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A density result for elliptic Dedekind sums

by

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1. Introduction. For coprime integers h and k with $k \neq 0$, consider the Dedekind sum

$$s(h,k) = \frac{1}{k} \sum_{0 \not\equiv \mu \,(\mathrm{mod}\, k)} \cot\left(\pi \frac{h\mu}{k}\right) \cot\left(\pi \frac{\mu}{k}\right).$$

Since $s(-h,-k) = s(h,k)$, we may define $s(h/k) = s(h,k)$ for an irreducible fraction h/k and get a function on the rational number field $\mathbb{Q}$. Hickerson has proved the following.

THEOREM 1 ([2]). *The set $\{(h/k, s(h/k)) : h/k \in \mathbb{Q}\}$ is dense in the plane.*

The purpose of the present note is to prove an elliptic analogue of this theorem. Let K be an imaginary quadratic field considered in the complex number field $\mathbb{C}$, d be the discriminant of K and $\mathfrak{O}$ be the ring of integers of K. Let L be a lattice in $\mathbb{C}$ such that $\mathfrak{O} = \{m \in \mathbb{C} : mL \subset L\}$ and, for each integer n, define the function $E_n(z)$ on $\mathbb{C}$ by

$$E_n(z) = E_n(z;L) = \sum_{l \in L,\, l+z \neq 0} (l+z)^{-n} |l+z|^{-s} \Big|_{s=0},$$

where the value at $s = 0$ should be considered in the sense of the analytic continuation. The function $E_n(z)$ is periodic with respect to L, $E_{2n}(z)$ is even and $E_{2n+1}(z)$ is odd. For two elements h and k of $\mathfrak{O}$ with $k \neq 0$, Sczech [6] has introduced the sum

$$D(h,k) = \frac{1}{k} \sum_{\mu \in L/kL} E_1\left(\frac{h\mu}{k}\right) E_1\left(\frac{\mu}{k}\right).$$

This is an analogue of $s(h,k)$ for the imaginary quadratic field K and we

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have the reciprocity formula

$$D(h,k)+D(k,h)=E_2(0)I\left(\frac{h}{k}+\frac{1}{hk}+\frac{k}{h}\right) \tag{1}$$

if h and k are coprime and $hk \neq 0$. Here, $I(z)=z-\bar{z}$. If the field K is $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, we always have $D(h,k)=0$. Otherwise, it is known that $E_2(0)\neq 0$ (cf. [6, p. 539]) and we let

$$\widetilde{D}(h,k)=(\sqrt{|d|}\,i\,E_2(0))^{-1}D(h,k).$$

The value $\widetilde{D}(h,k)$ depends only on the equivalence class of the lattice L and belongs to the field $\mathbb{Q}(j)$ of the j-invariant of L ([6], Ito [4]).

Suppose that the class number of K is one. Every number of K can be expressed as a fraction h/k with coprime h and k in $\mathfrak{O}$. Since $D(-h,-k)=D(h,k)$, we can define $D(h/k)=D(h,k)$ and obtain a function on K. When $K\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3})$, we put

$$\widetilde{D}\left(\frac{h}{k}\right)=(\sqrt{|d|}\,i\,E_2(0))^{-1}D\left(\frac{h}{k}\right).$$

This is a rational number.

Theorem 2. *If the field K is Euclidean and different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, the set $\{(h/k,\widetilde{D}(h/k)) : h/k\in K\}$ is dense in the space $\mathbb{C}\times\mathbb{R}$.*

The field K satisfying the conditions in the above theorem is one of the three fields $\mathbb{Q}(\sqrt{-2}),\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-11})$. Our proof for the above theorem is similar to the one for Theorem 1 given by Hickerson. It will be given in Section 2. A difficulty in our case is that the nature of continued fractions is not as simple as in the rational case. We use results due to Hurwitz [3], Lunz [5] and Fischer [1] concerning complex continued fraction expansions and prepare Lemma 1, on which the proof of Theorem 2 will be based.

Theorem 2 does not necessarily mean that the points $(h/k,\widetilde{D}(h/k))$ distribute uniformly in the space. From numerical calculations, we can observe some bias concerning the distribution of these points. Remarks including this observation will be made in Section 3.

2. Proof of Theorem 2. We begin with an analogue of Theorem 1 in [2], which is valid for any imaginary quadratic field K. Define a polynomial $[a_0,a_1,\ldots,a_n]$ of $a_0,a_1,\ldots,a_n$ by the relations

$$[a_0]=a_0,\qquad [a_0,a_1]=a_0a_1+1$$

and

$$[a_0,a_1,\ldots,a_m]=[a_0,a_1,\ldots,a_{m-1}]a_m+[a_0,a_1,\ldots,a_{m-2}]\qquad (m\geq 2). \tag{2}$$

It is known that

$$[a_0, a_1, \ldots, a_n] = a_0[a_1, \ldots, a_n] + [a_2, \ldots, a_n], \tag{3}$$

$$[a_0, a_1, \ldots, a_n] = [a_n, a_{n-1}, \ldots, a_0], \tag{4}$$

$$[a_0, \ldots, a_n][a_1, \ldots, a_{n-1}] - [a_0, \ldots, a_{n-1}][a_1, \ldots, a_n] = (-1)^{n+1}. \tag{5}$$

THEOREM 3. *Let $a_0, a_1, \ldots, a_n \in \mathfrak{O}$ be such that $[a_m, a_{m+1}, \ldots, a_n] \neq 0$ for every m with $1 \le m \le n$. Then*

$$\begin{aligned} &D([a_0, a_1, \ldots, a_n], [a_1, a_2, \ldots, a_n]) \\ &\quad = E_2(0)\, I\biggl(\frac{[0, a_1, \ldots, a_n]}{[a_1, \ldots, a_n]} + (-1)^{n+1}\,\frac{[0, a_n, \ldots, a_1]}{[a_n, \ldots, a_1]} + a_1 - a_2 \\ &\qquad + \ldots + (-1)^{n+1} a_n\biggr). \end{aligned}$$

Proof. We proceed by induction on n. If $n = 1$, by the reciprocity formula (1) and $D(a_1, 1) = 0$, we see that

$$D(a_0 a_1 + 1, a_1) = D(1, a_1) = E_2(0) I\biggl(\frac{2}{a_1} + a_1\biggr),$$

which proves the equation of the theorem.

Let n be greater than one and assume the result for $n-1$. From (3) and (1) it follows that

$$\begin{aligned} &D([a_0, a_1, \ldots, a_n], [a_1, a_2, \ldots, a_n]) \\ &\quad = D([a_2, \ldots, a_n], [a_1, a_2, \ldots, a_n]) \\ &\quad = E_2(0)\, I\biggl(\frac{[a_2, \ldots, a_n]}{[a_1, \ldots, a_n]} + \frac{1}{[a_1, \ldots, a_n][a_2, \ldots, a_n]} + \frac{[a_1, \ldots, a_n]}{[a_2, \ldots, a_n]}\biggr) \\ &\qquad - D([a_1, a_2, \ldots, a_n], [a_2, \ldots, a_n]), \end{aligned}$$

where we note that $[a_1, a_2, \ldots, a_n]$ and $[a_2, \ldots, a_n]$ are coprime by (5). The induction hypothesis gives

$$\begin{aligned} &D([a_1, a_2, \ldots, a_n], [a_2, \ldots, a_n]) \\ &= E_2(0)\, I\biggl(\frac{[0, a_2, \ldots, a_n]}{[a_2, \ldots, a_n]} + (-1)^n\,\frac{[0, a_n, \ldots, a_2]}{[a_n, \ldots, a_2]} + a_2 - a_3 + \ldots + (-1)^n a_n\biggr) \end{aligned}$$

and because

$$\begin{aligned} &\frac{[a_1, \ldots, a_n]}{[a_2, \ldots, a_n]} - \frac{[0, a_2, \ldots, a_n]}{[a_2, \ldots, a_n]} \\ &\qquad = \frac{1}{[a_2, \ldots, a_n]}\,(a_1[a_2, \ldots, a_n] + [a_3, \ldots, a_n] - [a_3, \ldots, a_n]) = a_1, \end{aligned}$$

we have

$$
\begin{aligned}
&D([a_0, a_1, \dots, a_n], [a_1, a_2, \dots, a_n]) \\
&\quad = E_2(0)\, I\biggl(\frac{[0, a_1, \dots, a_n]}{[a_1, \dots, a_n]} + \frac{1}{[a_1, \dots, a_n][a_2, \dots, a_n]} + (-1)^{n+1} \frac{[0, a_n, \dots, a_2]}{[a_n, \dots, a_2]} \\
&\qquad + a_1 - a_2 + \dots + (-1)^{n+1} a_n \biggr).
\end{aligned}
$$

Here, by (3)–(5), we see that

$$
\begin{aligned}
&\frac{[0, a_n, \dots, a_1]}{[a_n, \dots, a_1]} - \frac{[0, a_n, \dots, a_2]}{[a_n, \dots, a_2]} \\
&\qquad = \frac{[a_1, \dots, a_{n-1}]}{[a_1, \dots, a_n]} - \frac{[a_2, \dots, a_{n-1}]}{[a_2, \dots, a_n]} = \frac{(-1)^{n+1}}{[a_1, \dots, a_n][a_2, \dots, a_n]}
\end{aligned}
$$

and hence

$$
\frac{1}{[a_1, \dots, a_n][a_2, \dots, a_n]} + (-1)^{n+1} \frac{[0, a_n, \dots, a_2]}{[a_n, \dots, a_2]} = (-1)^{n+1} \frac{[0, a_n, \dots, a_1]}{[a_n, \dots, a_1]}.
$$

This proves the theorem.

To go further, we need to consider continued fraction expansions of complex numbers with respect to the integer ring $\mathfrak{O}$, and for this purpose the field K will be assumed to be Euclidean in the rest of this section. The continued fraction

$$
a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}
$$

will be denoted as $\langle a_0, a_1, \dots, a_n \rangle$. We have

$$
\langle a_0, a_1, \dots, a_n \rangle = \frac{[a_0, a_1, \dots, a_n]}{[a_1, \dots, a_n]} \tag{6}
$$

if the left hand side is well defined, which is the case when $[a_m, a_{m-1}, \dots, a_n] \neq 0$ for every m with $1 \leq m \leq n$. Let

$$
F = \{ z \in \mathbb{C} : |z| \leq |z + a| \, (0 \neq a \in \mathfrak{O}) \}.
$$

This set is contained in the unit circle. Fix a complete representative system R of $\mathbb{C}/\mathfrak{O}$ such that $R \subset F$ and, for a complex number z, denote by $\gamma(z)$ the element of $\mathfrak{O}$ satisfying $z \in R + \gamma(z)$. If z is not in K, we can define infinite sequences $\{a_n\}_{n \geq 0}$ and $\{z_n\}_{n \geq 0}$ by

$$
\begin{aligned}
&z_0 = z, \\
&a_n = \gamma(z_n), \qquad z_{n+1} = \frac{1}{z_n - a_n} \qquad (n \geq 0).
\end{aligned}
$$

We have $|z_n| > 1$ and $a_n \neq 0$ for $n \geq 1$. Put

$$p_n = [a_0, a_1, \ldots, a_n], \qquad q_n = [a_1, a_2, \ldots, a_n].$$

Then, by (2) and (6),

$$z = \langle a_0, a_1, \ldots, a_n, z_{n+1} \rangle = \frac{p_n z_{n+1} + p_{n-1}}{q_n z_{n+1} + q_{n-1}}$$

and

$$\langle a_0, a_1, \ldots, a_n \rangle = \frac{p_n}{q_n}.$$

By Hurwitz [3], Lunz [5] and Fischer [1], it is known that

$$z = \lim_{n\to\infty} \langle a_0, a_1, \ldots, a_n \rangle = \langle a_0, a_1, \ldots, a_n, \ldots \rangle, \tag{7}$$

and that

$$|q_{n+1}| > |q_n| \qquad (n \geq 1), \tag{8}$$

(cf. also Trinks [7, p. 133]). From (8) it follows that $|q_n|^2 \geq n$ $(n \geq 1)$, and considering z_{m-1} instead of z, we see that

$$|[a_m, a_{m+1}, \ldots, a_n]|^2 \geq n - m + 1 \quad (1 \leq m \leq n). \tag{9}$$

In particular, we note that the continued fraction $\langle a_0, a_1, \ldots, a_n \rangle$ is always well defined.

We need the following lemma.

Lemma 1. *Consider the continued fraction expansion* (7) *of a complex number z not in K. Let $0 < \delta < 1$ and $|w| \leq 1 - \delta$. Then, for any positive real number ε, there exists a natural number N independent of w such that*

$$|z - \langle a_0, a_1, \ldots, a_n + w \rangle| < \varepsilon \qquad \textit{if } n \geq N.$$

Proof. We have, by (2) and (6),

$$\langle a_0, a_1, \ldots, a_n + w \rangle = \frac{p_n + p_{n-1}w}{q_n + q_{n-1}w} \qquad (n \geq 2).$$

Here, we note that $q_n + q_{n-1}w \neq 0$ since

$$|q_n + q_{n-1}w| \geq |q_n| - |q_{n-1}w| > |q_{n-1}| - (1-\delta)|q_{n-1}| = \delta |q_{n-1}| \tag{10}$$

by (8) and $|w| \leq 1 - \delta$. Also, similar considerations imply that the continued fraction $\langle a_0, a_1, \ldots, a_n + w \rangle$ is well defined. We have

$$\begin{aligned} &|z - \langle a_0, a_1, \ldots, a_n + w \rangle| \\ &\qquad = \left| z - \frac{p_n + p_{n-1}w}{q_n + q_{n-1}w} \right| \leq \left| z - \frac{p_n}{q_n} \right| + \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}w}{q_n + q_{n-1}w} \right| \end{aligned}$$

and from (5), (9) and (10) it follows that

$$\left|\frac{p_n}{q_n} - \frac{p_n + p_{n-1}w}{q_n + q_{n-1}w}\right| = \frac{|w|}{|q_n(q_n + q_{n-1}w)|} < \frac{1-\delta}{\delta|q_n|\,|q_{n-1}|} \le \frac{1-\delta}{\delta\sqrt{n(n-1)}}.$$

Now, by (7), we can take N so that

$$\left|z - \frac{p_n}{q_n}\right| < \frac{\varepsilon}{2}, \qquad \frac{1-\delta}{\delta\sqrt{n(n-1)}} < \frac{\varepsilon}{2} \quad \text{if } n \ge N.$$

It follows that

$$|z - \langle a_0, a_1, \ldots, a_n + w\rangle| < \varepsilon \quad \text{if } n \ge N$$

and the lemma is proved.

We can now prove Theorem 2. Let K be different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. We shall prove that the set

$$\{(x, (\sqrt{|d|}\, i)^{-1} I(x-z)) : x, z \in \mathbb{C} - K\} \tag{11}$$

is contained in the closure of the set

$$\{(\alpha, \widetilde{D}(\alpha)) : \alpha \in K\}. \tag{12}$$

Then, since the set (11) is dense in $\mathbb{C} \times \mathbb{R}$, we get Theorem 2. Let x and z be complex numbers not belonging to K. Consider their continued fraction expansions as in (7) and write

$$x = \langle b_0, b_1, \ldots, b_n, \ldots\rangle, \qquad z = \langle c_0, c_1, \ldots, c_n, \ldots\rangle.$$

Let ε be an arbitrary positive real number. By the above lemma, if m and n are sufficiently large, we have

$$|x - \langle b_0, b_1, \ldots, b_{m-1} + w\rangle| < \varepsilon,$$
$$|z - \langle c_0, c_1, \ldots, c_{n-1} + w\rangle| < \varepsilon$$

for any complex number w with $|w| \le 1/2$. We take such m and n satisfying $m \equiv n \pmod 2$. Furthermore, take elements u and v in $\mathfrak{O}$ such that $|u| \ge 3$, $|v| \ge 3$ and

$$b_1 - b_2 + \ldots + (-1)^m b_{m-1} + (-1)^{m+1} u + (-1)^{m+2} v + (-1)^{n-1} c_{n-1} + \ldots - c_1 = b_0 - c_0. \tag{13}$$

If we put

$$w = \langle 0, u, v, c_{n-1}, \ldots, c_1\rangle, \qquad w' = \langle 0, v, u, b_{m-1}, \ldots, b_1\rangle,$$

then we have

$$|w| \le 1/2, \qquad |w'| \le 1/2.$$

In fact, letting $y = \langle c_{n-1}, \ldots, c_1 \rangle$, we see from (4) and (6) that

$$y = \frac{[c_1, \ldots, c_{n-1}]}{[c_1, \ldots, c_{n-2}]}$$

and from (8) it follows that $|y| > 1$. Hence,

$$\begin{aligned}
|\langle v, y \rangle| &= \left| v + \frac{1}{y} \right| \geq |v| - \left| \frac{1}{y} \right| > 3 - 1 = 2, \\
|\langle u, v, y \rangle| &= \left| u + \frac{1}{\langle v, y \rangle} \right| \geq |u| - \left| \frac{1}{\langle v, y \rangle} \right| > 3 - \frac{1}{2} = \frac{5}{2}, \\
|w| &= \left| \frac{1}{\langle u, v, y \rangle} \right| < \frac{2}{5} < \frac{1}{2}.
\end{aligned}$$

In the same way, we can see that $|w'| < 1/2$.

Now, let

$$\alpha = \langle b_0, b_1, \ldots, b_{m-1}, u, v, c_{n-1}, \ldots, c_1 \rangle, \qquad \alpha = x + \delta_1.$$

Since

$$\alpha = \langle b_0, b_1, \ldots, b_{m-1} + w \rangle,$$

we have $|\delta_1| < \varepsilon$ by the choice of m. Similarly, if we write

$$\langle c_0, c_1, \ldots, c_{n-1}, v, u, b_{m-1}, \ldots, b_1 \rangle = z + \delta_2,$$

then $|\delta_2| < \varepsilon$ by the choice of n. Recall that $E_2(0) \neq 0$ since $K \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$. By Theorem 3 and the formulae (6) and (13),

$$\begin{aligned}
\widetilde{D}(\alpha) &= (\sqrt{|d|}\, i)^{-1} I(\langle 0, b_1, \ldots, b_{m-1}, u, v, c_{n-1}, \ldots, c_1 \rangle \\
&\quad - \langle 0, c_1, \ldots, c_{n-1}, v, u, b_{m-1}, \ldots, b_1 \rangle \\
&\quad + b_1 - b_2 + \ldots + (-1)^m b_{m-1} + (-1)^{m+1} u + (-1)^{m+2} v \\
&\quad + (-1)^{n-1} c_{n-1} + \ldots - c_1) \\
&= (\sqrt{|d|}\, i)^{-1} I(\langle b_0, b_1, \ldots, b_{m-1}, u, v, c_{n-1}, \ldots, c_1 \rangle \\
&\quad - \langle c_0, c_1, \ldots, c_{n-1}, v, u, b_{m-1}, \ldots, b_1 \rangle) \\
&= (\sqrt{|d|}\, i)^{-1} I(x + \delta_1 - z - \delta_2).
\end{aligned}$$

Therefore,

$$|\alpha - x| < \varepsilon, \qquad |\widetilde{D}(\alpha) - (\sqrt{|d|}\, i)^{-1} I(x - z)| \leq 4 \sqrt{|d|}^{\,-1} \varepsilon.$$

It follows that the point $(x, (\sqrt{|d|}\, i)^{-1} I(x - z))$ is in the closure of the set (12). This concludes the proof of Theorem 2.

3. Remarks. 1. Even if the field K is not Euclidean, the question whether or not the set

$$\{(h/k, \widetilde{D}(h/k)) : h/k \in K\} \tag{14}$$

is dense in $\mathbb{C} \times \mathbb{R}$ makes sense if the class number of K is one, i.e., $d = -19, -43, -67, -163$. We would like to note why the treatment as in Section 2 does not work in the non-Euclidean case.

If a sequence $\{a_n\}_{n\geq 0}$ of elements of $\mathfrak{O}$ satisfies

$$[a_m, a_{m+1}, \ldots, a_n] \neq 0 \quad (1 \leq m \leq n),$$
$$|[a_1, \ldots, a_n, a_{n+1}]| > |[a_1, \ldots, a_n]| \quad (n \geq 1),$$

then the limit

$$z = \lim_{n\to\infty} \langle a_0, a_1, \ldots, a_n \rangle$$

exists as is shown in [3]. Denote by S the set of complex numbers which can be obtained in this way. Then, by the same argument as in Section 2, we see that the closure of the set (14) contains

$$\{(x, (\sqrt{|d|}\, i)^{-1} I(x - z)) : x, z \in S\}. \tag{15}$$

If K is Euclidean, the set S contains $\mathbb{C} - K$ by (8) and (9), the set (15) becomes dense in $\mathbb{C}\times\mathbb{R}$ and we get Theorem 2. However, if K is not Euclidean, we do not know well how large the sets S and (15) are and the discussion fails.

2. As Theorem 2 states, the set (14) is dense in the space if K is Euclidean and is different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. However, this does not necessarily mean that the points $(h/k, \widetilde{D}(h/k))$ distribute uniformly in the space. For instance, if we observe the distribution of these points for a fixed denominator k, then a certain kind of bias can be seen as will be explained in the following examples. Although we treat only the case $K = \mathbb{Q}(\sqrt{-2}), d = -8$, similar phenomena can be observed also in the cases $K = \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11})$.

Example 1. Let $K = \mathbb{Q}(\sqrt{-2})$ and $k = 41 + 47\sqrt{2}\, i$. The norm of k is $6099 = 3 \cdot 19 \cdot 107$ and the number of reduced residue classes modulo k is 3816. Let

$$B = \{x + iy : 0 \leq x < 1,\ 0 \leq y < \sqrt{2}\}$$

and plot the points

$$(h/k, \widetilde{D}(h/k)) \quad (h \in \mathfrak{O},\ (h,k) = 1,\ h/k \in B)$$

in the space. Using Mathematica, we get Figures 1 and 2. In Figure 2, the observation point is on the line which is parallel to the x-axis and passes through the center $(1/2, \sqrt{2}/2, 0)$ of the box. Very roughly speaking, the value $\widetilde{D}(h/k)$ tends to get larger as $\mathrm{Im}(h/k)$ grows. We can see a similar tendency for other values of k.

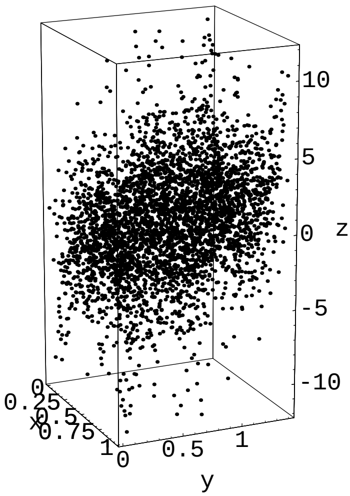

Fig. 1

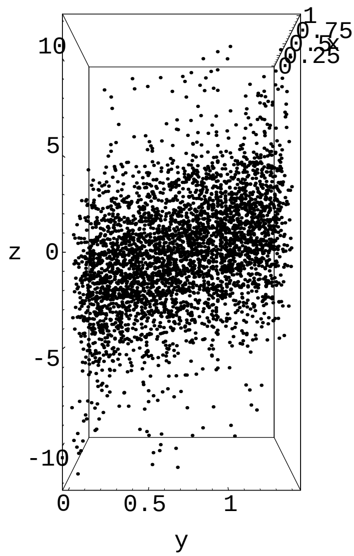

Fig. 2

EXAMPLE 2. Let $K = \mathbb{Q}(\sqrt{-2})$ as above. As one of the ways to describe the above-mentioned tendency of the value $\widetilde{D}(h/k)$, we compute the mean value for each k of the values of Dedekind sums

$$\widetilde{D}(h/k) \quad (h \in \mathfrak{O},\ (h,k) = 1,\ h/k \in B'),$$

where we put

$$B' = \{x + iy : 0 \le x < 1,\ 0 \le y < \sqrt{2}/2\}.$$

The mean values for 25 values

$$k = 200f + 1 + (200g - 1)\sqrt{2}\, i \quad (1 \le f \le 5,\ 1 \le g \le 5)$$

are listed in Table 1.

Table 1

$f \setminus g$	1	2	3	4	5
1	−1.14859	−1.17646	−1.18364	−1.18079	−1.15971
2	−1.18115	−1.18224	−1.17663	−1.17233	−1.17769
3	−1.18936	−1.17271	−1.19225	−1.17698	−1.18095
4	−1.18548	−1.18573	−1.18803	−1.17820	−1.18775
5	−1.18303	−1.18833	−1.18274	−1.18383	−1.18550

Combined with the result of some other numerical calculations, it is expected that the limit

$$\lim_{X\to\infty} \frac{1}{\sharp\{h/k \in K\cap B' : |k| < X\}} \sum_{h/k\in K\cap B',\, |k|<X} \widetilde{D}\left(\frac{h}{k}\right)$$

exists and is negative.

Numerical calculations also suggest that the mean value

$$\frac{1}{\sharp\{h/k \in \mathbb{Q}\cap(0,1/2) : 0 < k < X\}} \sum_{h/k\in\mathbb{Q}\cap(0,1/2),\, 0<k<X} s\left(\frac{h}{k}\right)$$

of the classical Dedekind sums $s(h/k)$ and the mean value

$$\frac{1}{\sharp\{h/k \in K\cap B' : |k| < X\}} \sum_{h/k\in K\cap B',\, |k|<X} \left|\widetilde{D}\left(\frac{h}{k}\right)\right|$$

of the absolute values of $\widetilde{D}(h/k)$ both tend to ∞ as X goes to ∞.

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