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On the non-existence of simple congruences for quotients of Eisenstein series

by

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1. Introduction. Define $p(n)$ to be the number of ways of writing n as a sum of non-increasing positive integers. Ramanujan famously established the congruences

$$p(5n+4) \equiv 0 \bmod 5,$$
$$p(7n+5) \equiv 0 \bmod 7,$$
$$p(11n+6) \equiv 0 \bmod 11,$$

and noted that there does not appear to be any other prime for which the partition function has equally simple congruences. Ahlgren and Boylan [1] build on the work of Kiming and Olsson [5] to prove that there truly are no other such primes. For large enough primes l, Sinick [7] and the author [3] prove the non-existence of simple congruences

$$a(ln+c) \equiv 0 \bmod l$$

for wide classes of functions $a(n)$ related to the coefficients of modular forms. However, all of the modular forms studied in [1], [7] and [3] are non-vanishing on the upper half-plane. Here we prove the non-existence of simple congruences (when l is large enough) for ratios of Eisenstein series.

Let $\sigma_m(n) := \sum_{d|n} d^m$ and define the Bernoulli numbers B_k by $t/(e^t-1) = \sum_{k=0}^{\infty} B_k t^k/k!$. For even $k \geq 2$, set

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Note that $E_2 \equiv E_4 \equiv E_6 \equiv 1$ modulo 2 and 3. Berndt and Yee [2] prove congruences for the quotients of Eisenstein series in Table 1, where $F(q) := \sum a(n)q^n$. An obviously necessary requirement for the congruences in the $n \equiv 2 \bmod 3$ column of Table 1 is that there are simple congruences of the

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Table 1. Congruences of Berndt and Yee [2]

$F(q)$	$n \equiv 2 \bmod 3$	$n \equiv 4 \bmod 8$
$1/E_2$	$a(n) \equiv 0 \bmod 3^4$	
$1/E_4$	$a(n) \equiv 0 \bmod 3^2$	
$1/E_6$	$a(n) \equiv 0 \bmod 3^3$	$a(n) \equiv 0 \bmod 7^2$
E_2/E_4	$a(n) \equiv 0 \bmod 3^3$	
E_2/E_6	$a(n) \equiv 0 \bmod 3^2$	$a(n) \equiv 0 \bmod 7^2$
E_4/E_6	$a(n) \equiv 0 \bmod 3^3$	
E_2^2/E_6	$a(n) \equiv 0 \bmod 3^5$	

form $a(3n+2) \equiv 0 \bmod 3$. All but the first form in Table 1 are covered by the following theorem.

Theorem 1.1. *Let $r \geq 0$ and $s, t \in \mathbb{Z}$. If $E_2^r E_4^s E_6^t = \sum a(n)q^n$ has a simple congruence $a(ln+c) \equiv 0 \bmod l$ for the prime l, then either $l \leq 2r + 8|s| + 12|t| + 21$ or $r = s = t = 0$.*

This theorem gives an explicit upper bound on primes l for which there can be congruences of the form $a(ln+c) \equiv 0 \bmod l^k$ as in the middle column of Table 1.

Remark 1.2. See Remark 4.1 for a slight improvement of Theorem 1.1 in some cases.

Example 1.3. The form E_6/E_4^{12} can only have simple congruences for $l \leq 129$. Of these, the primes $l = 2$ and 3 are trivial with $E_4 \equiv E_6 \equiv 1 \bmod l$. For the remaining primes, the only congruences are

$$a(ln+c) \equiv 0 \bmod 17, \qquad \text{where } \left(\frac{c}{17}\right) = -1.$$

Mahlburg [6] shows that for each of the forms in Table 1 except $1/E_2$, there are infinitely many primes l such that for any $i \geq 1$, the set of n with $a(n) \equiv 0 \bmod l^i$ has arithmetic density 1. On the other hand, our result shows that (for large enough l) every arithmetic progression modulo l has at least one non-vanishing coefficient modulo l.

Section 2 recalls certain definitions and tools from the theory of modular forms. Simple congruences are reinterpreted in terms of Tate cycles, which are reviewed in Section 3. Section 4 proves Theorem 1.1.

2. Preliminaries. A modular form of weight $k \in \mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and which is holomorphic at infinity. Modular forms have Fourier expansions in powers of $q = e^{2\pi i \tau}$. For any prime $l \geq 5$, let $\mathbb{Z}_{(l)} = \{a/b \in \mathbb{Q} : l \nmid b\}$. We denote by M_k the set of all weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with l-integral Fourier coefficients. Although E_k is a modular form of weight k whenever $k \geq 4$, E_2 is called a quasi-modular form since it satisfies the slightly different transformation rule

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6ic}{\pi}(c\tau + d).$$

DEFINITION 2.1. If l is a prime, then a Laurent series $f = \sum_{n \geq N} a(n) q^n \in \mathbb{Z}_{(l)}((q))$ has a *simple congruence* at $c \bmod l$ if $a(ln + c) \equiv 0 \bmod l$ for all n.

LEMMA 2.2. *Suppose that l is prime and that $f = \sum a(n) q^n$ and $g = \sum b(n) q^n \in \mathbb{Z}_{(l)}((q))$ with $g \not\equiv 0 \bmod l$. The series f has a simple congruence at $c \bmod l$ if and only if the series fg^l has a simple congruence at $c \bmod l$.*

Proof. It suffices to consider the reductions mod l of the series

$$\Big(\sum a(n) q^n\Big)\Big(\sum b(n) q^{ln}\Big) \equiv \sum_n \Big(\sum_m b(m) a(n - lm)\Big) q^n \bmod l.$$

If $a(n)$ vanishes when $n \equiv c \bmod l$, then the inner sum on the right hand side will also vanish for $n \equiv c \bmod l$. The converse follows via multiplication by $(\sum b(n) q^n)^{-l}$ and repetition of this argument. ∎

Our main tool is Ramanujan's Θ operator

$$\Theta := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}.$$

For any prime l and any Laurent series $f = \sum a(n) q^n \in \mathbb{Z}_{(l)}((q))$, by Fermat's Little Theorem

$$\Theta^l f = \sum a(n) n^l q^n \equiv \sum a(n) n q^n = \Theta f \bmod l.$$

We call the sequence $\Theta f, \ldots, \Theta^l f \bmod l$ the *Tate cycle* of f. Note that $\Theta^{l-1} f \equiv f \bmod l$ is equivalent to f having a simple congruence at $0 \bmod l$.

We now recall some facts about the reductions of modular forms mod l. See Swinnerton-Dyer [8, Section 3] for the details on this paragraph. There are polynomials $A(Q, R), B(Q, R) \in \mathbb{Z}_{(l)}[Q, R]$ such that

$$A(E_4, E_6) = E_{l-1}, \qquad B(E_4, E_6) = E_{l+1}.$$

Reduce the coefficients of these polynomials modulo l to get $\tilde{A}, \tilde{B} \in \mathbb{F}_l[Q, R]$. Then the polynomial $\tilde{A}$ has no repeated factor and is prime to $\tilde{B}$. Furthermore, the $\mathbb{F}_l$-algebra of reduced modular forms is naturally isomorphic to

$$\frac{\mathbb{F}_l[Q, R]}{\tilde{A} - 1} \tag{2.1}$$

via $Q \to E_4$ and $R \to E_6$. Whenever a power series f is congruent to a modular form, define the filtration of f by

$$\omega(f) := \inf\{k : f \equiv g \in M_k \bmod l\}.$$

If $f \in M_k$, then for some $g \in M_{k+l+1}$, $\Theta f \equiv g \bmod l$. The next lemma also follows from [8, Section 3].

Lemma 2.3. *Let $l \geq 5$ be prime, $f \in M_{k_1}$, $f \not\equiv 0 \bmod l$ and $g \in M_{k_2}$.*

(1) *If $f \equiv g \bmod l$, then $k_1 \equiv k_2 \bmod l-1$.*
(2) *$\omega(\Theta f) \leq \omega(f) + l + 1$ with equality if and only if $\omega(f) \not\equiv 0 \bmod l$.*
(3) *If $\omega(f) \equiv 0 \bmod l$, then for some $s \geq 1$, $\omega(\Theta f) = \omega(f) + (l+1) - s(l-1)$.*
(4) *$\omega(f^i) = i\omega(f)$.*

The natural grading induced by (2.1) provides a key step in the following lemma which is taken from the proof of [5, Proposition 2].

Lemma 2.4. *A form $f \in M_k$ with $\Theta f \not\equiv 0 \bmod l$ has a simple congruence at $c \not\equiv 0 \bmod l$ if and only if $\Theta^{(l+1)/2} f \equiv -\left(\frac{c}{l}\right)\Theta f \bmod l$.*

Proof. Since Θ satisfies the product rule, we have

$$\Theta^{l-1}(q^{-c}f) \equiv \sum_{i=0}^{l-1} \binom{l-1}{i} (-c)^{l-1-i} q^{-c}\Theta^i f \bmod l \equiv \sum_{i=0}^{l-1} c^{l-1-i} q^{-c}\Theta^i f \bmod l$$
$$\equiv c^{l-1}q^{-c}f + \sum_{i=1}^{l-1} c^{l-1-i}q^{-c}\Theta^i f \bmod l.$$

A simple congruence for f at $c \not\equiv 0 \bmod l$ is equivalent to a simple congruence for $q^{-c}f$ at $0 \bmod l$, which in turn is equivalent to $\Theta^{l-1}(q^{-c}f) \equiv q^{-c}f \bmod l$. This is equivalent to $0 \equiv \sum_{i=1}^{l-1} c^{l-1-i}q^{-c}\Theta^i f \bmod l$, by the computation above, and hence to $0 \equiv \sum_{i=1}^{l-1} c^{l-1-i}\Theta^i f \bmod l$. By Lemma 2.3(2) and (3), for $1 \leq i \leq (l-1)/2$ we have

$$\omega(\Theta^i f) \equiv \omega(\Theta^{i+(l-1)/2} f) \equiv \omega(f) + 2i \bmod l-1.$$

By Lemma 2.3(1) and the natural grading (filtration modulo $l-1$), the only way for the given sum to be zero is if for all $1 \leq i \leq (l-1)/2$ we have

$$c^{l-1-i}\Theta^i f + c^{l-1-(i+(l-1)/2)}\Theta^{i+(l-1)/2} f \equiv 0 \bmod l,$$

which happens if and only if

$$\Theta^{i+(l-1)/2} f \equiv -c^{(l-1)/2}\Theta^i f \equiv -\left(\frac{c}{l}\right)\Theta^i f \bmod l,$$

which happens if and only if

$$\Theta^{(l+1)/2} f \equiv -\left(\frac{c}{l}\right)\Theta f \bmod l. \quad \blacksquare$$

Lemma 2.5. *Let $a, b, c \geq 0$ be integers and let $l > 11$ be prime. Then $\omega(E_{l+1}^a E_4^b E_6^c) = al + a + 4b + 6c$.*

Proof. Since $E_{l+1}^a E_4^b E_6^c \in M_{al+a+4b+6c}$, it suffices to show that $\tilde{A}(Q,R)$ does not divide $\tilde{B}(Q,R)^a Q^b R^c$. However $\tilde{A}$ has no repeated factors and is prime to $\tilde{B}$ and so it suffices to show that $\tilde{A}$ does not divide QR. But QR has weight 10 and E_{l-1} has weight $l-1 > 10$ so this is impossible. ∎

3. The structure of Tate cycles. The framework we use below follows Jochnowitz [4]. Let $f \in M_k$ be such that $\Theta f \not\equiv 0 \bmod l$. Recall from Section 2 that the Tate cycle of f is the sequence $\Theta f, \ldots, \Theta^{l-1} f \bmod l$. With $s \geq 1$ as in (3) of Lemma 2.3, we have

$$\omega(\Theta^{i+1} f) \equiv \begin{cases} \omega(\Theta^i f) + 1 \bmod l & \text{if } \omega(\Theta^i f) \not\equiv 0 \bmod l, \\ s + 1 \bmod l & \text{if } \omega(\Theta^i f) \equiv 0 \bmod l. \end{cases}$$

In particular, when $\omega(\Theta^i f) \equiv 0 \bmod l$, the quantity s which determines the change in filtration also controls the time until the *next* occurrence of $\omega(\Theta^i f) \equiv 0 \bmod l$. We say that $\Theta^i f$ is a *high point* of the Tate cycle and $\Theta^{i+1} f$ is a *low point* of the Tate cycle whenever $\omega(\Theta^i f) \equiv 0 \bmod l$. Elementary considerations (see, for example, [4, Section 7] or [3, Section 3]) yield

Lemma 3.1. *Let $f \in M_k$ with $\Theta f \not\equiv 0 \bmod l$.*

(1) *If the Tate cycle has only one low point, then the low point has filtration* $2 \bmod l$.
(2) *The Tate cycle has one or two low points.*

Lemma 3.2. *Suppose $f \in M_k$ has a simple congruence at $c \not\equiv 0 \bmod l$, where $l \geq 5$ is prime, and $\Theta f \not\equiv 0 \bmod l$. Then the Tate cycle of f has two low points. Furthermore, if $\Theta^i f$ is a high point, then*

$$\omega(\Theta^{i+1} f) = \omega(\Theta^i f) + (l+1) - \left(\frac{l+1}{2}\right)(l-1) \equiv \frac{l+3}{2} \bmod l.$$

Proof. By Lemma 2.4, $\omega(\Theta f) = \omega(\Theta^{(l+1)/2} f)$. Hence, the filtration is not monotonically increasing between Θf and $\Theta^{(l+1)/2} f$, so there must be a fall in filtration (and hence a low point) somewhere in the first half of the Tate cycle. We also have $\omega(\Theta^{(l+1)/2} f) = \omega(\Theta f) = \omega(\Theta^l f)$ and so by the same reasoning there must be a low point somewhere in the second half of the Tate cycle. By Lemma 3.1, there are exactly two low points in the Tate cycle. Lemma 2.3(2) and (3) give

$$\omega(\Theta f) = \omega(\Theta^{(l+1)/2} f) = \omega(\Theta f) + \left(\frac{l-1}{2}\right)(l+1) - s(l-1)$$

for some $s \geq 1$. Hence $s = (l+1)/2$. By the same reasoning, the fall in filtration for the second half of the Tate cycle must also have $s = (l+1)/2$. The lemma follows. ∎

The proof of Theorem 1.1 uses the above lemma to determine how far the filtration falls, and the bounds of the next lemma to show a corresponding restriction on l.

LEMMA 3.3. *Let $l \geq 5$ be prime and suppose $f \in M_k$ has a simple congruence at $c \not\equiv 0 \bmod l$. If $\omega(f) = Al + B$ where $1 \leq B \leq l-1$, then*

$$\frac{l+1}{2} \leq B \leq A + \frac{l+3}{2}.$$

Proof. Since $B \neq 0$, $\omega(\Theta f) = (A+1)l + (B+1)$. From the proof of Lemma 3.2, the Tate cycle has a high point before $\Theta^{(l+1)/2} f$. By Lemma 3.2, the high point is $\Theta^i f$ with $1 \leq i \leq (l-1)/2$. Hence we have

$$\omega(\Theta^i f) = Al + B + i(l+1) \equiv B + i \equiv 0 \bmod l.$$

Together with the restrictions on B and i, this congruence implies that $B + i = l$ and $B \geq (l+1)/2$. Also, by Lemma 2.3 the high point has filtration

$$\omega(\Theta^{l-B} f) = \omega(f) + (l-B)(l+1) = (A + l - B + 1)l.$$

Lemma 3.2 implies that the corresponding low point has filtration

$$\omega(\Theta^{l-B+1} f) = \left(A - B + \frac{l+3}{2}\right)l + \left(\frac{l+3}{2}\right).$$

The fact that $\omega(\Theta^{l-B+1} f) \geq 0$ implies the second inequality. ∎

If $\Theta f \equiv 0 \bmod l$, then the Tate cycle is trivial and the above lemmas are not applicable. We dispense with this case now.

LEMMA 3.4. *Let $f = E_2^r E_4^s E_6^t$ where $r \geq 0$ and $s, t \in \mathbb{Z}$. If l is a prime such that $\Theta f \equiv 0 \bmod l$, then either $l \leq 13$ or $r \equiv s \equiv t \equiv 0 \bmod l$.*

EXAMPLE 3.5. We have $\Theta(E_4 E_6) \equiv 0 \bmod l$ for $l = 2, 3, 11$.

EXAMPLE 3.6. We have $\Theta(E_2^{144} E_4^{-15} E_6^{-14}) \equiv 0 \bmod l$ for $l = 2, 3, 5, 7, 13$.

Note that $\Theta f \equiv 0 \bmod l$ is equivalent to f having simple congruences at all $c \not\equiv 0 \bmod l$.

Proof of Lemma 3.4. Assume $l \geq 17$ and expand f as a power series to get

$$\begin{aligned} f = 1 &+ (-24r + 240s - 504t)q \\ &+ (288r^2 - 5760rs + 12096rt - 360r + 28800s^2 \\ &- 120960st - 26640s + 127008t^2 - 143640t)q^2 + \cdots. \end{aligned}$$

If $\Theta f \equiv 0 \bmod l$, then the coefficients of q and q^2 vanish modulo l. That is,

$$-24r + 240s - 504t \equiv 0 \bmod l, \tag{3.1}$$

and

$$288r^2 - 5760rs + 12096rt - 360r + 28800s^2 - 120960st - 26640s + 127008t^2 - 143640t \equiv 0 \bmod l. \tag{3.2}$$

Furthermore, by Lemmas 2.3(2) and 2.5 and the fact that $E_2 \equiv E_{l+1} \bmod l$, we have

$$\omega(E_{l+1}^r E_4^s E_6^t) \equiv r + 4s + 6t \equiv 0 \bmod l. \tag{3.3}$$

Solving the system of congruences given by (3.3) and (3.1) yields

$$7r \equiv -72t \bmod l, \tag{3.4}$$

$$14s \equiv 15t \bmod l. \tag{3.5}$$

Substituting (3.4) and (3.5) into 49 times (3.2) yields

$$-8255520t \equiv 0 \bmod l.$$

Since $8255520 = 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$, the lemma follows. ∎

4. Proof of Theorem 1.1. We begin with the trivial observation that $E_2^r E_4^s E_6^t = 1 + \cdots$ does not have a simple congruence at $0 \bmod l$. Hence, we assume that $E_2^r E_4^s E_6^t$ has a simple congruence at $c \not\equiv 0 \bmod l$, where $l \geq 5$. Since $E_2 \equiv E_{l+1} \bmod l$, $E_{l+1}^r E_4^s E_6^t$ has a simple congruence at $c \bmod l$. Recall that our goal is to show $l \leq 2r + 8|s| + 12|t| + 21$. Hence, if $l < |s|$ or $l < |t|$ then we are done. Thus we assume $l + s \geq 0$ and $l + t \geq 0$. We also assume $l > 11$. Lemma 3.4 allows us to take $\Theta(E_2^r E_4^s E_6^t) \not\equiv 0 \bmod l$ (otherwise we are done). By Lemma 2.2 we see that

$$E_{l+1}^r E_4^{l+s} E_6^{l+t} \in M_{(r+10)l+(r+4s+6t)}$$

has a simple congruence at $c \bmod l$. We work with this multiplied form $E_{l+1}^r E_4^{l+s} E_6^{l+t}$ because it is holomorphic (with positive weight) and so our filtration apparatus is applicable. By Lemma 2.5,

$$\omega(E_{l+1}^r E_4^{l+s} E_6^{l+t}) = (r + 10)l + (r + 4s + 6t). \tag{4.1}$$

We break into four cases depending on the size of $r + 4s + 6t$:

(1) If $l \leq |r + 4s + 6t|$ then we are done.

(2) If $0 < r + 4s + 6t < l$ then by equation (4.1) and the first inequality of Lemma 3.3, $(l + 1)/2 \leq r + 4s + 6t$ and we are done.

(3) If $r + 4s + 6t = 0$, then by Lemma 2.3,

$$\omega(\Theta E_{l+1}^r E_4^{l+s} E_6^{l+t}) = (r + 11)l + 1 - s'(l - 1)$$

for some $s' \geq 1$. If $l \leq r + 13$ then we are done, so it suffices to consider $l > r + 13$. Now in order for the filtration above to be non-negative, we must have $s' \leq r+11$. Now $\omega(\Theta E_{l+1}^r E_4^{l+s} E_6^{l+t}) \equiv s'+1 \bmod l$. By Lemma 2.4, there must be a high point of the Tate cycle before $\Theta^{(l+1)/2} E_{l+1}^r E_4^{l+s} E_6^{l+t}$. Let i be the index of the first high point, so $1 \leq i \leq (l-1)/2$. Then

$$\omega(\Theta^i E_{l+1}^r E_4^{l+s} E_6^{l+t}) \equiv s'+i \equiv 0 \bmod l.$$

Together with the restrictions on i and s' (namely $s' \leq r+11 < r+13 < l$), this congruence implies that

$$s' \geq \frac{l+1}{2}.$$

That is, $l \leq 2s'-1 \leq 2r+21$ and we are done.

(4) If $-l < r+4s+6t < 0$, then take $B = l+r+4s+6t$ and $A = r+9$. Equation (4.1) and the second inequality of Lemma 3.3 give

$$l+r+4s+6t \leq r+9+\frac{l+3}{2},$$

which is equivalent to $l \leq 21-8s-12t$ and we are done. ■

Remark 4.1. Combining these four cases and recalling that the proof assumed $l+s \geq 0$, $l+t \geq 0$ and $l > 11$, we see that if $r+4s+6t > 0$, then

$$l \leq \max\{|s|-1, |t|-1, 11, 2r+8s+12t-1\},$$

and if $r+4s+6t \leq 0$, then

$$l \leq \max\{|s|-1, |t|-1, 11, 21-8s-12t\}.$$

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