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The 8-rank of tame kernels of quadratic number fields

by

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1. Introduction. The purpose of this paper is to prove the following theorem.

Theorem 1.1. *For any finite abelian group G of exponent* 8*, there are infinitely many imaginary quadratic fields E such that*

$$K_2\mathcal{O}_E/(K_2\mathcal{O}_E)^8 \simeq G.$$

For any finite abelian group H of exponent 8 *with* $\mathrm{rk}_2(H) \geq 2 + \mathrm{rk}_4(H)$*, there are infinitely many real quadratic fields F such that*

$$K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq H.$$

Note that $\mathrm{rk}_2(K_2\mathcal{O}_F) \geq [F : \mathbb{Q}]$ for all totally real fields F.

Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with d a square free integer. Let $\mathrm{Cl}(F)$ be the class group of F, and $\mathrm{Cl}^+(F)$ the narrow class group of F. The study of the 2-Sylow subgroup of $\mathrm{Cl}^+(F)$ has a very long history. Gauss's genus theory gives the 2-rank formula of $\mathrm{Cl}^+(F)$ (see [10] and [11] for details). Then Rédei studied the 2-, 4-, 8-rank of $\mathrm{Cl}^+(F)$ in a series of papers ([26], [27]). Stevenhagen's paper [29] contains a nice review of Rédei's methods. In particular, Rédei proved that for any nonnegative integers $r_8 \leq r_4 \leq r_2$, there are infinitely many real quadratic number fields such that r_2, r_4 and r_8 are the 2-, 4-, 8-rank of $\mathrm{Cl}^+(F)$ respectively.

Later, Morton [17] proved that Rédei's theorem holds for imaginary quadratic fields, i.e., there are infinitely many imaginary quadratic fields E for which the 2-, 4-, 8 ranks of $\mathrm{Cl}(E)$ have arbitrarily assigned values. He also gave a much simpler proof of Rédei's theorem for real quadratic fields (see [18] and [16]). Morton's results were generalized by Stevenhagen [30] by using the theory of governing fields. Kolster [14] gave an algorithm to compute the 2^n-rank of $\mathrm{Cl}^+(F)$ for every n. In this paper, we will mainly use Kolster's algorithm.

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One should note that the study of the 8-rank of $\mathrm{Cl}^+(F)$ is much more difficult than that of the 4-rank. The reason is that the 8-rank formulas involve solutions of certain Diophantine equations which cannot be solved effectively.

By Tate's Theorem 6.2 of [31], one can get a 2-rank formula for $K_2\mathcal{O}_F$ (see [3] for a more explicit formula). Rédei's theorem gives a formula for the 4-rank of $\mathrm{Cl}^+(F)$ by means of the rank of a matrix whose entries are the local Hilbert symbols $(p_i, d)_{p_j}$, where p_i, p_j are prime divisors of the discriminant of F. Formulas for the 4-rank of $K_2\mathcal{O}_F$ are much more involved. If $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$, we have to deal with solutions of certain Diophantine equations. This is the difference between class groups and K_2 groups.

By Qin's methods of [21]–[23] and [25], we can determine the 2^n-rank of $K_2\mathcal{O}_F$ for $n = 2$ and 3. One can find the explicit structure of the tame kernels of quadratic fields F whose discriminant has few prime divisors in [21]–[25], [34], [35]. Qin's method is generalized to relatively quadratic extensions in [12]. The 4-rank density of the tame kernels of quadratic fields whose discriminant has less than 3 prime divisors can be found in [19], [20] and [5]. The 4-rank density for general quadratic fields can be found in [8].

In [32], Vazzana proved that the 8-rank of the tame kernels of quadratic fields can be arbitrarily large. He also studied certain cases where the 8-rank of the tame kernel of a quadratic field is exactly the 8-rank of the narrow class group.

In [25], Qin made the following conjecture.

CONJECTURE 1.2. *Let $k \geq 2$ and $n \in \mathbb{N}$. Given $k-1$ integers $r_4, r_8, \ldots, r_{2^k}$ satisfying $n \geq r_4 \geq r_8 \geq \cdots \geq r_{2^k} \geq 0$. Then there exist infinitely many quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ such that $d > 0$ square free has exactly n prime divisors, all of them $\equiv 1 \pmod 8$ and the 2^j-rank of $K_2\mathcal{O}_F$ is r_{2^j} $(2 \leq j \leq k)$.*

The same assertion should be true for $F = \mathbb{Q}(\sqrt{d})$ with $d = -d'$ or $d = 2d'$ or $d = -2d'$, where d' has exactly n prime divisors, all of them $\equiv 1 \pmod 8$.

In [24], Qin proved the above conjecture for $k = 2$ and $n-1 \geq r_4 \geq 0$. In our main theorem, there is a prime divisor q of d with $q \equiv 3$ or $5 \pmod 8$. Hence Conjecture 1.2 remains open. We put $q \equiv 3$ or $5 \pmod 8$ for a technical reason (to avoid the case $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$ in which even the 4-rank of $K_2\mathcal{O}_F$ is very complicated).

This paper is organized as follows. In Section 2, we briefly review the well known results on the 2^n-rank of the narrow class groups of quadratic number fields in the language of [14]. In Section 3, we briefly review Qin's theorems on the 2^n-rank $(n \leq 3)$ of the tame kernels of quadratic number fields which we will use in the next two sections. In Section 3, we prove that for any finite abelian group G of exponent 8, there are infinitely many imaginary quadratic fields E such that $K_2\mathcal{O}_E/(K_2\mathcal{O}_E)^8 \simeq G$. In Section 5, we prove that for any finite abelian group H of exponent 8 with $\mathrm{rk}_2(H) \geq 2 + \mathrm{rk}_4(H)$, there are infinitely many real quadratic fields F such that $K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq H$.

Although we cannot prove that the imaginary quadratic fields E (resp. real quadratic fields F) with

$$K_2\mathcal{O}_E/(K_2\mathcal{O}_E)^8 \simeq G \quad (\text{resp. } K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq H)$$

have a positive density among all imaginary (resp. real) quadratic fields, our results show that for any G (resp. H) there exists a P (resp. Q) such that the primes q with

$$K_2\mathcal{O}_{\mathbb{Q}(\sqrt{-Pq})}/(K_2\mathcal{O}_{\mathbb{Q}(\sqrt{-Pq})})^8 \simeq G$$
$$(\text{resp. } K_2\mathcal{O}_{\mathbb{Q}(\sqrt{Qq})}/(K_2\mathcal{O}_{\mathbb{Q}(\sqrt{Qq})})^8 \simeq H)$$

have a positive density by Morton's Density Theorem in [17] and [18].

In the case of real quadratic fields, we assume in this paper that $\mathrm{rk}_2(H) \geq 2 + \mathrm{rk}_4(H)$. However one should note that there are many examples of real quadratic fields F with $\mathrm{rk}_2(K_2\mathcal{O}_F) = \mathrm{rk}_4(K_2\mathcal{O}_F) + 1$. Our construction depends on Morton's explicit construction of certain quadratic fields. While in those cases one always has $\mathrm{rk}_2(K_2\mathcal{O}_F) \geq \mathrm{rk}_4(K_2\mathcal{O}_F) + 2$, we believe that for any finite abelian group H of exponent 8 with $\mathrm{rk}_2(H) \geq 1 + \mathrm{rk}_4(H)$, there are infinitely many real quadratic fields F such that $K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq H$ (see Conjecture 5.8).

2. The 2^n-rank of the class groups of quadratic fields. In this section, we will briefly review the well known results on the 2^n-rank of the class groups of quadratic fields. We will use Kolster's method and notation of [14] to deal with the 2^n-rank of the class groups of quadratic fields for $n = 1, 2, 3$.

Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field, where d is a square free integer. Let $\mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}$. Let D be the discriminant of F. For each nontrivial positive square free divisor m of D, let $[m]$ be the product of the distinct ramified primes above the prime divisors of m. Let $\mathrm{Cl}(F)$ be the class group of F and $\mathrm{Cl}^+(F)$ the narrow class group of F. Let

$$\alpha = \begin{cases} \sqrt{d} & \text{if } d \equiv 2 \text{ or } 3 \pmod 4, \\ (1+\sqrt{d})/2 & \text{if } d \equiv 1 \pmod 4. \end{cases} \tag{2.1}$$

Then $\{1, \alpha\}$ is a basis of $\mathcal{O}_F$. An element $a + b\alpha \in \mathcal{O}_F$ is called *primitive* if $\mathrm{GCD}(a, b) = 1$ (see [9] and [14] for some equivalent descriptions). An integral ideal J is called *primitive* if

$$J = I[m],$$

where m is a square free positive divisor of D and I is an integral ideal such that I is a product of powers of unramified primes $\mathfrak{p}$ and $\mathfrak{p} \mid I$ implies $\mathfrak{p}^\sigma \nmid I$.

Let p be a rational prime. Let $a = p^\alpha u,\ b = p^\beta v$ be two nonzero rational numbers, where u and v are p-adic units. Then the *local Hilbert symbol* $(a,b)_p$ is defined to be

$$(a,b)_p = \begin{cases} (-1)^{\alpha\beta\varepsilon(p)} \left(\dfrac{u}{p}\right)^\beta \left(\dfrac{v}{p}\right)^\alpha & \text{if } p \text{ is odd,} \\ (-1)^{\varepsilon(u)\varepsilon(v)+\alpha\omega(v)+\beta\omega(u)} & \text{if } p = 2, \end{cases} \tag{2.2}$$

where

$$\varepsilon(x) = \frac{x-1}{2}, \qquad \omega(x) = \frac{x^2-1}{8}$$

(see [28] for details).

Let A be a matrix whose entries are local Hilbert symbols. Following Kolster's notation of [14], we can view A as a matrix $\varphi(A)$ over $\mathbb{F}_2$ if we replace 1 by 0 and -1 by 1. The *rank* of A is understood as the $\mathbb{F}_2$-rank of $\varphi(A)$.

Let k be the number of primes which are ramified in F, and $p_1, \ldots, p_k$ the prime divisors of the discriminant D. Let

$$R_F^{(1)} = \begin{pmatrix} (p_1,d)_{p_1} & (p_1,d)_{p_2} & \cdots & (p_1,d)_{p_k} \\ (p_2,d)_{p_1} & (p_2,d)_{p_2} & \cdots & (p_2,d)_{p_k} \\ \vdots & \vdots & \ddots & \vdots \\ (p_k,d)_{p_1} & (p_k,d)_{p_2} & \cdots & (p_k,d)_{p_k} \end{pmatrix} = \begin{pmatrix} (p_1,d) \\ \vdots \\ (p_k,d) \end{pmatrix},$$

where

$$(m,d) = ((m,d)_{p_1}, \ldots, (m,d)_{p_k})$$

for any $m \mid D$.

Theorem 2.1 (Rédei, [26]). *Let F be a quadratic number field. Then*

$$\mathrm{rk}_4(\mathrm{Cl}^+(F)) = k - 1 - \mathrm{rank}(R_F^{(1)}).$$

Let $k_1 = \mathrm{rank}(R^{(1)})$. Without loss of generality, we may assume that the first k_1 rows $\varphi((p_1,d)), \ldots, \varphi((p_{k_1},d))$ are linearly independent. Let $S^{(1)} = \{p_1, \ldots, p_{k_1}\}$ and

$$N_F^{(1)} = \begin{pmatrix} (p_1,d) \\ \vdots \\ (p_{k_1},d) \end{pmatrix}.$$

For any j with $k_1 + 1 \le j \le k$, one can find $p_{j1}, \ldots, p_{jl_j} \in S^{(1)}$ such that

$$(p_j p_{j1} \cdots p_{jl_j}, d) = (1, \ldots, 1).$$

Let $m_j = p_j p_{j1} \cdots p_{jl_j}$. As $(m_j, d) = (1, \ldots, 1)$, we have $m_j \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. By Proposition 2.1 and Corollary 2.3 of [14], there exists a primitive integral ideal I_j of norm less than $\sqrt{|d|}$ such that

$$I_j^2[m_j] = (z_j)$$

for some primitive element $z_j \in \mathcal{O}_F^+$, where $\mathcal{O}_F^+$ is the set of totally positive elements of $\mathcal{O}_F$. Let $t_j = \mathrm{Norm}(I_j)$ and

$$R_F^{(2)} = \begin{pmatrix} N_F^{(1)} \\ (t_{k_1+1}, d) \\ \vdots \\ (t_k, d) \end{pmatrix}.$$

Note that the rank of $R_F^{(2)}$ does not depend on the choice of I_j. The following theorem was proved by Waterhouse.

Theorem 2.2 (Waterhouse, [33]). *Let F be a quadratic number field. Then*

$$\mathrm{rk}_8(\mathrm{Cl}^+(F)) = k - 1 - \mathrm{rank}(R_F^{(2)}).$$

3. The 2-Sylow subgroups of the tame kernels of quadratic fields. In this section, we briefly review the known results on the 2-Sylow subgroups of the tame kernels of quadratic fields. Let F be a number field, r_1 the number of real embeddings of F, $g_2(F)$ the number of distinct prime ideals of $\mathcal{O}_F$ above 2, and $\mathrm{Cl}_2(F)$ the subgroup of $\mathrm{Cl}(F)$ generated by the prime ideals of $\mathcal{O}_F$ above 2. Then by Theorem 6.2 of [31],

$$\mathrm{rk}_2(K_2\mathcal{O}_F) = \mathrm{rk}_2(\mathrm{Cl}(F)/\mathrm{Cl}_2(F)) + g_2(F) + r_1 - 1 \tag{3.1}$$

(see also [3] and [2] for more details).

Let $F = \mathbb{Q}(\sqrt{d})$, where d is a square free integer (d is allowed to be negative), $E = \mathbb{Q}(\sqrt{-d})$, $\delta_F = \mathrm{rk}_2(\mathrm{Cl}^+(F)/\mathrm{Cl}_2^+(F)) - \mathrm{rk}_2(\mathrm{Cl}(F)/\mathrm{Cl}_2(F))$, where $\mathrm{Cl}_2^+(F)$ is the subgroup of $\mathrm{Cl}^+(F)$ generated by the prime ideals of $\mathcal{O}_F$ above 2.

Theorem 3.1 (Boldy, [1]). *Let $F = \mathbb{Q}(\sqrt{d})$ and $E = \mathbb{Q}(\sqrt{-d})$ with d a square free integer. Then*

$$\mathrm{rk}_4(K_2\mathcal{O}_F) = \mathrm{rk}_4(\mathrm{Cl}^+(E)/\mathrm{Cl}_2^+(E)) + g_2(E) + \delta_F - 1.$$

See also Theorem 3.4 of [4]. The following theorem can be used to tell if $\{-1, m\} \in (K_2\mathcal{O}_F)^2$, where $m \mid d$. Note that the theorem is only a special case ($2 \notin \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$) of Qin's theorems. In our explicit construction, we will always make F satisfy the condition $2 \notin \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$.

THEOREM 3.2 (Qin, [21], [22], [25]). *Let $F = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ square free. Suppose $m \mid d$ ($m > 0$ if $d > 0$) and $2 \notin \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. The set $S(d)$ is defined to be $\{\pm 1, \pm 2\}$ if $d > 0$ or $\{1, 2\}$ if $d < 0$. Then the Steinberg symbol $\{-1, m\}$ is in $(K_2\mathcal{O}_F)^2$ if and only if one can find an $\varepsilon \in S(d)$ such that for any odd prime $p \mid d$,*

$$(m, -d)_p = \left(\frac{\varepsilon}{p}\right).$$

The 8-rank of the tame kernels of quadratic number fields involves the solution of certain Diophantine equations. We know that a necessary condition for $\{-1, m\} \in (K_2\mathcal{O}_F)^4$ is that there is an $\epsilon \in \{1, 2\}$ such that

$$\epsilon m Z^2 = X^2 + dY^2 \tag{3.2}$$

is solvable. For a square free integer n and $i = 1, 3, 5, 7$, denote by n_i the product of all prime divisors of n which are $\equiv i \pmod 8$ ($n_i = 1$ if d has no prime divisor which is congruent to i modulo 8). We use the notation $(a, b) \overset{2}{=} 1$ to mean that the integers a and b have no common odd divisors. We let $\sigma(l) = 1$ or 0 according to whether $l \mid m_5$ or not. The following theorem is a special case of Qin's Theorem 2.4 of [25].

THEOREM 3.3 (Qin, [23], [25]). *Let $F = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}$ square free. Suppose $m \mid d$. Write $m = \pm m_1 m_3 m_5 m_7$ with $m_i \mid d_i$ for $i = 1, 3, 5, 7$. Assume that* (3.2) *is solvable and let $X_m, Y_m, Z_m \in \mathbb{N}$ with $(X_m, Y_m) = 1$ and $(Z_m, d) \overset{2}{=} 1$ be a solution of* (3.2).

Suppose that $2 \notin \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. Then $\{-1, m\} \in (K_2\mathcal{O}_F)^4$ if and only if for $i = 1, 3, 5, 7$, there are $h_i \mid d_i$, in particular, $h_i = 1$ is permitted, and $\varepsilon \in \{\pm 1, \pm 2\}$ such that for any odd prime $l \mid d$,

$$(d, m_3 h_1 h_5)_l (-2^{\sigma(l)} d, m_5 h_3 h_7)_l = \left(\frac{\varepsilon Z_m}{l}\right).$$

4. Tame kernels of imaginary quadratic fields. Let G be any finite abelian group of exponent 8. In this section, we will prove that there are infinitely many imaginary quadratic fields E such that

$$K_2\mathcal{O}_E/(K_2\mathcal{O}_E)^8 \simeq G.$$

By using Qin's method of [21]–[23] and [25], we will reduce the problem to showing that there are infinitely many real quadratic number fields F of certain types such that

$$\mathrm{Cl}^+(F)/(\mathrm{Cl}^+(F))^8 \simeq G,$$

while the existence of infinitely many such real quadratic number fields F can be proved by Morton's Theorem [18].

Let $s \le r$ be nonnegative integers. Then there exist $r + 1$ primes $p_1, \ldots, p_r, p_{r+1} = q$ such that

$$(4.1)\qquad \begin{aligned}&(1)\ p_i \equiv 1 \pmod 8 \quad \text{for } 1 \le i \le r;\\ &(2)\ \left(\frac{p_i}{p_j}\right) = 1 \quad \text{for } 1 \le i \ne j \le r;\\ &(3)\ q \equiv 5 \pmod 8;\\ &(4)\ \left(\frac{p_i}{q}\right) = \begin{cases} 1 & \text{if } 1 \le i \le s,\\ -1 & \text{if } s+1 \le i \le r.\end{cases}\end{aligned}$$

The existence can be proved easily. One can define primes p_j inductively by applying well known properties of the Legendre symbol.

Let $d = p_1 \cdots p_r q$ and $F = \mathbb{Q}(\sqrt{d})$. Recall that in Section 2, we defined

$$R_F^{(1)} = ((p_i, d)_{p_j})_{(r+1)\times(r+1)}.$$

By (2.1), we have

$$\varphi(R_F^{(1)}) = \begin{pmatrix} O_{s\times s} & O_{s\times(r-s+1)} \\ O_{(r-s+1)\times s} & A_F \end{pmatrix},$$

where the O's are zero matrices and

$$A_F = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1\\ 0 & 1 & \cdots & 0 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 1\\ 1 & 1 & \cdots & 1 & a \end{pmatrix}_{(r-s+1)\times(r-s+1)}$$

where $a \equiv r - s \pmod 2$. It is easy to see that $\operatorname{rank}(A_F) = r - s$. Then by Rédei's Theorem,

$$\operatorname{rk}_4(\mathrm{Cl}^+(F)) = r - \operatorname{rank}(A_F) = s.$$

By Gauss's genus theory, ${}_2\mathrm{Cl}^+(F)$ is generated by $[p_1], \ldots, [p_r], [q]$. And there is a unique nontrivial relation among these $r+1$ elements. We assume that this relation is

$$[p_1^{a_1} \cdots p_r^{a_r} q^b] = 1 \in \mathrm{Cl}^+(F), \qquad \text{where } a_i, b \in \{0, 1\}.$$

Since $[p_1^{a_1} \cdots p_r^{a_r} q^b] = (\alpha)$ for some $\alpha \in \mathcal{O}_F^+$ (the totally real elements of $\mathcal{O}_F$), we have $p_1^{a_1} \cdots p_r^{a_r} q^b = \mathrm{Norm}_{F/\mathbb{Q}}(\alpha)$. Hence $(p_1^{a_1} \cdots p_r^{a_r} q^b, d) = 1$. Since $(p_i, d) = 1$ for any $1 \le i \le s$, we have $(p_{s+1}^{a_{s+1}} \cdots p_r^{a_r} q^b, d) = 1$. Hence for any $s+1 \le j \le r$, we have $(p_{s+1}^{a_{s+1}} \cdots p_r^{a_r} q^b, d)_{p_j} = 1$, i.e., $\left(\frac{q}{p_j}\right)^{b+a_j} = 1$. Since $\left(\frac{q}{p_j}\right) = -1$ for $s+1 \le j \le r$, we have $a_{s+1} = \cdots = a_r = b$. The subgroup ${}_2\mathrm{Cl}^+(F) \cap (\mathrm{Cl}^+(F))^2$ is generated by the elements

$$[p_1], \ldots, [p_s], [p_{s+1} \cdots p_r q]$$

by Proposition 2.1 of [14], and there is exactly one nontrivial relation among these $s+1$ elements. By Proposition 2.1 of [14], for $1 \le i \le s+1$, there are $t_i \in \mathbb{Z}$ and $\alpha_i \in \mathcal{O}_F^+$ such that

$$\begin{aligned} p_1 t_1^2 &= \mathrm{Norm}_{F/\mathbb{Q}}(\alpha_1), \\ &\ \vdots \\ p_s t_s^2 &= \mathrm{Norm}_{F/\mathbb{Q}}(\alpha_s), \\ p_{s+1}\cdots p_r q t_{s+1}^2 &= \mathrm{Norm}_{F/\mathbb{Q}}(\alpha_{s+1}), \end{aligned} \tag{4.2}$$

where t_i $(1 \le i \le s+1)$ are the norms of some primitive integral ideals of $\mathcal{O}_F$.

By Lemma 2.5 of [14], $(t_i, d)_l$ is trivial for all primes l which are unramified in F. Let

$$(t_i, d) = ((t_i, d)_{p_1}, \ldots, (t_i, d)_{p_r}, (t_i, d)_q).$$

Note that $(t_i, d)_{p_1} \cdots (t_i, d)_{p_r} (t_i, d)_q = 1$ by the product formula.

Let

$$N_F^{(1)} = \begin{pmatrix} (p_{s+1}, d) \\ \vdots \\ (p_r, d) \end{pmatrix}, \qquad R_F^{(2)} = \begin{pmatrix} N_F^{(1)} \\ (t_1, d) \\ \vdots \\ (t_{s+1}, d) \end{pmatrix}.$$

By Theorem 2.2, the 8-rank of $\mathrm{Cl}^+(F)$ is

$$r_8 = r - \mathrm{rank}(R_F^{(2)}).$$

Let m be a divisor of d such that $[m] \in {}_2\mathrm{Cl}^+(F) \cap (\mathrm{Cl}^+(F))^2$. Note that if $q \mid m$, then $p_{s+1}\cdots p_r \mid m$ also. We assume that

$$m = p_1^{a_1} \cdots p_s^{a_s} (p_{s+1} \cdots p_r q)^b,$$

where $a_i, b \in \{0, 1\}$. We define

$$t_{(m)} = t_1^{a_1} \cdots t_s^{a_s} t_{s+1}^b.$$

Then there is a primitive element $\alpha \in \mathcal{O}_F^+$ such that $t_{(m)}^2 m = \mathrm{Norm}_{F/\mathbb{Q}}(\alpha)$ and $t_{(m)}$ is the norm of some primitive integral ideal of $\mathcal{O}_F$. By Theorem 2.6 of [14], $[m] \in {}_2\mathrm{Cl}^+(F) \cap (\mathrm{Cl}^+(F))^4$ if and only if there is an integral ideal I' whose class in $\mathrm{Cl}^+(F)$ is of exponent 2 such that for $t' = \mathrm{Norm}_{F/\mathbb{Q}}(I')$ the product $t_{(m)} \cdot t'$ is a norm from F, i.e., there is a divisor t' of $p_{s+1} \cdots p_r$ such that $(t_{(m)} t', d)$ is trivial. We write this fact as a proposition.

Proposition 4.1 (Kolster, Theorem 2.6 of [14]). *Let the notation be as above. Assume that* $[m] \in {}_2\mathrm{Cl}^+(F) \cap (\mathrm{Cl}^+(F))^2$ *and* $t_{(m)} \in \mathbb{Z}^+$ *such that*

$$t_{(m)}^2 m = \mathrm{Norm}_{F/\mathbb{Q}}(\alpha) \quad \textit{for some primitive } \alpha \in \mathcal{O}_F^+$$

and $t_{(m)}$ is the norm of some primitive integral ideal of $\mathcal{O}_F$. Then $[m] \in {}_2\mathrm{Cl}^+(F) \cap (\mathrm{Cl}^+(F))^4$ if and only if there is a divisor t' of $p_{s+1}\cdots p_r$ such that $(t_{(m)}t', d)$ is trivial.

Let $E = \mathbb{Q}(\sqrt{-d})$, where $d = p_1\cdots p_r q$ and p_i, q satisfy the four conditions of (4.1).

THEOREM 4.2. *With the notation as above, we have*

$$\mathrm{rk}_2(K_2\mathcal{O}_E) = r, \qquad \mathrm{rk}_4(K_2\mathcal{O}_E) = s.$$

Let $m \mid d$, where m is allowed to be negative. Then $\{-1, m\} \in (K_2\mathcal{O}_E)^2$ if and only if $[|m|] \in (\mathrm{Cl}^+(F))^2$.

Proof. By (3.1), we have $\mathrm{rk}_2(K_2\mathcal{O}_E) = r$. Let $F = \mathbb{Q}(\sqrt{d})$. Then by Theorem 3.1, we have $\mathrm{rk}_4(K_2\mathcal{O}_E) = s$.

Since $p_i \equiv 1 \pmod 8$ and $q \equiv 5 \pmod 8$, we have

$$\left(\frac{2}{p_i}\right) = 1 \quad \text{for} \quad 1 \le i \le r \quad \text{and} \quad \left(\frac{2}{q}\right) = -1.$$

Hence we can always choose $\varepsilon \in \{1, 2\}$ such that

$$(m, -d)_q = \left(\frac{\varepsilon}{q}\right)$$

and $\left(\frac{\varepsilon}{p_i}\right) = 1$. By Theorem 3.2, $\{-1, m\} \in (K_2\mathcal{O}_E)^2$ if and only if

$$(m, d)_{p_i} = 1$$

for any $1 \le i \le r$. Note that $(-1, d)_{p_i} = 1$ for $p_i \equiv 1 \pmod 8$ for any $1 \le i \le r$. Hence $(m, d)_{p_i} = 1$ if and only if $(|m|, d)_{p_i} = 1$. By Corollary 2.3 of [14], $[|m|] \in (\mathrm{Cl}^+(F))^2$ if and only if $(|m|, d) = 1$ for any prime p. By Lemma 2.5 and the product formula, $(|m|, d) = 1$ for all primes p if and only if $(|m|, d)_{p_i} = 1$ for all $1 \le i \le r$. So $\{-1, m\} \in (K_2\mathcal{O}_E)^2$ if and only if $[|m|] \in (\mathrm{Cl}^+(F))^2$. ∎

THEOREM 4.3. *Let the notation be as above. Let $F = \mathbb{Q}(\sqrt{d})$ and $E = \mathbb{Q}(\sqrt{-d})$. Let $m \in \mathbb{Z}$ with $m \mid d$ and $[|m|] \in (\mathrm{Cl}^+(F))^2$. Then $\{-1, m\} \in (K_2\mathcal{O}_E)^4$ if and only if $[|m|] \in (\mathrm{Cl}^+(F))^4$.*

Proof. Since $-d \equiv 3 \pmod 8$, we have $2 \notin \mathrm{Norm}_{E/\mathbb{Q}}(E^\times)$. Since $[|m|] \in (\mathrm{Cl}^+(F))^2$, there is a primitive element $\alpha \in \mathcal{O}_F^+$ such that

$$|m|\widetilde{Z}_m^2 = \mathrm{Norm}_{F/\mathbb{Q}}(\alpha),$$

where $\widetilde{Z}_m$ is the norm of a primitive integral ideal of $\mathcal{O}_F$. If $\alpha = X_m + Y_m\sqrt{d}$ with $X_m, Y_m \in \mathbb{Z}$, then $|m|\widetilde{Z}_m^2 = X_m^2 - dY_m^2$. If $\alpha = (X_m + Y_m\sqrt{d})/2$ with $X_m, Y_m \in \mathbb{Z}$ odd integers, then $|m|(2\widetilde{Z}_m)^2 = X_m^2 - dY_m^2$. We define

$$Z_m = \begin{cases} \widetilde{Z}_m & \text{if } \alpha \in \mathbb{Z} + \mathbb{Z}\sqrt{d}, \\ 2\widetilde{Z}_m & \text{otherwise,} \end{cases} \qquad \varepsilon_0 = \begin{cases} 1 & \text{if } \alpha \in \mathbb{Z} + \mathbb{Z}\sqrt{d}, \\ 2 & \text{otherwise.} \end{cases}$$

These X_m, Y_m, Z_m satisfy the conditions of Theorem 3.3. By Theorem 3.3, $\{-1, m\} \in (K_2\mathcal{O}_E)^4$ if and only if there exist $h_1 \mid p_1 \cdots p_r$, $h_3 \mid q$ and $\varepsilon \in \{\pm 1, \pm 2\}$ such that for any odd prime $l \mid d$,

$$(-d, m_3 h_1 h_5)_l (2^{\sigma(l)} d, m_5 h_3 h_7)_l = \left(\frac{\varepsilon Z_m}{l}\right). \tag{4.3}$$

Note that $m_3 = h_3 = h_7 = 1$. Since Z_m is prime to d, we have

$$\left(\frac{\varepsilon Z_m}{l}\right) = (d, \varepsilon Z_m)_l$$

for any primitive prime divisor $l \mid d$. Note that $\left(\frac{-1}{p_i}\right) = \left(\frac{2}{p_i}\right) = \left(\frac{-1}{q}\right) = 1$ and $\left(\frac{2}{q}\right) = -1$. Hence $(-1, h_1 h_5)_l = 1$ and $(\pm d, -1)_l = 1$ for any prime l. So we can assume that $\varepsilon = 1$ or 2.

Hence (4.3) holds if and only if we can find $h_1 \mid p_1 \cdots p_r$, $h_5 \mid q$ and $\varepsilon \in \{1, 2\}$ such that for any prime $l \mid d$, we have

$$\begin{aligned} &(1) \text{ if } q \nmid m, \text{ then } (h_1 h_5 \varepsilon Z_m, d)_l = 1 \text{ for all } l \mid d; \\ &(2) \text{ if } q \mid m, \text{ then } (h_1 h_5 q \varepsilon Z_m, d)_l = \begin{cases} 1 & \text{if } l = p_i,\ 1 \le i \le r, \\ -1 & \text{if } l = q. \end{cases} \end{aligned} \tag{4.4}$$

Since $(2, d)_{p_i} = 1$ $(1 \le i \le r)$ and $(2, d)_q = -1$, we have $(\varepsilon, d)_{p_i} = 1$ $(1 \le i \le r)$ and $(\varepsilon, d)_q = \pm 1$. Hence (4.4) holds if and only if we can find $h_1 \mid p_1 \cdots p_r$, $h_5 = 1$ or q and $\varepsilon = 1$ or 2 such that

$$\begin{aligned} &(1)\ (h_1 h_5 m_5 Z_m, d)_{p_i} = 1, \text{ where } 1 \le i \le r; \\ &(2)\ (h_1 h_5 m_5 Z_m, d)_q = (\varepsilon, d)_q. \end{aligned} \tag{4.5}$$

If $2 \nmid Z_m$, then $Z_m = \widetilde{Z}_m$. We know that $(h_1 h_5 m_5 Z_m, d)_{p_i} = 1$ for $1 \le i \le r$ implies $(h_1 h_5 m_5 Z_m, d)_q = 1$ by the product formula. Hence $\varepsilon = 1$. If $2 \mid Z_m$, then $Z_m = 2\widetilde{Z}_m$. Hence $\varepsilon = 2$ by the product formula. Item (2) of (4.5) is now $(h_1 h_5 m_5 \widetilde{Z}_m, d)_q = 1$. So (4.5) holds if and only if we can find $h_1 \mid p_1 \cdots p_r$ and $h_5 = 1$ or q such that

$$(h_1 h_5 m_5 \widetilde{Z}_m, d)_l = 1 \quad \text{for any } l \mid d. \tag{4.6}$$

Since $h_5 m_5 = 1$, q or q^2, and $(p_1, d)_l = \cdots = (p_s, d)_l = (p_{s+1} \cdots p_r q, d)_l = 1$ for any $l \mid d$, we see that (4.6) holds if and only if we can find $h_1' \mid p_{s+1} \cdots p_r$ such that

$$(h_1' \widetilde{Z}_m, d)_l = 1 \quad \text{for any } l \mid d. \tag{4.7}$$

By Theorem 2.6 of [14], $[|m|] \in (\mathrm{Cl}^+(F))^4$ is equivalent to the existence of an integral ideal $I' \in {}_2\mathrm{Cl}^+(F)$ such that for $t' = \mathrm{Norm}_{F/\mathbb{Q}}(I')$ we have $Z_m t' \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. Recall that ${}_2\mathrm{Cl}^+(F)$ is generated by $[p_i]$ $(1 \le i \le r)$ and $[q]$. Since $p_1, \ldots, p_s$ and $p_{s+1} \cdots p_r q$ are in $\mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$, we can assume that $t' \mid p_{s+1} \cdots p_r$. So $[|m|] \in (\mathrm{Cl}^+(F))^4$ is equivalent to the existence of an integral ideal $I' \in {}_2\mathrm{Cl}^+(F)$ such that $t' = \mathrm{Norm}_{F/\mathbb{Q}}(I') \mid p_{s+1} \cdots p_r$ and $Z_m t' \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$, i.e.,

$$(t' Z_m, d)_l = 1 \quad \text{for any } l \mid d. \tag{4.8}$$

It is easy to see that (4.8) is equivalent to (4.7). Hence $\{-1, m\} \in (K_2\mathcal{O}_E)^4$ if and only if $[|m|] \in (\mathrm{Cl}^+(F))^4$. ∎

We define

$$A_4 = \{m : m \in \mathbb{Z},\ m \mid d,\ \{-1, m\} \in (K_2\mathcal{O}_E)^4\},$$
$$B_4 = \{n : n \in \mathbb{Z}_{>0},\ n \mid d,\ [n] \in (\mathrm{Cl}^+(F))^4\},$$
$$G_4 = A_4(E^\times)^2/(E^\times)^2.$$

Then G_4 is a finite elementary 2-group. Since $-d \in A_4 \cap (E^\times)^2$, we have $\#G_4 = (\#A_4)/2$. Let $T_E = \{x \in E^\times : \{-1, x\} = 1\}$ be the Tate kernel of E. Then by Theorem 6.3 of [31], we have

$$T_E/(E^\times)^2 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Since $2 \in T_E$ but $2 \notin A_4(E^\times)^2$, we have $T_E \not\subseteq A_4(E^\times)^2$. Consider the map

$$f : G_4 \to K_2\mathcal{O}_E, \quad x \mapsto \{-1, x\}.$$

Lemma 4.4. *With the notation as above,* $\ker f = (T_E/(E^\times)^2) \cap G_4 \simeq \mathbb{Z}/2\mathbb{Z}$.

Proof. Since $T_E \not\subseteq A_4(E^\times)^2$, $\ker f$ must be trivial or $\mathbb{Z}/2\mathbb{Z}$. Let

$$A_2 = \{m : m \in \mathbb{Z},\ m \mid d,\ \{-1, m\} \in (K_2\mathcal{O}_E)^2\}, \quad G_2 = A_2(E^\times)^2/(E^\times)^2.$$

Let g be the map

$$g : G_2 \to K_2\mathcal{O}_E, \quad y \mapsto \{-1, y\}.$$

Then by Theorem 4.2, $\mathrm{rk}_4(K_2\mathcal{O}_E) = s$. Hence the cardinality of the image of g is 2^s. And the cardinality of G_2 is 2^{s+1}. Hence there is exactly one nontrivial $y_0 \in G_2$ such that $\{-1, y_0\} = 1$. Obviously $y_0 \in G_4$. Hence $\ker f \simeq \mathbb{Z}/2\mathbb{Z}$. ∎

Theorem 4.5. *With the notation as above,* $\mathrm{rk}_8(\mathrm{Cl}^+(F)) = \mathrm{rk}_8(K_2\mathcal{O}_E)$.

Proof. By Lemma 4.4,

$$\mathrm{rk}_8(K_2\mathcal{O}_E) = \mathrm{rk}_2(G_4) - 1 = \log_2(\#A_4) - 2 = \log_2(\#B_4) - 1.$$

Recall that there is exactly one nontrivial $n \mid d$ such that $[n]$ is trivial in $\mathrm{Cl}^+(F)$ by Gauss's genus theory. Hence $\mathrm{rk}_8(\mathrm{Cl}^+(F)) = \log_2(\#B_4) - 1$. Therefore $\mathrm{rk}_8(\mathrm{Cl}^+(F)) = \mathrm{rk}_8(K_2\mathcal{O}_E)$. ∎

For any $1 \le i \le s$, let K_i be the unique quartic cyclic extension of $\mathbb{Q}$ with conductor p_i. Note that $K_i \supset \mathbb{Q}(\sqrt{p_i})$. For any i,j such that $1 \le i \ne j \le s$, let L_{ij} be the unique quartic cyclic extension of $\mathbb{Q}(\sqrt{p_i p_j})$ which is unramified at finite primes. Let $M = p_1 \cdots p_s$. Let $\overline{\Lambda}_M$ be the class field over

$\mathbb{Q}(\sqrt{-M})$ corresponding to the subgroup $(\mathrm{Cl}(\mathbb{Q}(\sqrt{-M})))^4$ of fourth powers in $\mathrm{Cl}(\mathbb{Q}(\sqrt{-M}))$. Let

$$K_M = \prod_{1\le i\le s} K_i, \qquad \Sigma_M = K_M \Lambda_M,$$
$$\Lambda_M = \prod_{1\le i\ne j\le s} L_{ij}, \quad \overline{\Sigma}_M = \Sigma_M \overline{\Lambda}_M.$$

THEOREM 4.6 (Morton, [18]). *With the notation as above, the structure of* $\mathrm{Cl}^+(F)/(\mathrm{Cl}^+(F))^8$ *is completely determined by the Frobenius symbol* $\big(\frac{\overline{\Sigma}_M/\mathbb{Q}}{q}\big)$. *Moreover, for any nonnegative integer* $\rho \le s$, *there are infinitely many primes* $q \equiv 1 \pmod 4$ *such that*

$$\mathrm{Cl}^+(F)/(\mathrm{Cl}^+(F))^8 \simeq (\mathbb{Z}/2\mathbb{Z})^{r-s} \oplus (\mathbb{Z}/4\mathbb{Z})^{s-\rho} \oplus (\mathbb{Z}/8\mathbb{Z})^{\rho}.$$

COROLLARY 4.7. *For any nonnegative integer* $\rho \le s$, *there are infinitely many primes* q *such that* $q \equiv 5 \pmod 8$ *and*

$$\mathrm{Cl}^+(F)/(\mathrm{Cl}^+(F))^8 \simeq (\mathbb{Z}/2\mathbb{Z})^{r-s} \oplus (\mathbb{Z}/4\mathbb{Z})^{s-\rho} \oplus (\mathbb{Z}/8\mathbb{Z})^{\rho}.$$

Proof. Let $G = (\mathbb{Z}/2\mathbb{Z})^{r-s} \oplus (\mathbb{Z}/4\mathbb{Z})^{s-\rho} \oplus (\mathbb{Z}/8\mathbb{Z})^{\rho}$. Note that $i = \sqrt{-1} \in \overline{\Sigma}_M$. By considering the ramification index of 2 in the extension $\overline{\Sigma}_M/\mathbb{Q}$, it is easy to see that $\zeta_8 \notin \overline{\Sigma}_M$. Let $K = \overline{\Sigma}_M\mathbb{Q}(\zeta_8) = \overline{\Sigma}_M(\sqrt{2})$. Choose a $\tau_0 \in \mathrm{Gal}(\overline{\Sigma}_M/\mathbb{Q})$ such that there is a $q \equiv 1 \pmod 4$ satisfying

$$\left(\frac{\overline{\Sigma}_M/\mathbb{Q}}{q}\right) = \tau_0 \quad \text{and} \quad \mathrm{Cl}^+(F)/(\mathrm{Cl}^+(F))^8 \simeq G.$$

Then there is a $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\tau|_{\overline{\Sigma}_M} = \tau_0$ and $\tau(\sqrt{2}) = -\sqrt{2}$. By Chebotarev's density theorem, there are infinitely many q such that $\big(\frac{K/\mathbb{Q}}{q}\big) = \tau$. Hence $\tau_{\overline{\Sigma}_M} = \big(\frac{\overline{\Sigma}_M/\mathbb{Q}}{q}\big) = \tau_0$ and $\tau(\sqrt{2}) = -\sqrt{2}$. So q is inert in $\mathbb{Q}(\sqrt{2})$, which implies that $q \equiv 5 \pmod 8$. Hence there are infinitely many q such that $q \equiv 5 \pmod 8$ and $\mathrm{Cl}^+(F)/(\mathrm{Cl}^+(F))^8 \simeq G$. ∎

By Theorem 4.5 and Corollary 4.7, we have

THEOREM 4.8. *For any finite abelian group* G *of exponent* 8, *there are infinitely many imaginary quadratic fields* E *such that*

$$K_2\mathcal{O}_E/(K_2\mathcal{O}_E)^8 \simeq G.$$

5. Tame kernels of real quadratic fields. Let ρ, s, $\widetilde{r}$ be three nonnegative integers such that $\rho \le s \le \widetilde{r}$ and $\widetilde{r} \ge 2 + s$. In this section, we will prove that there are infinitely many real quadratic fields F such that

$$K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq (\mathbb{Z}/2\mathbb{Z})^{\widetilde{r}-s} \oplus (\mathbb{Z}/4\mathbb{Z})^{s-\rho} \oplus (\mathbb{Z}/8\mathbb{Z})^{\rho}.$$

Note that we always have $\mathrm{rk}_2(K_2\mathcal{O}_F) \ge 2$ for real quadratic fields F by (3.1). See [3], [13, Lemma 2.4] or [6, p. 325] for more details. All real quadratic fields with $K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^2 \simeq (\mathbb{Z}/2\mathbb{Z})^2$ are determined by Browkin and Schinzel [3]. All totally real number fields L with $K_2\mathcal{O}_L \simeq (\mathbb{Z}/2\mathbb{Z})^{[L:\mathbb{Q}]}$ are determined in [15] and [7].

Let p, q be two different primes. The biquadratic residue symbol $\left(\frac{p}{q}\right)_4$ is defined to be

$$\left(\frac{p}{q}\right)_4 = \begin{cases} 1 & \text{if } p \equiv a^4 \pmod{q} \text{ for some integer } a, \\ -1 & \text{if } p \not\equiv a^4 \pmod{q} \text{ for any integer } a \text{ and } \left(\frac{p}{q}\right) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $r = \widetilde{r} - 2$. We choose primes $p_1, \ldots, p_r$ and q (q will vary to create infinitely many real quadratic fields F) such that

$$\begin{aligned} &(1)\ p_i \equiv 1 \pmod 8 \quad \text{for } 1 \le i \le r; \\ &(2)\ \left(\frac{p_i}{p_j}\right) = 1 \quad \text{for } 1 \le i \ne j \le r; \\ &(3)\ \left(\frac{p_i}{p_j}\right)_4 \left(\frac{p_j}{p_i}\right)_4 = 1 \ \text{ for } i \ne j, \\ &(4)\ q \equiv 3 \pmod 8), \\ &(5)\ \left(\frac{p_i}{q}\right) = \begin{cases} 1 & \text{if } 1 \le i \le s, \\ -1 & \text{if } s+1 \le i \le r. \end{cases} \end{aligned} \tag{5.1}$$

Let $d = p_1 \cdots p_r q$, $F = \mathbb{Q}(\sqrt{d})$ and $E = \mathbb{Q}(\sqrt{-d})$. Recall that

$$R_E^{(1)} = ((p_i, -d)_{p_j})_{(r+1)\times(r+1)}.$$

By (2.1), we have

$$\varphi(R_E^{(1)}) = \begin{pmatrix} O_{s\times s} & O_{s\times(r-s+1)} \\ O_{(r-s+1)\times s} & A_E \end{pmatrix}, \tag{5.2}$$

where the O's are zero matrices,

$$A_E = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & a \end{pmatrix}_{(r-s+1)\times(r-s+1)} \tag{5.3}$$

and $a \equiv r - s \pmod 2$. It is easy to see that $\operatorname{rank}(A_E) = r - s$. Then by Rédei's Theorem,

$$\operatorname{rk}_4(\operatorname{Cl}(E)) = r - \operatorname{rank}(A_E) = s.$$

By Gauss's genus theory, ${}_2\operatorname{Cl}(E)$ is generated by $[p_1], \ldots, [p_r]$, and these r elements are linearly independent. By (3.1), we have $\operatorname{rk}_2(K_2\mathcal{O}_F) = r + 2$.

Note that $(-2,d)_{p_j}=1$ for any prime p_j. Hence $d\in \mathrm{Norm}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{Q}(\sqrt{-2})^\times)$, i.e., there exist $u,w\in\mathbb{N}$ such that $d=u^2+2w^2$. By [3], ${}_2(K_2\mathcal{O}_F)$ is generated by linearly independent elements

$$\{-1,p_1\},\ldots,\{-1,p_r\},\{-1,-1\},\{-1,u+\sqrt{d}\}.$$

The linear independence follows from Theorem 6.3 of [31].

We will show that ${}_2(K_2\mathcal{O}_F)\cap(K_2\mathcal{O}_F)^2$ is contained in the subgroup generated by $\{-1,p_1\},\ldots,\{-1,p_r\}$. We suppose that $\{-1,m(u+\sqrt{d})\}\in(K_2\mathcal{O}_F)^2$. Then we see that the real Hilbert symbols $(-1,m(u+\sqrt{d}))_{\mathbb{R}}=(-1,m(u-\sqrt{d}))_{\mathbb{R}}$ are 1. Hence $u+\sqrt{d}>0$ and $u-\sqrt{d}>0$. However this is impossible for $(u+\sqrt{d})(u-\sqrt{d})=u^2-d=-2w^2<0$. So ${}_2(K_2\mathcal{O}_F)\cap(K_2\mathcal{O}_F)^2$ is contained in the subgroup generated by $\{-1,p_1\},\ldots,\{-1,p_r\}$.

Theorem 5.1. *With the notation as above, let m be a positive integer with $m\mid d$. Then $\{-1,m\}\in(K_2\mathcal{O}_F)^2$ if and only if $[m]\in(\mathrm{Cl}(E))^2$.*

Proof. By Theorem 3.2, $\{-1,m\}\in(K_2\mathcal{O}_F)^2$ if and only if one can find an $\varepsilon\in\{\pm1,\pm2\}$ such that for any odd prime $l\mid d$,

$$(m,-d)_l=\left(\frac{\varepsilon}{l}\right). \tag{5.4}$$

By the product formula, we need only show that there exists $\varepsilon\in\{\pm1,\pm2\}$ such that

$$(m,-d)_{p_i}=\left(\frac{\varepsilon}{p_i}\right)\quad\text{for } 1\le i\le r. \tag{5.5}$$

Since $p_i\equiv1 \pmod 8$, we have $\left(\frac{\varepsilon}{p_i}\right)=1$. Hence (5.5) is equivalent to

$$(m,-d)_{p_i}=1. \tag{5.6}$$

By Corollary 2.3 of [14], we know that (5.6) holds if and only if $[m]\in\mathrm{Cl}(E)^2$. ∎

By (5.2), (5.3) and Corollary 2.3 of [14], $[m]\in(\mathrm{Cl}(E))^2$ if and only if $m=p_1^{a_1}\cdots p_s^{a_s}(p_{s+1}\cdot p_rq)^b$ for some $a_1,\ldots,a_s,b\in\{0,1\}$. Hence $\{-1,m\}\in(K_2\mathcal{O}_F)^2$ if and only $m=p_1^{a_1}\cdots p_s^{a_s}(p_{s+1}\cdot p_rq)^b$ for some $a_1,\ldots,a_s,b\in\{0,1\}$.

Theorem 5.2. *Assume $m=p_1^{a_1}\cdots p_s^{a_s}(p_{s+1}\cdot p_rq)^b$ for some $a_1,\ldots,a_s,b\in\{0,1\}$. Then $\{-1,m\}\in(K_2\mathcal{O}_F)^4$ if and only if $[m]\in(\mathrm{Cl}(E))^4$.*

Proof. Since $d\equiv3\pmod 8$ and $q\equiv3\pmod 8$, we have $(2,d)_q=-1$. Hence $2\notin\mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. Since $\{-1,d\}=1$ and $[d]=1\in\mathrm{Cl}(E)$, we can always assume that $b=0$. Hence $m\mid p_1\cdots p_s$ and the following Diophantine equation is solvable in $\mathbb{Z}$:

$$mZ^2=X^2+dY^2.$$

We assume that (X_m,Y_m,Z_m) is a solution with $Z_m>0$ and Z_m prime to d.

By Lemma 3.3, $\{-1,m\} \in (K_2\mathcal{O}_F)^4$ if and only if there exist $h_1 \mid p_1\cdots p_r$, $h_3 = 1$ or q, and $\varepsilon \in \{\pm 1, \pm 2\}$ such that for any odd prime $l \mid d$,

$$(d, m_3h_1h_5)_l(-2^{\sigma(l)}d, m_5h_3h_7)_l = \left(\frac{\varepsilon Z_m}{l}\right). \tag{5.7}$$

Note that $m_5 = h_5 = h_7 = 1$, $\sigma(l) = 0$ and $\left(\frac{\varepsilon Z_m}{l}\right) = (-d, \varepsilon Z_m)$ for any odd prime $l \mid d$. Hence (5.7) is equivalent to the existence of $h_1|p_1\cdots p_r$, $h_3 = 1$ or q, and $\varepsilon \in \{\pm 1, \pm 2\}$ such that for any odd prime $l \mid d$,

$$(-d, m_3h_1h_3Z_m\varepsilon)_l = (-1, m_3)_l. \tag{5.8}$$

Let

$$h_3' = \begin{cases} q & \text{if } m_3h_3 = q, \\ 1 & \text{otherwise.} \end{cases} \tag{5.9}$$

Hence (5.8) is equivalent to the existence of $h_1 \mid p_1\cdots p_r$, $h_3' = 1$ or q, and $\varepsilon \in \{\pm 1, \pm 2\}$ such that

$$\begin{aligned} &(1)\ (-d, h_1h_3'Z_m)_{p_i} = 1 \text{ for all } 1 \le i \le r, \\ &(2)\ (-d, h_1h_3'Z_m\varepsilon)_q = -1. \end{aligned} \tag{5.10}$$

Since $(-d, 2)_q = -1$, we can always find $\varepsilon \in \{\pm 1, \pm 2\}$ such that (2) of 5.10 holds. Hence (5.10) holds if and only if we can find $h_1 \mid p_1\cdots p_r$ and $h_3' = 1$ or q such that

$$(-d, h_1h_3'Z_m)_{p_i} = 1 \quad \text{for all } 1 \le i \le r. \tag{5.11}$$

By the product formula, (5.11) implies $(-d, h_1h_3'Z_m)_q = 1$.

By the same argument as in the proof of Theorem 4.3 and [14, Theorem 2.6], (5.11) is equivalent to $[m] \in (\mathrm{Cl}(E))^4$. ■

By Tate's Theorem 6.3 of [31], the Tate kernel T_F is $(F^\times)^2 \cup 2(F^\times)^2$. Hence if $m \mid d$, then $\{-1,m\} = 1$ if and only if $m = 1$. Hence

$$\#\{m : m \mid p_1\cdots p_s \text{ and } \{-1,m\} \in (K_2\mathcal{O}_F)^4\} = 2^{\mathrm{rk}_8(K_2\mathcal{O}_F)}. \tag{5.12}$$

Let m be a divisor of $p_1\cdots p_s$. Since $[p_1],\ldots,[p_s]$ are linearly independent, we have $[m] = 1 \in \mathrm{Cl}(E)$ if and only $m = 1$. Hence

$$\#\{m : m \mid p_1\cdots p_s \text{ and } [m] \in (\mathrm{Cl}(E))^4\} = 2^{\mathrm{rk}_8(\mathrm{Cl}(E))}. \tag{5.13}$$

Thus we get the following theorem.

THEOREM 5.3. *With the notation as above,* $\mathrm{rk}_8(\mathrm{Cl}(E)) = \mathrm{rk}_8(K_2\mathcal{O}_F)$.

Proof. This follows from (5.12), (5.13) and Theorem 5.2. ■

Hence we have the following theorem.

THEOREM 5.4. *Let* $p_1, \dots, p_r, q$ *be primes satisfying conditions* (1)–(5) *of* (5.1). *Let* $d = p_1 \cdots p_r q$, $F = \mathbb{Q}(\sqrt{d})$, *and* $E = \mathbb{Q}(\sqrt{-d})$. *Then*

$$\begin{aligned} &\mathrm{rk}_2(\mathrm{Cl}(E)) = r, && \mathrm{rk}_4(\mathrm{Cl}(E)) = s, \\ &\mathrm{rk}_2(K_2\mathcal{O}_F) = r+2, && \mathrm{rk}_4(K_2\mathcal{O}_F) = s, \\ &\mathrm{rk}_8(\mathrm{Cl}(E)) = \mathrm{rk}_8(K_2\mathcal{O}_F). \end{aligned}$$

For any $1 \le i \le s$, let K_i be the unique quartic cyclic extension of $\mathbb{Q}$ with conductor p_i. Note that $K_i \supset \mathbb{Q}(\sqrt{p_i})$. For any i, j such that $1 \le i \ne j \le s$, let L_{ij} be the unique quartic cyclic extension of $\mathbb{Q}(\sqrt{p_i p_j})$ which is unramified at finite primes. Let

$$\Sigma = \Big(\prod_{1\le i\le s}\Big)\Big(\prod_{1\le i\ne j\le s} L_{ij}\Big).$$

THEOREM 5.5 (Morton, Theorems 1 and 4 of [17]). *With the notation as above, let* $E = \mathbb{Q}(\sqrt{-d})$, *where* $d = p_1 \cdots p_r q$ *and* $p_1, \dots, p_r, q$ *satisfy the five conditions of* (5.1). *Then the structure of* $\mathrm{Cl}(E)/(\mathrm{Cl}(E))^8$ *is completely determined by the Frobenius symbol* $\big(\frac{\Sigma/\mathbb{Q}}{q}\big)$. *Let* G *be a finite abelian group whose exponent divides* 8. *Then there are infinitely many imaginary quadratic fields* $E = \mathbb{Q}(\sqrt{-p_1 \cdots p_r q})$ (*i.e., infinitely many* q) *such that*

$$\mathrm{Cl}(E)/(\mathrm{Cl}(E))^8 \simeq G.$$

THEOREM 5.6. *For any nonnegative integer* $\rho \le s$, *there are infinitely many* q *such that* $q \equiv 3 \pmod 8$ *and*

$$\mathrm{Cl}(E)/(\mathrm{Cl}(E))^8 \simeq (\mathbb{Z}/2\mathbb{Z})^{r-s} \oplus (\mathbb{Z}/4\mathbb{Z})^{s-\rho} \oplus (\mathbb{Z}/8\mathbb{Z})^{\rho}.$$

Proof. Note that $\sqrt{-1} \notin \Sigma$ and $\sqrt{2} \notin \Sigma$. This theorem can be proved by the same argument as Corollary 4.7. ∎

Hence by Theorems 5.4 and 5.6, we get

THEOREM 5.7. *For any finite abelian group* H *of exponent* 8 *with* $\mathrm{rk}_2(H) \ge 2 + \mathrm{rk}_4(H)$, *there are infinitely many real quadratic fields* F *such that*

$$K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq H.$$

As mentioned in the Introduction, our proof depends on Morton's explicit construction of certain quadratic fields. In these cases one always gets $\mathrm{rk}_2(K_2\mathcal{O}_F) \ge \mathrm{rk}_4(K_2\mathcal{O}_F) + 2$. However there are many examples of real quadratic fields F with $\mathrm{rk}_2(K_2\mathcal{O}_F) = \mathrm{rk}_4(K_2\mathcal{O}_F) + 1$. Note that since $\{-1,-1\}$ is not a square in $K_2\mathcal{O}_F$, $\mathrm{rk}_2(K_2\mathcal{O}_F) \ge \mathrm{rk}_4(K_2\mathcal{O}_F) + 1$ always holds. In our cases, $2 \notin \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$. However, if $\mathrm{rk}_2(K_2\mathcal{O}_F) = \mathrm{rk}_4(K_2\mathcal{O}_F) + 1$, then one might have to deal with the cases when $2 \in \mathrm{Norm}_{F/\mathbb{Q}}(F^\times)$ which are much more difficult.

Conjecture 5.8. *For any finite abelian group H of exponent 8 with $\mathrm{rk}_2(H) \geq 1 + \mathrm{rk}_4(H)$, there are infinitely many real quadratic fields F such that $K_2\mathcal{O}_F/(K_2\mathcal{O}_F)^8 \simeq H$.*

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