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A localized uniformly Jarník set in continued fractions

by

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1. Introduction. For any $x \in [0,1]$, let $[a_1(x), a_2(x), \dots]$ be its continued fraction expansion and $\{p_n(x)/q_n(x)\}_{n\geq 1}$ be the sequence of the convergents of x. Legendre's theorem states that once $|x - p/q| < 1/(2q^2)$, then p/q must be a convergent of x. In this sense, the classical Jarník set [2, 9] can be expressed in terms of continued fractions as

$$J(\tau) := \left\{ x \in [0,1) : \left| x - \frac{p_n(x)}{q_n(x)} \right| < \left(\frac{1}{q_n(x)} \right)^{\tau+2} \text{ i.o. } n \in \mathbb{N} \right\}$$

for $\tau > 0$, where *i.o.* stands for *infinitely often*.

The Jarník set $J(\tau)$ represents the set of points which can be well approximated by their convergents infinitely often. Instead of infinitely often, we consider the set of points which can be well approximated by their convergents eventually:

$$U(\tau) := \left\{ x \in [0,1) : \left| x - \frac{p_n(x)}{q_n(x)} \right| < \left(\frac{1}{q_n(x)} \right)^{\tau+2} \text{ for } n \in \mathbb{N} \text{ ultimately} \right\}.$$

We call $U(\tau)$ a *uniformly Jarník set*. In view of Legendre's theorem, $U(\tau)$ represents the set of points x such that for every rational p/q,

$$|x - p/q| \geq 1/(2q^2) \quad \text{or} \quad |x - p/q| < 1/q^{\tau+2}.$$

In this paper, instead of a constant function τ in $U(\tau)$, we consider the following localized version:

$$U_{\mathrm{loc}}(\tau) := \left\{ x \in [0,1) : \left| x - \frac{p_n(x)}{q_n(x)} \right| < \left(\frac{1}{q_n(x)} \right)^{\tau(x)+2} \text{ for } n \in \mathbb{N} \text{ ultimately} \right\},$$

where $\tau : [0,1] \to \mathbb{R}^+ = [0,\infty)$ is a continuous function, and call $U_{\mathrm{loc}}(\tau)$ a *localized uniformly Jarník set*.

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Define

$$F(\tau)=\left\{x\in[0,1):\lim_{n\to\infty}\frac{\log a_{n+1}(x)}{\log q_n(x)}=\tau(x)\right\}.$$

When τ is a constant function, the dimension of $F(\tau)$ is given in [12].

In this paper, we determine the Hausdorff dimensions of $F(\tau)$ and $U_{\mathrm{loc}}(\tau)$:

Theorem 1.1. *Let $\tau:[0,1]\to\mathbb{R}^+$ be a strictly positive continuous function. Then*

$$\dim_{\mathrm{H}} F(\tau)=\dim_{\mathrm{H}} U_{\mathrm{loc}}(\tau)=\frac{1}{\min\{\tau(x):x\in[0,1]\}+2},$$

where $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension.*

The strict positivity of τ implies that $\min\{\tau(x):x\in[0,1]\}>0$. In this case, as Theorem 1.1 shows, the Hausdorff dimension of $F(\tau)$ depends only on the minimal value of $\tau(x)$. But when $\min\{\tau(x):x\in[0,1]\}=0$, the situation will be quite different, as can be seen in the following result.

Define $\mathcal{D}_0$ to be the set of Hausdorff dimensions that can be realized when $\min\{\tau(x):x\in[0,1]\}=0$, i.e.

$$\mathcal{D}_0=\Big\{\dim_{\mathrm{H}} F(\tau):\tau\text{ continuous on }[0,1]\text{ and }\min_{x\in[0,1]}\tau(x)=0\Big\}.$$

Theorem 1.2. $\overline{\mathcal{D}_0}=[1/2,1]$, *where* $\overline{\mathcal{D}_0}$ *denotes the closure of* $\mathcal{D}_0$.

It should be mentioned that the localized version of the Diophantine analysis was first carried out in the work of Barral & Seuret [1], who considered the localized version of the Jarník theorem. For more dimensional results about the set of points with restrictions on their partial quotients, see e.g. Good [6], D. Hensley [7, 8], Bugeaud [4], Wang & Wu [13, 14], Wu [15].

2. Preliminaries. In this section, we collect some known facts and establish some elementary properties of continued fractions for later use.

Any irrational number $x\in[0,1)$ has a simple infinite continued fraction expansion

$$x=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\cfrac{1}{a_3(x)+\cdots}}}=[a_1(x),a_2(x),a_3(x),\ldots],\tag{2.1}$$

where $a_n(x)$, $n\geq 1$, are called the *partial quotients* of x. Finite truncations of (2.1) give the *convergents* of x:

$$\frac{p_n(x)}{q_n(x)}=[a_1(x),\ldots,a_n(x)],\qquad n\geq 1.$$

With the conventions $p_{-1}(x) = 1$, $q_{-1}(x) = 0$, $p_0(x) = 0$, $q_0(x) = 1$, we have (see [11])

$$p_{n+1}(x) = a_{n+1}(x) \cdot p_n(x) + p_{n-1}(x), \qquad n \geq 0, \tag{2.2}$$

$$q_{n+1}(x) = a_{n+1}(x) \cdot q_n(x) + q_{n-1}(x), \qquad n \geq 0. \tag{2.3}$$

The following proposition collects some basic properties of q_n.

PROPOSITION 2.1 ([11]). *For any $a_1, \ldots, a_n \in \mathbb{N}$, let $p_n = p_n(a_1, \ldots, a_n)$ and $q_n = q_n(a_1, \ldots, a_n)$ be recursively defined by* (2.2)–(2.3). *Then*

(i) $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$;
(ii) $a_{n+1}q_n \leq q_{n+1} \leq 2a_{n+1}q_n$;
(iii) $q_n \geq 2^{(n-1)/2}$, $\prod_{k=1}^n a_k \leq q_n \leq \prod_{k=1}^n (a_k + 1)$;
(iv) *for any integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ with $n, m \geq 1$,*

$$1 \leq \frac{q_{n+m}(a_1, \ldots, a_n, b_1, \ldots, b_m)}{q_n(a_1, \ldots, a_n) q_m(b_1, \ldots, b_m)} \leq 2; \tag{2.4}$$

(v)
$$\frac{1}{2a_{n+1}(x)q_n(x)^2} \leq \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{a_{n+1}(x)q_n(x)^2}.$$

For any $a_1, \ldots, a_n \in \mathbb{N}$, an *nth order cylinder* is defined as

$$I_n(a_1, \ldots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\},$$

which is the collection of points whose expansions begin with $(a_1, \ldots, a_n)$.

The following lemma gives the length of a cylinder.

LEMMA 2.2 ([11]). *For any $a_1, \ldots, a_n \in \mathbb{N}$, the nth order cylinder $I_n(a_1, \ldots, a_n)$ is the interval with endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. As a consequence, the length of $I_n(a_1, \ldots, a_n)$ is equal to*

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}.$$

Now we give some consequences of Lemma 2.2 which will be used in proving Theorem 1.1. Let $a_1, \ldots, a_n \in \mathbb{N}$ and $a \geq 2$. Write

$$J_n(a_1, \ldots, a_n) = \bigcup_{a \leq a_{n+1} < 2a} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$$

Clearly, $J_n(a_1, \ldots, a_n)$ is a subinterval of $I_n(a_1, \ldots, a_n)$ and depends on a. But to ease the notation, we hide this dependence.

COROLLARY 2.3.

(i) *The length of $J_n(a_1, \ldots, a_n)$ satisfies*

$$\frac{1}{8aq_n^2} \leq |J_n(a_1, \ldots, a_n)| \leq \frac{1}{aq_n^2}. \tag{2.5}$$

(ii) *If we denote by* $g_n(a_1,\dots,a_n)$ *the minimal distance between the endpoints of* $J_n(a_1,\dots,a_n)$ *and those of* $I_n(a_1,\dots,a_n)$*, then*

$$g_n(a_1,\dots,a_n) \ge \tfrac{1}{2}|J_n(a_1,\dots,a_n)|.$$

(iii) *For any* $a \le a_{n+1}, a'_{n+1} < 2a$*, we have*

$$\frac{1}{4} \le \frac{|I_{n+1}(a_1,\dots,a_n,a_{n+1})|}{|I_{n+1}(a_1,\dots,a_n,a'_{n+1})|} \le 4.$$

Proof. We consider the case of n even. For n odd, the argument is similar.

(i) By Lemma 2.2 and the definition of $J_n(a_1,\dots,a_n)$, we have

$$J_n(a_1,\dots,a_n) = \left(\frac{2ap_n+p_{n-1}}{2aq_n+q_{n-1}}, \frac{ap_n+p_{n-1}}{aq_n+q_{n-1}}\right].$$

Hence

$$|J_n(a_1,\dots,a_n)| = \frac{a}{(aq_n+q_{n-1})(2aq_n+q_{n-1})}. \tag{2.6}$$

By a simple calculation,

$$\frac{1}{8aq_n^2} \le |J_n(a_1,\dots,a_n)| \le \frac{1}{aq_n^2}.$$

(ii) By (2.6), the distance between the left endpoint of $J_n(a_1,\dots,a_n)$ and that of $I_n(a_1,\dots,a_n)$ is

$$\frac{2ap_n+p_{n-1}}{2aq_n+q_{n-1}} - \frac{p_n}{q_n} = \frac{1}{(2aq_n+q_{n-1})q_n} \ge \frac{1}{2}|J_n(a_1,\dots,a_n)|,$$

while the distance between the right endpoint of $I_n(a_1,\dots,a_n)$ and that of $J_n(a_1,\dots,a_n)$ is

$$\frac{p_n+p_{n+1}}{q_n+q_{n+1}} - \frac{ap_n+p_{n-1}}{aq_n+q_{n-1}} = \frac{a-1}{(aq_n+q_{n-1})(q_n+q_{n-1})} \ge \frac{1}{2}|J_n(a_1,\dots,a_n)|;$$

hence

$$g_n(a_1,\dots,a_n) \ge \tfrac{1}{2}|J_n(a_1,\dots,a_n)|.$$

(iii) By Lemma 2.2 and the assumption $a \le a_{n+1}, a'_{n+1} < 2a$,

$$\begin{aligned}
\frac{1}{4} &\le \frac{(aq_n+q_{n-1})((a+1)q_n+q_{n-1})}{((2a-1)q_n+q_{n-1})(2aq_n+q_{n-1})} \\
&\le \frac{|I_{n+1}(a_1,\dots,a_n,a_{n+1})|}{|I_{n+1}(a_1,\dots,a_n,a'_{n+1})|} = \frac{(a'_{n+1}q_n+q_{n-1})((a'_{n+1}+1)q_n+q_{n-1})}{(a_{n+1}q_n+q_{n-1})((a_{n+1}+1)q_n+q_{n-1})} \\
&\le \frac{((2a-1)q_n+q_{n-1})(2aq_n+q_{n-1})}{(aq_n+q_{n-1})((a+1)q_n+q_{n-1})} \le 4. \quad \blacksquare
\end{aligned}$$

We write $u_n \approx v_n$ when there exist absolute positive constants c_1, c_2 such that $c_1 v_n \leq u_n \leq c_2 v_n$ for any $n \geq 1$. From Corollary 2.3(i),

$$|J_n(a_1, \dots, a_n)| \approx \frac{1}{a}|I_n(a_1, \dots, a_n)|,$$

because by Lemma 2.2,

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \dots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}.$$

We will denote by $I_n(x)$ the nth order cylinder that contains x, i.e. $I_n(x) = I_n(a_1(x), \dots, a_n(x))$. Let $B(x, r)$ denote the ball centered at x with radius r. For any $x \in I_n(a_1, \dots, a_n)$, we have the following relationship between the ball $B(x, |I_n(a_1, \dots, a_n)|)$ and $I_n(a_1, \dots, a_n)$, which is known as the *regular property* [3].

Lemma 2.4 ([3]). *Let $x = [a_1, a_2, \dots]$. If $a_n \neq 1$, then*

$$B(x, |I_n(x)|) \subset \bigcup_{j=-1}^{3} I_n(a_1, \dots, a_{n-1}, a_n + j).$$

The following lemmas give some dimension results.

Lemma 2.5 ([12]). *For any $\alpha > 0$, the set*

$$G(\alpha) = \{x \in [0,1) : a_{n+1}(x) \geq q_n(x)^\alpha \text{ for } n \in \mathbb{N} \text{ ultimately}\}$$

has Hausdorff dimension $1/(\alpha + 2)$.

Lemma 2.6 ([6]).

$$\dim_{\mathrm{H}}\Big\{x \in [0,1] : \lim_{n\to\infty} a_n(x) = \infty\Big\} = 1/2,$$
$$\dim_{\mathrm{H}}\big\{x \in [0,1] : \{a_n(x)\}_{n\geq 1} \text{ is bounded}\big\} = 1.$$

Finally, we state a result which is known as the *Texan conjecture* [10].

Lemma 2.7 ([10]). *For any $\Lambda \subset \mathbb{N}$, define*

$$J_\Lambda := \{x \in [0,1] : a_i(x) \in \Lambda,\ i \in \mathbb{N}\}.$$

Then $\{\dim_{\mathrm{H}} J_\Lambda : \Lambda \subset \mathbb{N} \text{ finite}\}$ is dense in $[0,1]$.

3. Proof of Theorem 1.1. The proof is divided into two parts: upper bound and lower bound. Recall that

$$F(\tau) = \left\{x \in [0,1) : \lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = \tau(x)\right\},$$

and define

$$\tau_0 = \min\{\tau(x) : x \in [0,1]\}.$$

Since $\tau : [0,1] \to (0, \infty)$ is continuous, we have $\tau_0 > 0$.

By a simple calculation (using Proposition 2.1), we get

$$U_{\mathrm{loc}}(\tau) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} \ge \tau(x) \right\}.$$

3.1. Upper bound. Since $\tau_0 > 0$ is the minimal value of $\tau(x)$, it follows that

$$U_{\mathrm{loc}}(\tau) \subset G(\tau_0 - \epsilon) \quad \text{for all } \epsilon > 0.$$

Thus, by Lemma 2.5 and since ϵ is arbitrary, we have

$$\dim_{\mathrm{H}} U_{\mathrm{loc}}(\tau) \le \frac{1}{\tau_0 + 2}.$$

As a consequence,

$$\dim_{\mathrm{H}} F(\tau) \le \dim_{\mathrm{H}} U_{\mathrm{loc}}(\tau) \le \frac{1}{\tau_0 + 2},$$

because $F(\tau) \subset U_{\mathrm{loc}}(\tau)$.

3.2. Lower bound. The lower bound is obtained by using the following mass distribution principle (see [5, Proposition 4.2]), which is a classical tool to estimate the Hausdorff dimension of a set from below.

PROPOSITION 3.1 (Falconer [5]). *Let $E \subset [0,1]$ be a Borel set and μ be a measure with $\mu(E) > 0$. Suppose that*

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge s \quad \textit{for all } x \in E,$$

where $B(x,r)$ denotes the ball with center at x and radius r. Then $\dim_{\mathrm{H}} E \ge s$.

Thus in what follows, we will first construct a Cantor set D_∞ inside $F(\tau)$, then a probability measure μ supported on D_∞, and finally, we will estimate the Hölder exponent of μ.

3.2.1. *Cantor set.* Fix $\underline{\tau} \in \{\tau(x) : x \in [0,1]\}$. We construct a Cantor set D_∞ such that for each $x \in D_\infty$,

$$\lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = \tau(x) \tag{3.1}$$

and $\tau(x)$ is close to $\underline{\tau}$. It should be emphasized that since the limit in (3.1) depends on the individual points, in the construction of the Cantor set, each generation depends on its predecessor.

Fix $0 < \epsilon \le \underline{\tau}/2$. By the continuity of τ, one can choose a cylinder $I_N(a_1, \ldots, a_N)$ such that

$$|\tau(x) - \underline{\tau}| < \epsilon \quad \text{for any } x \in \overline{I_N(a_1, \ldots, a_N)}, \tag{3.2}$$

where $\overline{E}$ denotes the closure of E.

We define

$$D_0 = \{I_N(a_1, \ldots, a_N)\}$$

to be the zeroth generation of the Cantor set.

Look at (3.1). To ensure that a point x in the Cantor set to be constructed fulfills (3.1), its $(n+1)$th partial quotients $a_{n+1}(x)$ should be close to $q_n(x)^{\tau(x)}$ for all n large enough. In view of this, the other generations of the Cantor set will be constructed in the following way.

The first generation D_1 of the Cantor set. Let $I_N(a_1, \ldots, a_N) \in D_0$. Write

$$\tau([a_1, \ldots, a_N]) = \tau(p_N/q_N), \quad \text{where} \quad p_N/q_N = [a_1, \ldots, a_N].$$

Then define

$$\begin{aligned} D_1(I_N(a_1, \ldots, a_N)) = \big\{ & I_{N+1}(a_1, \ldots, a_N, a_{N+1}) : \\ & q_N(a_1, \ldots, a_N)^{\tau([a_1, \ldots, a_N])} \le a_{N+1} < 2q_N(a_1, \ldots, a_N)^{\tau([a_1, \ldots, a_N])} \big\}, \end{aligned}$$

where $q_N(a_1, \ldots, a_N)$ is the denominator of the convergent.

The first generation D_1 is defined as

$$D_1 = \bigcup_{I_N(a_1, \ldots, a_N) \in D_0} D_1(I_N(a_1, \ldots, a_N)).$$

The inductive step of the construction. Suppose that the kth generation D_k is already defined, which is a collection of cylinders of order $N+k$. Fix $I_{N+k}(a_1, \ldots, a_{N+k}) \in D_k$. Write

$$\tau([a_1, \ldots, a_{N+k}]) = \tau(p_{N+k}/q_{N+k}), \quad \text{where} \quad p_{N+k}/q_{N+k} = [a_1, \ldots, a_{N+k}].$$

Then define

$$\begin{aligned} D_{k+1}(I_{N+k}(a_1, \ldots, a_{N+k})) = \big\{ & I_{N+k+1}(a_1, \ldots, a_{N+k}, a_{N+k+1}) : \\ & q_{N+k}(a_1, \ldots, a_{N+k})^{\tau([a_1, \ldots, a_{N+k}])} \le a_{N+k+1} \\ & < 2q_{N+k}(a_1, \ldots, a_{N+k})^{\tau([a_1, \ldots, a_{N+k}])} \big\}. \end{aligned} \tag{3.3}$$

The $(k+1)$th generation D_{k+1} is defined as

$$D_{k+1} = \bigcup_{I_{N+k}(a_1, \ldots, a_{N+k}) \in D_k} D_{k+1}(I_{N+k}(a_1, \ldots, a_{N+k})).$$

The desired Cantor set is defined as

$$D_\infty = \bigcap_{k=0}^{\infty} \bigcup_{I_{N+k}(a_1, \ldots, a_{N+k}) \in D_k} I_{N+k}(a_1, \ldots, a_{N+k}).$$

For each $k \ge 0$, we call the elements in D_k *basic intervals* of order k.

Finally, we are going to express the Cantor set D_∞ in another way.

For each $k \geq 0$ and $I_{N+k}(a_1, \dots, a_{N+k}) \in D_k$, define

$$J_{N+k}(a_1, \dots, a_{N+k}) = \bigcup_{A \leq a < 2A} I_{N+k+1}(a_1, \dots, a_{N+k}, a),$$

where $A = q_{N+k}(a_1, \dots, a_{N+k})^{\tau([a_1, \dots, a_{N+k}])}$, and call it a *fundamental interval* of order k. By Corollary 2.3(i),

$$\begin{aligned} (3.4) \qquad \frac{1}{8} \cdot \left(\frac{1}{q_{N+k}(a_1, \dots, a_{N+k})} \right)^{2+\tau([a_1, \dots, a_{N+k}])} &\leq |J_{N+k}(a_1, \dots, a_{N+k})| \\ &\leq \left(\frac{1}{q_{N+k}(a_1, \dots, a_{N+k})} \right)^{2+\tau([a_1, \dots, a_{N+k}])}. \end{aligned}$$

In fact, $J_{N+k}(a_1, \dots, a_{N+k})$ is just the union of all basic intervals of order $k+1$ contained in $I_{N+k}(a_1, \dots, a_{N+k})$. As a result,

$$D_\infty = \bigcap_{k=0}^{\infty} \bigcup_{I_{N+k}(a_1, \dots, a_{N+k}) \in D_k} J_{N+k}(a_1, \dots, a_{N+k}).$$

But Corollary 2.3 says that there are some gaps between J_{N+k}. This technical adjustment will simplify the argument on the Hölder exponent of the mass distribution μ (defined later).

Proposition 3.2. $D_\infty \subset F(\tau)$.

Proof. For each $x \in D_\infty$, let $x = [a_1, a_2, \dots]$ be its continued fraction expansion. Then by the construction of D_∞,

$$I_{N+k}(x) \in D_k \quad \text{ for all } k \geq 1.$$

This implies that for each $n \geq N$,

$$q_n(x)^{\tau([a_1(x), \dots, a_n(x)])} \leq a_{n+1}(x) < 2 q_n(x)^{\tau([a_1(x), \dots, a_n(x)])}.$$

Then by the continuity of τ,

$$\lim_{n \to \infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = \lim_{n \to \infty} \tau([a_1(x), \dots, a_n(x)]) = \tau(x). \ \blacksquare$$

3.2.2. *A probability measure supported on D_∞.* Now we define inductively a probability measure μ supported on D_∞, which is defined by distributing measure among fundamental intervals.

Let

$$\mu(J_N(a_1, \dots, a_N)) = 1.$$

Assume that μ has been well defined on all fundamental intervals of order $k-1$. Now we distribute the measure μ on fundamental intervals of order k. Fix a basic interval $I_{N+k}(a_1, \dots, a_{N+k}) \in D_k$. Then $I_{N+k-1}(a_1, \dots, a_{N+k-1})$ is in D_{k-1}. We define

$$(3.5)\quad \mu(J_{N+k}(a_1,\dots,a_{N+k}))$$
$$= \left(\frac{1}{q_{N+k-1}(a_1,\dots,a_{N+k-1})}\right)^{\tau([a_1,\dots,a_{N+k-1}])} \times \mu(J_{N+k-1}(a_1,\dots,a_{N+k-1}))$$
$$= \prod_{j=1}^{k-1}\left(\frac{1}{q_{N+j}(a_1,\dots,a_{N+j})}\right)^{\tau([a_1,\dots,a_{N+j}])}.$$

Note that by (3.3) the number of fundamental subintervals of order k contained in the $(k-1)$th fundamental interval $J_{N+k-1}(a_1,\dots,a_{N+k-1})$ is just

$$q_{N+k-1}(a_1,\dots,a_{N+k-1})^{\tau([a_1,\dots,a_{N+k-1}])}.$$

Thus in other words, the measure on $J_{N+k-1}(a_1,\dots,a_{N+k-1})$ is uniformly distributed among its offsprings. Hence, μ satisfies Kolmogorov's consistency condition and so can be uniquely extended to a probability measure supported on D_∞.

According to the distribution of the mass on D_∞, the mass of a basic interval $I_{N+k}(a_1,\dots,a_{N+k}) \in D_k$ concentrates on one of its subintervals $J_{N+k}(a_1,\dots,a_{N+k})$, i.e.

$$\mu(I_{N+k}(a_1,\dots,a_{N+k})) = \mu(J_{N+k}(a_1,\dots,a_{N+k})).$$

Finally, we make a remark on (3.5): for each $j \ge 1$,

$$(3.6)\quad q_{N+j}(a_1,\dots,a_{N+j})^{\tau([a_1,\dots,a_{N+j}])} \le a_{N+j+1} < 2q_{N+j}(a_1,\dots,a_{N+j})^{\tau([a_1,\dots,a_{N+j}])}.$$

Thus, by Proposition 2.1(ii), (iv), we have

$$(3.7)\quad \mu(J_{N+k}(a_1,\dots,a_{N+k})) \le 2^k \prod_{j=N+1}^{N+k} \frac{1}{a_j} \le 4^k \frac{c}{q_{N+k}(a_1,\dots,a_{N+k})},$$

where the absolute constant c can be taken to be $q_N(a_1,\dots,a_N)$.

3.2.3. *Hölder exponent of μ.* Now we estimate the measure of an arbitrary ball $B(x,r)$ with center $x \in D_\infty$ and radius $r > 0$ small enough. Let $x = [a_1, a_2, \dots]$ be the continued fraction expansion. Let $k \ge 1$ be the integer such that

$$(3.8)\quad \tfrac{1}{2}|J_{N+k}(a_1,\dots,a_{N+k})| \le r < \tfrac{1}{2}|J_{N+k-1}(a_1,\dots,a_{N+k-1})|.$$

Then $B(x,r) \subset I_{N+k-1}(a_1,\dots,a_{N+k-1})$ by Corollary 2.3(ii).

To estimate the measure of $B(x,r)$ we distinguish two cases.

Case (i): $\frac{1}{2}|J_{N+k}(a_1,\dots,a_{N+k})| \le r < |I_{N+k}(a_1,\dots,a_{N+k})|$. By Lemma 2.4 and the fact that $a_{N+k} \ne 1$, $B(x,r)$ can intersect at most five kth order basic intervals. Since these basic intervals have the same μ-measure,

$$\mu(B(x,r)) \le 5\mu(I_{N+k}(a_1,\dots,a_{N+k})) = 5\mu(J_{N+k}(a_1,\dots,a_{N+k})).$$

By the estimates (3.7) on the measure of J_{N+k} and (3.4) on its length, we have

$$\begin{aligned}\mu(B(x,r)) &\le 5\cdot c\cdot 4^k\cdot \frac{1}{q_{N+k}(a_1,\ldots,a_{N+k})}\\ &= 5\cdot c\cdot 4^k\cdot \left[\left(\frac{1}{q_{N+k}(a_1,\ldots,a_{N+k})}\right)^{\tau([a_1,\ldots,a_{N+k}])+2}\right]^{\frac{1}{\tau([a_1,\ldots,a_{N+k}])+2}}\\ &\le 5\cdot c\cdot 4^k\cdot (8\cdot |J_{N+k}(a_1,\ldots,a_{N+k})|)^{\frac{1}{\tau([a_1,\ldots,a_{N+k}])+2}}\\ &\le 5\cdot c\cdot 4^{k+2}\cdot |J_{N+k}(a_1,\ldots,a_{N+k})|^{\frac{1}{\tau([a_1,\ldots,a_{N+k}])+2}}\\ &\le 5\cdot c\cdot 4^{k+2}\cdot r^{\frac{1}{\underline{\tau}-\epsilon+2}}.\end{aligned}$$

Case (ii): $|I_{N+k}(a_1,\ldots,a_{N+k})| \le r < \frac{1}{2}|J_{N+k-1}(a_1,\ldots,a_{N+k-1})|$. As μ is supported on basic intervals, we only need to count the basic intervals of order k intersecting $B(x,r)$. Moreover, by (3.6) and Corollary 2.3(iii), all the basic intervals of order k contained in $I_{N+k-1}(a_1,\ldots,a_{N+k-1})$ are of equivalent length in the sense that

$$1/4 \le \frac{|I_{N+k}(a_1,\ldots,a_{N+k-1},a)|}{|I_{N+k}(a_1,\ldots,a_{N+k-1},a')|} \le 4,$$

where $I_{N+k}(a_1,\ldots,a_{N+k-1},a), I_{N+k}(a_1,\ldots,a_{N+k-1},a')$ are any two basic intervals of order k contained in $I_{N+k-1}(a_1,\ldots,a_{N+k-1})$.

Note that

$$|I_{N+k}(a_1,\ldots,a_{N+k})| \ge \left(\frac{1}{q_{N+k}(a_1,\ldots,a_{N+k})}\right)^2.$$

Hence the ball $B(x,r)$ which is contained in $I_{N+k-1}(a_1,\ldots,a_{N+k-1})$ can intersect at most $8rq^2_{N+k}(a_1,\ldots,a_{N+k})$ basic intervals of order k. As a consequence,

$$\begin{aligned}\mu(B(x,r)) &\le \min\big\{\mu(J_{N+k-1}(a_1,\ldots,a_{N+k-1})),\\ &\qquad\qquad 8rq^2_{N+k}(a_1,\ldots,a_{N+k})\mu(J_{N+k}(a_1,\ldots,a_{N+k}))\big\}\\ &\le c\cdot 4^{k+2}\cdot \min\left\{\frac{1}{q_{N+k-1}(a_1,\ldots,a_{N+k-1})}, rq_{N+k}(a_1,\ldots,a_{N+k})\right\}.\end{aligned}$$

Notice that by Proposition 2.1(ii) and (3.6),

$$\begin{aligned}q_{N+k}(a_1,\ldots,a_{N+k}) &\le 2a_{N+k}q_{N+k-1}(a_1,\ldots,a_{N+k-1})\\ &\le 4(q_{N+k-1}(a_1,\ldots,a_{N+k-1}))^{\tau([a_1,\ldots,a_{N+k-1}])+1}\\ &\le 4(q_{N+k-1}(a_1,\ldots,a_{N+k-1}))^{\underline{\tau}+\epsilon+1}.\end{aligned}$$

Then using the elementary inequality $\min\{a,b\} \le a^{1-s}b^s$ with $a,b>0$ and

$0 < s < 1$, and letting $s = \frac{1}{\underline{\tau}+\epsilon+2}$, we get

$$\mu(B(x,r)) \le c \cdot 4^{k+3} \cdot \frac{1}{q_{N+k-1}(a_1,\dots,a_{N+k-1})} \min\{1, r(q_{N+k-1}(a_1,\dots,a_{N+k-1}))^{\underline{\tau}+\epsilon+2}\} \le c \cdot 4^{k+3} \cdot r^{\frac{1}{\underline{\tau}+\epsilon+2}}.$$

In order to use the mass distribution principle (Proposition 3.1) to get the desired upper bound of the Hausdorff dimension, we should show that r is much smaller than 4^k. First we prove that $q_n(x)$ grows much faster than an exponential function. By (3.6) and Proposition 2.1(ii),

$$\begin{aligned} q_{N+k}(a_1,\dots,a_{N+k}) &\ge a_{N+k} q_{N+k-1}(a_1,\dots,a_{N+k-1}) \\ &\ge q_{N+k-1}(a_1,\dots,a_{N+k-1})^{\tau([a_1,\dots,a_{N+k-1}])+1} \\ &\ge q_{N+k-1}(a_1,\dots,a_{N+k-1})^{\underline{\tau}-\epsilon+1} \\ &\ge \dots \ge q_N(a_1,\dots,a_N)^{(\underline{\tau}-\epsilon+1)^k} = c^{(\underline{\tau}-\epsilon+1)^k}. \end{aligned}$$

Then by (3.4) and (3.8),

$$\begin{aligned} r &< \frac{1}{2}|J_{N+k-1}(a_1,\dots,a_{N+k-1})| \\ &\le \frac{1}{2}\left(\frac{1}{q_{N+k-1}(a_1,\dots,a_{N+k-1})}\right)^{2+\tau([a_1,\dots,a_{N+k-1}])} \\ &\le c^{-(\underline{\tau}-\epsilon+1)^{k-1}(\underline{\tau}-\epsilon+2)}. \end{aligned}$$

Finally, by Proposition 3.1,

$$\dim_{\mathrm{H}} D_\infty \ge \min\left\{\frac{1}{\underline{\tau}+\epsilon+2}, \frac{1}{\underline{\tau}-\epsilon+2}\right\}.$$

Since ϵ is arbitrary and $\underline{\tau}$ can be taken close to $\min\{\tau(x) : x \in [0,1]\}$, we arrive at

$$\dim_{\mathrm{H}} F(\tau) \ge \frac{1}{\min\{\tau(x) : x \in [0,1]\} + 2} = \frac{1}{\tau_0 + 2}.$$

With almost the same argument, one can also show that a similar result holds when the set $F(\tau)$ is restricted in a closed interval.

THEOREM 3.3. *Let U be a closed interval, and $\tau : U \to \mathbb{R}^+$ be a strictly positive continuous function. Then*

$$\dim_{\mathrm{H}}(F(\tau) \cap U) = \frac{1}{2 + \min\{\tau(x) : x \in U\}}.$$

Proof. Instead of choosing a cylinder $I_N(a_1,\dots,a_N)$ such that $\tau(x)$ is close to $\min\{\tau(x) : x \in [0,1]\}$ for all $x \in I_N(a_1,\dots,a_N)$ as in (3.2), we choose a cylinder $I_N(a_1,\dots,a_N)$ such that $\tau(x)$ is close to $\min\{\tau(x) : x \in U\}$ for all $x \in I_N(a_1,\dots,a_N)$. The remaining argument can be carried out similarly. ∎

4. Proof of Theorem 1.2. First, we show that $\dim_{\mathrm{H}} F(\tau) \geq 1/2$ for any τ with $\min\{\tau(x) : x \in [0,1]\} = 0$.

Case (i): $\tau(x) = 0$ for all $x \in [0,1]$. In this case, it is trivial that the set of points with bounded partial quotients is a subset of $F(\tau)$. So, by Lemma 2.6, we have $\dim_{\mathrm{H}} F(\tau) = 1$.

Case (ii): $\tau(x) \neq 0$ for some $x \in [0,1]$. By the continuity of τ, the closed set

$$\{x \in [0,1] : \tau(x) = 0\}$$

is nonempty and not dense. Thus for any $\eta > 0$, we can find $0 < \delta < \eta$ and a closed interval U such that

$$\delta/2 \leq \tau(x) \leq \delta \quad \text{for all } x \in U.$$

Applying Theorem 3.3, we arrive at

$$\dim_{\mathrm{H}} F(\tau) \geq \frac{1}{2+\eta}.$$

Since η is arbitrary, we get $\dim_{\mathrm{H}} F(\tau) \geq 1/2$.

Secondly, we prove that there is a dense subset $\mathcal{S}$ of $[1/2,1]$ such that for any $s \in \mathcal{S}$, one can construct a continuous function τ such that $\dim_{\mathrm{H}} F(\tau) = s$.

Recall the Texan conjecture (Lemma 2.7):

$$\{\dim_{\mathrm{H}} J_\Lambda : \Lambda \text{ is a finite subset of } \mathbb{N}\} \text{ is dense in } [0,1].$$

For any finite subset $\Lambda \subset \mathbb{N}$ with $\dim_{\mathrm{H}} J_\Lambda \geq 1/2$, we will construct a continuous function τ with $\min\{\tau(x) : x \in [0,1]\} = 0$ but

$$\dim_{\mathrm{H}} F(\tau) = \dim_{\mathrm{H}} J_\Lambda.$$

Since Λ is a finite set, we have two obvious facts: the compactness of J_Λ and the boundedness of the partial quotients of each element in J_Λ. Define

$$\tau(x) = d(x, J_\Lambda), \quad x \in [0,1],$$

where $d(x,E)$ is the distance from the point x to the set E. The compactness of J_Λ implies that

$$\tau(x) = 0 \iff x \in J_\Lambda. \tag{4.1}$$

On the one hand, the boundedness of the partial quotients for each $x \in J_\Lambda$ implies that

$$\lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = 0 = \tau(x) \quad \text{for all } x \in J_\Lambda.$$

Therefore $J_\Lambda \subset F(\tau)$. Hence

$$\dim_{\mathrm{H}} J_\Lambda \le \dim_{\mathrm{H}} F(\tau). \tag{4.2}$$

On the other hand, we divide the set $F(\tau)$ into two parts according to whether $\tau(x) = 0$ or not. By (4.1), it follows that

$$F(\tau) \subset \left\{ x \in [0,1) \setminus J_\Lambda : \lim_{n\to\infty} \frac{\log a_{n+1}(x)}{\log q_n(x)} = \tau(x) > 0 \right\} \cup J_\Lambda.$$

Clearly, the first set on the right side is contained in

$$\left\{ x \in [0,1] : \lim_{n\to\infty} a_n(x) = \infty \right\},$$

so has Hausdorff dimension less than 1/2 by Lemma 2.6. As a result,

$$\dim_{\mathrm{H}} F(\tau) \le \max\{1/2, \dim_{\mathrm{H}} J_\Lambda\} = \dim_{\mathrm{H}} J_\Lambda. \tag{4.3}$$

Combining (4.2) and (4.3), we arrive at $\dim_{\mathrm{H}} F(\tau) = \dim_{\mathrm{H}} J_\Lambda$.

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