in \Omega : \text{4-rank tuple is } (2,1,0,1)\},\\
I_5 &= \{l \in \Omega : \text{4-rank tuple is } (1,2,1,0)\},\\
I_6 &= \{l \in \Omega : \text{4-rank tuple is } (1,2,0,1)\},\\
I_7 &= \{l \in \Omega : \text{4-rank tuple is } (2,2,0,0)\},\\
I_8 &= \{l \in \Omega : \text{4-rank tuple is } (2,2,1,1)\}.
\end{aligned}$$

Table 1

Primes	$p = 7$		$p = 23$		$p = 31$	
Cardinality	$N = 1000000$	%	$N = 1000000$	%	$N = 1000000$	%
$\lvert\Omega\rvert$	9730		9742		9754	
$\lvert\Omega_1\rvert$	4866	50.01	4905	50.35	4916	50.40
$\lvert\Omega_2\rvert$	4864	49.99	4837	49.65	4838	49.60
$\lvert\Omega_3\rvert$	4866	50.01	4911	50.41	4851	49.73
$\lvert\Omega_4\rvert$	4864	49.99	4831	49.59	4903	50.27
$\lvert\Lambda_1\rvert$	4878	50.13	4912	50.42	4930	50.54
$\lvert\Lambda_2\rvert$	4852	49.87	4830	49.58	4824	49.46
$\lvert\Lambda_3\rvert$	4878	50.13	4876	50.05	4943	50.68
$\lvert\Lambda_4\rvert$	4852	49.87	4866	49.95	4811	49.32

Table 2

Primes	$p = 7$		$p = 23$		$p = 31$	
Cardinality	$N = 1000000$	%	$N = 1000000$	%	$N = 1000000$	%
$\lvert\Omega\rvert$	9730		9742		9754	
$\lvert I_1\rvert$	1215	12.49	1246	12.79	1246	12.77
$\lvert I_2\rvert$	1213	12.46	1229	12.62	1203	12.33
$\lvert I_3\rvert$	1228	12.62	1211	12.43	1214	12.45
$\lvert I_4\rvert$	1210	12.44	1225	12.57	1188	12.18
$\lvert I_5\rvert$	1210	12.44	1204	12.36	1227	12.58
$\lvert I_6\rvert$	1228	12.62	1226	12.58	1240	12.71
$\lvert I_7\rvert$	1225	12.59	1215	12.47	1256	12.88
$\lvert I_8\rvert$	1201	12.34	1186	12.17	1180	12.10

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