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Small prime solutions of ternary linear equations

by

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1. Introduction. For any integer n, we consider the ternary linear equation

$$a_1p_1 + a_2p_2 + a_3p_3 = n, \tag{1.1}$$

where p_j are prime variables and the coefficients a_j are non-zero integers. A necessary condition for the solubility of (1.1) is

$$a_1 + a_2 + a_3 \equiv n \bmod 2. \tag{1.2}$$

We also suppose

$$(a_i, a_j) = 1, \quad (a_j, n) = 1 \qquad \text{for } 1 \le i < j \le 3, \tag{1.3}$$

and write $A = \max\{2, |a_1|, |a_2|, |a_3|\}$. The main result of this paper is the following.

Theorem 1.1. *Assume* (1.2) *and* (1.3).

(i) *If* a_1, a_2, a_3 *are not all of the same sign*, *then* (1.1) *has solutions in primes* p_j *satisfying*

$$|a_j|p_j \ll |n| + A|a_1a_2a_3|^{5/2}\log^{26} A.$$

(ii) *If* a_1, a_2, a_3 *are all positive*, *then* (1.1) *is soluble whenever*

$$n \gg A(a_1a_2a_3)^{5/2}\log^{26} A.$$

It follows from the above theorem that, in case (i), (1.1) has prime solutions satisfying $p_j \ll |n| + A^{15/2}\log^{26} A$, and in case (ii), (1.1) is soluble in primes p_j if $n \gg A^{17/2}\log^{26} A$.

This problem was first raised and investigated by Baker in his well known work [1]. In the case when condition (1.3) is relaxed to that *any three of* a_1, a_2, a_3, n *are relatively prime*, the problem was settled qualitatively by

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M. C. Liu and Tsang [10]. In Choi [2] the bound A^{4190} was obtained in place of those in our Theorem 1.1. The number 4190 was subsequently reduced to 45 by M. C. Liu and Wang [12], and then to 38 by Li [8]. Under the Generalized Riemann Hypothesis, Choi, M. C. Liu, and Tsang [3] reduced the constant to $5+\varepsilon$.

We prove our theorem by the circle method, and the idea will be explained in §2. At this stage, we point out that in contrast to the earlier works [2], [10], [11], [12], which treated the enlarged major arcs by the Deuring–Heilbronn phenomenon, we show that under the stronger condition (1.3), the possible existence of Siegel's zero does not have special influence and hence the Deuring–Heilbronn phenomenon can be avoided. This observation enables us to get better results without using heavy numerical computations.

Notation. As usual, $\varphi(n)$, $\mu(n)$ and $\Lambda(n)$ stand for the functions of Euler, Möbius and von Mangoldt respectively, and $\tau(n)$ is the divisor function. We use χ mod q and χ^0 mod q to denote a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. $r\sim R$ means $R<r\le 2R$. The letters c and c_j denote absolute positive constants, but the value of c without subscript may vary at different places. The letter ε denotes a positive constant which is arbitrarily small.

2. Outline of the method. Denote by $r(n)$ the weighted number of solutions of (1.1), that is

$$r(n)=\sum_{\substack{n=a_1p_1+a_2p_2+a_3p_3\\ M<|a_j|p_j\le N}}(\log p_1)(\log p_2)(\log p_3),$$

where $M=N/200$. We will estimate $r(n)$ by the circle method. To this end, we set

$$P=(N/A)^{2/5},\qquad L=\log N,\qquad Q=N/(PL^2). \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha\in[1/Q,1+1/Q]$ may be written in the form

$$\alpha=a/q+\lambda,\qquad |\lambda|\le 1/(qQ), \tag{2.2}$$

for some integers a,q with $1\le a\le q\le Q$ and $(a,q)=1$. We denote by $\mathfrak{M}(q,a)$ the set of α satisfying (2.2) and write $\mathfrak{M}$ for the union of all these major arcs, that is, those $\mathfrak{M}(q,a)$ with $1\le a\le q\le P$ and $(a,q)=1$. It follows from $2P\le Q$ that these major arcs $\mathfrak{M}(a,q)$ are mutually disjoint. Define, as usual, the minor arcs $\mathfrak{m}$ to be the complement of $\mathfrak{M}$ in $[1/Q,1+1/Q]$. Let

$$S_j(\alpha) = \sum_{M<|a_j|p\le N} (\log p) e(a_j p\alpha).$$

Then we have

$$r(n) = \int_{1/Q}^{1+1/Q} S_1(\alpha)S_2(\alpha)S_3(\alpha)e(-n\alpha)\,d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}. \tag{2.3}$$

The integral over the major arcs $\mathfrak{M}$ causes the main difficulty, which is handled by the following.

THEOREM 2.1. *Assume* (1.3). *Let* P, Q *be defined by* (2.1). *Then*

$$\int_{\mathfrak{M}} S_1(\alpha)S_2(\alpha)S_3(\alpha)e(-n\alpha)\,d\alpha = \mathfrak{S}(n,P)\mathfrak{J}(n) + O\left(\frac{N^2}{|a_1a_2a_3|L}\right), \tag{2.4}$$

where $\mathfrak{S}(n,P)$ *and* $\mathfrak{J}(n)$ *are defined in* (2.5) *and* (2.6) *respectively.*

The proof of this theorem forms the bulk of this paper, in §§3–5. The quality of our bounds in Theorem 1.1 depends on the size of our major arcs which, as can be seen in (2.1), are quite large. The key observation is that under the assumption (1.3), we can save one negative power of r_0 in Lemma 3.1 below. With this saving, (2.4) can be derived from a hybrid estimate for Dirichlet polynomials (Lemma 3.4 below), Heath-Brown's identity, Gallagher's lemma, and classical results on the distribution of the zeros of L-functions.

To derive Theorem 1.1 from Theorem 2.1, we need to bound $\mathfrak{S}(n,P)$ and $\mathfrak{J}(n)$ from below. For χ mod q, define

$$C(\chi,a) = \sum_{h=1}^{q} \overline{\chi}(h) e\left(\frac{ah}{q}\right), \qquad C(q,a) = C(\chi^0,a).$$

If χ_1,χ_2,χ_3 are characters modulo q, we write

$$B(q,\chi_1,\chi_2,\chi_3) = \sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(-\frac{hn}{q}\right) C(\chi_1,a_1h)C(\chi_2,a_2h)C(\chi_3,a_3h),$$

$$B(q) = B(q,\chi^0,\chi^0,\chi^0), \qquad A(q) = \frac{B(q)}{\varphi^3(q)}.$$

Note that the functions $B(q)$ and $A(q)$ depend also on a_1, a_2, a_3 and n, which are fixed throughout, but we do not make this explicit for simplicity. Finally, define

$$\mathfrak{S}(n,x) = \sum_{q\le x} A(q). \tag{2.5}$$

The following two results are Lemmas 4.4 and 7.2 of [10].

LEMMA 2.2. *Assuming* (1.2), *we have* $\mathfrak{S}(n,P) \geq c_1$ *for some absolute constant* $c_1 > 0$.

LEMMA 2.3. *Suppose* (1.3) *and*

(i) *the* a_j*'s are not all of the same sign and* $N \geq 3|n|$; *or*
(ii) *the* a_j*'s are all positive and* $N = n$.

Then

$$\mathfrak{J}(n) := \sum_{\substack{a_1m_1+a_2m_2+a_3m_3=n \\ M<|a_j|m_j\leq N}} 1 \asymp \frac{N^2}{|a_1a_2a_3|}. \tag{2.6}$$

We now derive Theorem 1.1 from Theorem 2.1 and Lemmas 2.2 and 2.3.

Proof of Theorem 1.1. We start from (2.3) and let $N_j = N/|a_j|$. To estimate the integral over $\mathfrak{m}$, we appeal to Lemma 7.1 in [10]:

$$S_3(\alpha) \ll L^4(N_3P^{-1/2}|a_3|^{1/2} + N_3^{4/5} + N_3^{1/2}Q^{1/2}) \ll NL^4/\sqrt{|a_3|P}. \tag{2.7}$$

Also, we have the following mean-value estimate:

$$\int_{1/Q}^{1+1/Q} |S_j(\alpha)|^2\,d\alpha \ll LN_j,$$

which in combination with Schwarz's inequality gives

$$\int_{1/Q}^{1+1/Q} |S_1(\alpha)S_2(\alpha)|\,d\alpha \ll \frac{LN}{\sqrt{|a_1a_2|}}. \tag{2.8}$$

It therefore follows from (2.7) and (2.8) that

$$\Big|\int_{\mathfrak{m}}\Big| \ll \frac{N^2L^5}{\sqrt{|a_1a_2a_3|P}}. \tag{2.9}$$

The contribution from the major arcs is estimated in Theorem 2.1 and, together with (2.9), gives

$$r(n) = \mathfrak{S}(n,P)\mathfrak{J}(n) + O\bigg(\frac{N^2}{|a_1a_2a_3|L} + \frac{N^2L^5}{\sqrt{|a_1a_2a_3|P}}\bigg).$$

With conditions (i) or (ii) in Lemma 2.3, we deduce from Lemmas 2.2, 2.3 and the above formula that

$$r(n) \gg \frac{N^2}{|a_1a_2a_3|}$$

provided that $P \gg L^{10.4}|a_1a_2a_3|$, or equivalently, $N \gg AL^{26}|a_1a_2a_3|^{5/2}$. This proves Theorem 1.1. ∎

3. Proof of Theorem 2.1: preliminaries. In this section, we give four lemmas pertaining to the proof of Theorem 2.1.

LEMMA 3.1. *Let q and m be positive integers.*

(i) *If* $\chi \bmod q$ *is a character*, *then*

$$|C(\chi,m)| \le (q,m)^{1/2}q^{1/2}.$$

(ii) *Let* $\chi_j \bmod r_j$ *with* $j=1,2,3$ *be primitive characters*, $r_0=[r_1,r_2,r_3]$, *and* χ^0 *be the principal character modulo q. Then*

$$\sum_{\substack{q\le x\\ r_0\mid q}} \frac{1}{\varphi^3(q)}\,|B(q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)| \ll \frac{r_0^2}{\varphi^3(r_0)}\sqrt{(\varpi,r_0)}\log^2 x;$$

here and throughout $\varpi=|a_1a_2a_3n|$.

Proof. (i) Let $\chi \bmod q$ be induced by the primitive character $\chi^* \bmod q^*$. Write $q=q_1q_2$ with $(q_2,q^*)=1$ and $p\mid q_1\Rightarrow p\mid q^*$. Then, by [6, p. 450],

$$C(\chi,m)=\chi^*\bigg(\frac{m}{(m,q)}\bigg)\overline{\chi}^*\bigg(\frac{q}{q^*(m,q)}\bigg)\mu\bigg(\frac{q}{q^*(m,q)}\bigg)\frac{\varphi(q)}{\varphi(q/(m,q))}\,C(\chi^*,1)$$

if $q^*=q_1/(q_1,m)$; otherwise $C(\chi,m)=0$.

We first establish our assertion in the special case that q is a power of a prime, say $q=p^\alpha$. In this case we must have $q_1=p^\alpha$ and $q_2=1$. Let $q^*=p^\beta$ and $p^\gamma\,\|\,m$. We may suppose $\gamma\le\alpha$, since otherwise p divides $m/(m,q)$ and hence

$$\chi^*\bigg(\frac{m}{(m,q)}\bigg)=0,$$

which gives $C(\chi,m)=0$. Also, we only have to consider the case $q^*=q_1/(q_1,m)$, that is,

$$\beta=\alpha-\gamma,$$

since otherwise we have $C(\chi,m)=0$ again. Finally, we have

$$|C(\chi,m)|\le\frac{\varphi(p^\alpha)}{\varphi(p^{\alpha-\gamma})}\,|C(\chi^*,1)|\le p^{\gamma+\beta/2}=p^{\gamma/2+\alpha/2}=(q,m)^{1/2}q^{1/2}.$$

This proves our assertion in the special case $q=p^\alpha$.

The general case can be established by decomposing $C(\chi,m)$ according to the canonical factorization of $\chi \bmod q$ and then using the Chinese remainder theorem.

(ii) By Lemma 4.5 in [10], we have

$$B(q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)=B(r_0,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)B(q/r_0)$$

if $(r_0,q/r_0)=1$; and it vanishes otherwise. Here we note that the moduli of the principal characters in the above functions $B(\cdot)$ on the right hand side are r_0 and q/r_0 respectively. It therefore follows that

$$\sum_{\substack{q\le x\\ r_0|q}} \frac{1}{\varphi^3(q)} |B(q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)| = \frac{1}{\varphi^3(r_0)} |B(r_0,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)| \sum_{\substack{q\le x/r_0\\ (r_0,q)=1}} |A(q)|. \tag{3.1}$$

Now the argument of Lemma 4.4(1) in [10] gives

$$\sum_{q\le x} |A(q)| \ll \log^2 x. \tag{3.2}$$

It remains, therefore, to estimate $B(r_0,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)$.

By the definition of $C(\chi,m)$, we have $C(\chi,mh)=\chi(h)C(\chi,m)$. Consequently,

$$\begin{aligned} &B(r_0,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)\\ &\quad= \sum_{\substack{h=1\\ (h,r_0)=1}}^{r_0} e\Big(-\frac{nh}{r_0}\Big) C(\chi_1\chi^0,a_1h)C(\chi_2\chi^0,a_2h)C(\chi_3\chi^0,a_3h)\\ &\quad= C(\chi_1\chi^0,a_1)C(\chi_2\chi^0,a_2)C(\chi_3\chi^0,a_3)C(\overline{\chi}_1\overline{\chi}_2\overline{\chi}_3\chi^0,-n). \end{aligned}$$

Now (i) and (1.3) gives

$$|B(r_0,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)| \le r_0^2\sqrt{(a_1,r_0)(a_2,r_0)(a_3,r_0)(n,r_0)} = r_0^2\sqrt{(\varpi,r_0)},$$

which together with (3.2) and (3.1) yields the desired result. ■

Recall $N_j=N/|a_j|$ for $j=1,2,3$, and set

$$M_j=\frac{M}{|a_j|},\qquad V_j(\lambda)=\sum_{M<|a_j|m\le N} e(a_jm\lambda),$$

and

$$W_j(\chi,\lambda)=\sum_{M<|a_j|p\le N} (\log p)\chi(p)e(a_jp\lambda) - \delta_\chi \sum_{M<|a_j|m\le N} e(a_jm\lambda), \tag{3.3}$$

where $\delta_\chi=1$ or 0 according as χ is principal or not. Define

$$J_j(R)=\sum_{r\sim R}\frac{\sqrt{(\varpi,r)}}{r}\sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\le 1/(rQ)} |W_j(\chi,\lambda)|, \tag{3.4}$$

and for any positive integer g,

$$K_j(g,R)=\sum_{r\sim R}\frac{\sqrt{(\varpi,[g,r])}}{[g,r]}\sideset{}{^*}\sum_{\chi \bmod r}\Big\{\int_{|\lambda|\le 1/(rQ)} |W_j(\chi,\lambda)|^2\,d\lambda\Big\}^{1/2}, \tag{3.5}$$

where $\sum^*_{\chi \bmod r}$ is over all the primitive characters modulo r. To prove our Theorem 2.1, we need the following two key lemmas which will be proved in §5.

LEMMA 3.2. *For P, Q satisfying* (2.1), *we have*

(i) $K_j(g,R) \ll g^{-1}\sqrt{(\varpi,g)}\,\tau(g)\tau(\varpi)\sqrt{N}\,|a_j|^{-1}L^c$ *if* $R \ll P$,
(ii) $K_j(g,R) \ll g^{-1}\sqrt{(\varpi,g)}\,\tau(g)\sqrt{N}\,|a_j|^{-1}L^c$ *if* $R \ll N^{1/10}$.

LEMMA 3.3. *Let P, Q be as in* (2.1). *We have*

(i) $J_j(R) \ll \tau(\varpi)R^{-1/4}N|a_j|^{-1}L^c$ *if* $R \ll P$,
(ii) $J_j(R) \ll R^{-1/4}N|a_j|^{-1}L^{-c_2}$ *for any large constant c_2 if* $R \ll N^{1/10}$.

These two lemmas depend on a hybrid estimate for Dirichlet polynomials (Lemma 3.4 below). Let $X^{2/5} < Y \le X$ and $D_1,\dots,D_{10}$ be positive integers such that

$$2^{-10}Y \le D_1\cdots D_{10} < X \quad \text{and} \quad 2D_6,\dots,2D_{10} \le X^{1/5}. \tag{3.6}$$

For $j=1,\dots,10$, define

$$b_j(m) = \begin{cases} \log m & \text{if } j=1, \\ 1 & \text{if } j=2,\dots,5, \\ \mu(m) & \text{if } j=6,\dots,10, \end{cases} \tag{3.7}$$

where $\mu(n)$ is the Möbius function. For any Dirichlet character χ and complex variable s, define the functions

$$f_j(s,\chi) = \sum_{m\sim D_j} \frac{b_j(m)\chi(m)}{m^s}$$

and

$$F_{\mathbf{D}}(s,\chi) = f_1(s,\chi)\cdots f_{10}(s,\chi), \tag{3.8}$$

where $\mathbf{D} = (D_1,\dots,D_{10})$. The following hybrid estimate for $F_{\mathbf{D}}(s,\chi)$ is Lemma 2.1 in [9]. The parameter d in (3.9) is crucial for our iterative argument in §4.

LEMMA 3.4. *Let $F_{\mathbf{D}}(s,\chi)$ be defined as above. Then for any $1 \le R \le X^2$ and $T > 0$,*

$$\sum_{\substack{r\sim R \\ d\mid r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_T^{2T} \left| F_{\mathbf{D}}\left(\frac{1}{2}+it,\chi\right)\right| dt \ll \left\{\frac{R^2}{d}T + \frac{R}{d^{1/2}}T^{1/2}X^{3/10} + X^{1/2}\right\}\log^c X. \tag{3.9}$$

4. Proof of Theorem 2.1: an iterative method. Introducing Dirichlet characters, we can express the exponential sum $S_j(\alpha)$ as (see for example [4, §26, (2)])

$$S_j\left(\frac{h}{q}+\lambda\right) = \frac{C(q,a_jh)}{\varphi(q)}V_j(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C(\chi,a_jh)W_j(\chi,\lambda) =: T_j + U_j,$$

say. Thus,

$$\int_{\mathfrak{M}} S_1(\alpha)S_2(\alpha)S_3(\alpha)e(-n\alpha)\,d\alpha = I_0+I_1+I_2+I_3, \tag{4.1}$$

where I_0, I_1, I_2, I_3 are the contributions from, respectively,

$$\begin{aligned}&T_1T_2T_3,\\ &U_1T_2T_3+T_1U_2T_3+T_1T_2U_3,\\ &U_1U_2T_3+U_1T_2U_3+T_1U_2U_3,\\ &U_1U_2U_3.\end{aligned}$$

We shall now show that I_0 contains the main term for $\int_{\mathfrak{M}}$ and I_1, I_2, I_3 constitute the error term.

Lemma 4.1. *For $j=1,2,3$, we have*

(i) $\displaystyle\int_{|\lambda|\le 1/|2a_j|} |V_j(\lambda)|^2\,d\lambda \ll N|a_j|^{-2}$,

(ii) $\displaystyle\int_{|\lambda|\le 1/2} |V_j(\lambda)|^2\,d\lambda \ll N|a_j|^{-1}$.

Proof. By definition of $V_j(\lambda)$,

$$\int_{|\lambda|\le 1/|2a_j|} |V_j(\lambda)|^2\,d\lambda = \sum_{M_j<m,\,m'\le N_j}\ \int_{|\lambda|\le 1/|2a_j|} e((m-m')a_j\lambda)\,d\lambda \ll \frac{N_j}{|a_j|}.$$

Part (ii) follows from (i) and the fact that $V_j(\lambda)$ has period $|a_j|^{-1}$. ∎

Lemma 4.2. *For $(2A)^{-1}<|\lambda|<1/2$, we have*

$$\max_{j=1,2,3}(\|a_j\lambda\|) \ge 1/(2A).$$

Proof. For $j=1,2,3$, let $\beta_j=\|a_j\lambda\|$ and $a_j\lambda = n_j\pm\beta_j$, where n_j are integers. If the three rational numbers n_j/a_j are all the same, then since $(a_1,a_2,a_3)=1$ they must be all equal to an integer k. Furthermore k must be equal to zero, for otherwise, from

$$\lambda = \frac{n_1}{a_1}\pm\frac{\beta_1}{a_1} = k\pm\frac{\beta_1}{a_1}$$

we have $|\lambda| \geq |k| - 1/2 \geq 1/2$, which is a contradiction. Hence

$$\beta_j = |a_j\lambda| > \frac{|a_j|}{2A}, \qquad j = 1, 2, 3.$$

This yields the desired bound.

On the other hand, if the three rational numbers n_j/a_j are not all the same, $n_1/a_1 \neq n_2/a_2$, say, then

$$\left|\frac{\beta_1}{a_1} \pm \frac{\beta_2}{a_2}\right| = \left|\frac{n_1}{a_1} - \frac{n_2}{a_2}\right| \geq \frac{1}{|a_1a_2|}.$$

Hence

$$\max(\beta_1, \beta_2)\left(\frac{1}{|a_1|} + \frac{1}{|a_2|}\right) \geq \frac{1}{|a_1a_2|}.$$

The desired bound again follows in this case. ∎

By definition, I_0 is equal to

$$\sum_{q\leq P} \frac{B(q)}{\varphi^3(q)} \int\limits_{|\lambda|\leq 1/(qQ)} V_1(\lambda)V_2(\lambda)V_3(\lambda)e(-n\lambda)\, d\lambda. \tag{4.2}$$

We begin by extending the above integral to the interval $[-1/2, 1/2]$ in two steps. First, by the obvious bound $V_j(\lambda) \ll \|a_j\lambda\|^{-1} = |a_j\lambda|^{-1}$ for $|\lambda| \leq 1/(2A)$, we have

$$\int\limits_{1/(qQ)<|\lambda|\leq 1/(2A)} V_1(\lambda)V_2(\lambda)V_3(\lambda)e(-n\lambda)\, d\lambda \ll \int\limits_{1/(qQ)<|\lambda|\leq 1/(2A)} \frac{d\lambda}{|a_1a_2a_3|\lambda^3} \ll \frac{(qQ)^2}{|a_1a_2a_3|}.$$

In view of (2.1) and (3.2), the error this contributes to I_0 in (4.2) is $\ll N^2(L|a_1a_2a_3|)^{-1}$, which is acceptable. Next, by the bound $V_j(\lambda) \ll \|a_j\lambda\|^{-1}$ and Lemma 4.2, we have

$$\int\limits_{1/(2A)<|\lambda|\leq 1/2} V_1(\lambda)V_2(\lambda)V_3(\lambda)e(-n\lambda)\, d\lambda \ll A \sum_{1\leq i<j\leq 3} \int\limits_{|\lambda|\leq 1/2} |V_i(\lambda)V_j(\lambda)|\, d\lambda \ll AN,$$

by Lemma 4.1(ii) and Schwarz's inequality. Clearly, the contribution of this to (4.2) is negligible. Hence we find that

$$I_0 = \sum_{q\leq P} \frac{B(q)}{\varphi^3(q)} \int\limits_{|\lambda|\leq 1/2} V_1(\lambda)V_2(\lambda)V_3(\lambda)e(-n\lambda)\, d\lambda + O\left(\frac{N^2}{L|a_1a_2a_3|}\right).$$

In view of (2.6) and (2.5), this yields the main term on the right hand side of (2.4).

We now turn to the terms in I_1, I_2, and I_3. The main feature in these terms is that each of them has at least one factor of U_j and this is precisely the source of the saving of a factor L^{-c_2} in Lemma 3.3(ii). We explain our strategy below by treating in detail the most complicated case, viz. I_3, and then indicate briefly the treatment for I_2 and I_1.

Reducing the characters in I_3 into primitive characters, we have

$$\begin{aligned}
|I_3| &= \bigg| \sum_{q\le P} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} \frac{B(q,\chi_1,\chi_2,\chi_3)}{\varphi^3(q)} \\
&\qquad \times \int_{|\lambda|\le 1/(qQ)} W_1(\chi_1,\lambda)W_2(\chi_2,\lambda)W_3(\chi_3,\lambda)e(-n\lambda)\,d\lambda \bigg| \\
&\le \sum_{r_i\le P} \sideset{}{^*}\sum_{\chi_i \bmod r_i} \sum_{\substack{q\le P\\ r_0|q}} \frac{|B(q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)|}{\varphi^3(q)} \\
&\qquad \times \int_{|\lambda|\le 1/(qQ)} |W_1(\chi_1\chi^0,\lambda)W_2(\chi_2\chi^0,\lambda)W_3(\chi_3\chi^0,\lambda)|\,d\lambda,
\end{aligned}$$

where

$$\sum_{r_i\le P} = \sum_{r_1\le P}\sum_{r_2\le P}\sum_{r_3\le P}, \qquad \sideset{}{^*}\sum_{\chi_i \bmod r_i} = \sideset{}{^*}\sum_{\chi_1 \bmod r_1}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3},$$

χ^0 is the principal character modulo q and $r_0 = [r_1, r_2, r_3]$. For $q \le P$ and $M < |a_j|p \le N$, we have $(q,p) = 1$. Using this and (3.3), we see that $W_j(\chi_j\chi^0,\lambda) = W_j(\chi_j,\lambda)$ for the primitive characters χ_j above. Consequently, by Lemma 3.1(ii) we have

$$\begin{aligned}
(4.3)\qquad |I_3| &\le \sum_{r_i\le P} \sideset{}{^*}\sum_{\chi_i \bmod r_i} \int_{|\lambda|\le 1/(r_0Q)} |W_1(\chi_1,\lambda)W_2(\chi_2,\lambda)W_3(\chi_3,\lambda)|\,d\lambda \\
&\qquad \times \sum_{\substack{q\le P\\ r_0|q}} \frac{|B(q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)|}{\varphi^3(q)} \\
&\ll L^3 \sum_{r_i\le P} \frac{\sqrt{(\varpi, r_0)}}{r_0} \\
&\qquad \times \sideset{}{^*}\sum_{\chi_i \bmod r_i} \int_{|\lambda|\le 1/(r_0Q)} |W_1(\chi_1,\lambda)W_2(\chi_2,\lambda)W_3(\chi_3,\lambda)|\,d\lambda \\
&\ll L^6 \max_{R_1,R_2,R_3\le P} I_3(R_1,R_2,R_3),
\end{aligned}$$

where

$$(4.4)\qquad I_3(R_1,R_2,R_3)=\sum_{r_1\sim R_1}\sum_{r_2\sim R_2}\sum_{r_3\sim R_3}\frac{\sqrt{(\varpi,r_0)}}{r_0}\times\sideset{}{^*}\sum_{\chi_i \bmod r_i}\int_{|\lambda|\le 1/(r_0Q)}|W_1(\chi_1,\lambda)W_2(\chi_2,\lambda)W_3(\chi_3,\lambda)|\,d\lambda.$$

Without loss of generality, suppose $R_1 \ge R_2, R_3$. In the integral in $I_3(R_1,R_2,R_3)$, we take out $|W_1(\chi_1,\lambda)|$ and then use Schwarz's inequality to get

$$(4.5)\qquad |I_3(R_1,R_2,R_3)| \ll \sum_{r_1\sim R_1}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le 1/(r_1Q)}|W_1(\chi_1,\lambda)| \times\sum_{r_2\sim R_2}\sideset{}{^*}\sum_{\chi_2 \bmod r_2}\Big\{\int_{|\lambda|\le 1/(r_2Q)}|W_2(\chi_2,\lambda)|^2\,d\lambda\Big\}^{1/2} \times\sum_{r_3\sim R_3}\frac{\sqrt{(\varpi,r_0)}}{r_0}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\Big\{\int_{|\lambda|\le 1/(r_3Q)}|W_3(\chi_3,\lambda)|^2\,d\lambda\Big\}^{1/2}.$$

We now consider two cases.

Case (I): $R_1 \gg N^{1/10}$. The innermost sum over r_3 in $I_3(R_1,R_2,R_3)$ is $K_3([r_1,r_2],R_3)$. Applying the bound in Lemma 3.2(i) twice then yields

$$(4.6)\qquad I_3(R_1,R_2,R_3) \ll \sum_{r_1\sim R_1}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le 1/(r_1Q)}|W_1(\chi_1,\lambda)| \times\sum_{r_2\sim R_2}\sideset{}{^*}\sum_{\chi_2 \bmod r_2}\Big\{\int_{|\lambda|\le 1/(r_2Q)}|W_2(\chi_2,\lambda)|^2\,d\lambda\Big\}^{1/2} \times\frac{\sqrt{(\varpi,[r_1,r_2])}}{[r_1,r_2]}\tau(\varpi)\tau([r_1,r_2])\frac{\sqrt{N}}{|a_3|}L^c$$
$$\ll \tau(\varpi)R_1^{\varepsilon}\frac{\sqrt{N}}{|a_3|}\sum_{r_1\sim R_1}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le 1/(r_1Q)}|W_1(\chi_1,\lambda)|K_2(r_1,R_2)$$
$$\ll \tau^2(\varpi)R_1^{\varepsilon}\frac{N}{|a_2a_3|}\sum_{r_1\sim R_1}\frac{\sqrt{(\varpi,r_1)}}{r_1}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le 1/(r_1Q)}|W_1(\chi_1,\lambda)|.$$

In the above, we have used the fact that $\tau([r_1,r_2]) \ll [r_1,r_2]^{\varepsilon} \ll R_1^{\varepsilon}$. The last double sum above is $J_1(R_1)$, which we estimate by the bound in Lemma 3.3(i). This leads to

$$I_3(R_1, R_2, R_3) \ll \tau^3(\varpi) R_1^{\varepsilon - 1/4} \frac{N^2}{|a_1 a_2 a_3|} \ll \frac{N^{2-\varepsilon}}{|a_1 a_2 a_3|},$$

in view of $\tau(\varpi) \ll \varpi^\varepsilon \ll N^\varepsilon$ and our assumption that $R_1 \gg N^{1/10}$ in this case.

Case (II): $R_1 \ll N^{1/10}$. The procedure here is the same as in Case (I), except that we use the alternative bounds in Lemmas 3.2, 3.3 in which $\tau(\varpi)$ does not appear. In this way, we get

$$I_3(R_1, R_2, R_3) \ll \frac{N^2}{|a_1 a_2 a_3| L^{c_3}}$$

for any constant $c_3 > 0$.

Inserting these two cases into (4.5) and then into (4.3), we obtain

$$I_3 \ll \frac{N^2}{|a_1 a_2 a_3| L},$$

as desired.

The treatment for the terms in I_2 is similar. For instance, the contribution of $U_1 T_2 U_3$ is

$$\begin{aligned} (4.7) \quad & \ll L^3 \sum_{r_1 \le P} \sum_{r_3 \le P} \frac{\sqrt{(\varpi, r')}}{r'} \\ & \quad \times \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \int_{|\lambda| \le 1/(r'Q)} |W_1(\chi_1, \lambda) V_2(\lambda) W_3(\chi_3, \lambda)| \, d\lambda \\ & \ll L^5 \max_{R_1, R_3 \le P} \sum_{r_1 \sim R_1} \sum_{r_3 \sim R_3} \frac{\sqrt{(\varpi, r')}}{r'} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\ & \quad \times \int_{|\lambda| \le 1/(r'Q)} |W_1(\chi_1, \lambda) V_2(\lambda) W_3(\chi_3, \lambda)| \, d\lambda, \end{aligned}$$

where $r' = [r_1, r_3]$. Without loss of generality, assume $R_1 \ge R_3$. Then the inner integral is

$$\begin{aligned} (4.8) \quad & \le \max_{|\lambda| \le 1/(r_1 Q)} |W_1(\chi_1, \lambda)| \Big\{ \int_{|\lambda| \le 1/Q} |V_2(\lambda)|^2 \, d\lambda \Big\}^{1/2} \\ & \quad \times \Big\{ \int_{|\lambda| \le 1/(r_3 Q)} |W_3(\chi_3, \lambda)|^2 \, d\lambda \Big\}^{1/2}. \end{aligned}$$

By Lemma 4.1(i) (note that $1/Q < 1/|2a_2|$) we see that the right hand side of (4.7) is

$$\ll L^5 \frac{\sqrt{N}}{|a_2|} \max_{R_3 \le R_1 \le P} \sum_{r_1 \sim R_1} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \le 1/(r_1 Q)} |W_1(\chi_1, \lambda)| K_3(r_1, R_3),$$

which can be handled as before by considering separately the cases $R_1 \ll N^{1/10}$ and $R_1 \gg N^{1/10}$.

The treatment of the three terms in I_1 is even simpler, requiring only Lemma 4.1(i) twice and Lemma 3.3. This completes the proof of Theorem 2.1.

5. Bounds for K_j and J_j. Let

$$\widehat{W}_j(\chi, \lambda) = \sum_{M < |a_j| m \le N} (\Lambda(m)\chi(m) - \delta_\chi) e(a_j m \lambda).$$

Then

$$\begin{aligned} (5.1) \qquad W_j(\chi,\lambda) - \widehat{W}_j(\chi,\lambda) \\ = -\sum_{k \ge 2} \sum_{M < |a_j| p^k \le N} (\log p) \chi(p) e(a_j p^k \lambda) \ll \sqrt{N_j}. \end{aligned}$$

The contributions of this error term $\sqrt{N_j}$ to J_j and K_j are

$$\ll \sqrt{N_j} \sum_{r \sim R} \sqrt{(\varpi, r)} \quad \text{and} \quad \sqrt{\frac{R N_j}{Q}} \sum_{r \sim R} \frac{\sqrt{(\varpi, [g, r])}}{[g, r]}$$

respectively. Estimating these by using (5.8), (5.9), (5.20) and (5.22) below, we see that these are negligible in comparison with our bounds for J_j and K_j. We shall henceforth replace W_j by $\widehat{W}_j$ in J_j and K_j.

Proof of Lemma 3.3. To the sum

$$(5.2) \qquad \sum_{M_j < m \le u} \Lambda(m)\chi(m), \qquad u \le N_j,$$

in J_j, we apply Heath-Brown's identity (see Lemma 1 in [7]) with $k = 5$, which states that

$$\Lambda(m) = \sum_{j=1}^{5} \binom{5}{j} (-1)^{j-1} \sum_{\substack{m_1 \cdots m_{2j} = m \\ m_{j+1}, \ldots, m_{2j} \le u^{1/5}}} (\log m_1) \mu(m_{j+1}) \cdots \mu(m_{2j}).$$

Dividing the summation range for each m_i into dyadic intervals of the form $(D_i, 2D_i]$, where $D_1, \ldots, D_{10}$ satisfy the conditions in (3.6) with $Y = M_j$

and $X = N_j$, we see that the sum in (5.2) is equal to $\sum_{\mathbf{D}} \sigma(u;\mathbf{D})$ with

$$\sigma(u;\mathbf{D}) := \sum_{\substack{m_1\sim D_1 \\ M_j<m_1\cdots m_{10}\le u}} \cdots \sum_{m_{10}\sim D_{10}} b_1(m_1)\chi(m_1)\cdots b_{10}(m_{10})\chi(m_{10}).$$

Here the functions b_i are defined in (3.7) and $\sum_{\mathbf{D}}$ is the summation over all the vectors $\mathbf{D} = (D_1,\ldots,D_{10})$ which satisfy (3.6). By the definition of $F_{\mathbf{D}}(s,\chi)$ in (3.8) and by using Perron's summation formula (see, for example, Lemma 3.12 in [14]), we have

$$\sigma(u;\mathbf{D}) = \frac{1}{2\pi i}\int_{1+1/L-iT}^{1+1/L+iT} F(s,\chi)\,\frac{u^s - M_j^s}{s}\,ds + O(L^2)$$

where $T = N_j$.

As usual, we shift the path of integration to the vertical line $\mathrm{Re}(s) = 1/2$ (note that $F_{\mathbf{D}}(s,\chi)$ is a Dirichlet polynomial and hence has no poles) and estimate the contributions on the two horizontal segments as

$$\max_{1/2\le\sigma\le1+1/L} |F_{\mathbf{D}}(\sigma\pm iT,\chi)|\,\frac{u^\sigma}{T} \ll \max_{1/2\le\sigma\le1+1/L} N_j^{1-\sigma}L\,\frac{u^\sigma}{T} \ll L,$$

on using the trivial estimate

$$\begin{aligned} F_{\mathbf{D}}(\sigma\pm iT,\chi) &\ll |f_1(\sigma\pm iT,\chi)|\cdots|f_{10}(\sigma\pm iT,\chi)| \\ &\ll (D_1^{1-\sigma}L)D_2^{1-\sigma}\cdots D_{10}^{1-\sigma} \ll N_j^{1-\sigma}L. \end{aligned}$$

Then we find that

$$\sigma(u;\mathbf{D}) = \frac{1}{2\pi}\int_{-T}^{T} F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\frac{u^{1/2+it}-M_j^{1/2+it}}{1/2+it}\,dt + O(L^2). \tag{5.3}$$

We may assume that $R \ge 1$ so that the primitive character χ mod r is not principal and $\delta_\chi = 0$. Then

$$\begin{aligned} \widehat{W}_j(\chi,\lambda) &= \sum_{M_j<m\le N_j} \Lambda(m)\chi(m)e(a_jm\lambda) = \sum_{\mathbf{D}}\int_{M_j}^{N_j} e(a_ju\lambda)\,d(\sigma(u;\mathbf{D})) \\ &= \sum_{\mathbf{D}}\frac{1}{2\pi}\int_{-T}^{T} F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\int_{M_j}^{N_j} u^{-1/2+it}e(a_ju\lambda)\,du\,dt \\ &\quad + O\{(1+|\lambda|N)L^{12}\}. \end{aligned}$$

The inner integral over u is equal to

$$\int_{M_j}^{N_j} u^{-1/2}e\Big(\frac{t}{2\pi}\log u + a_j\lambda u\Big)\,du,$$

which we now estimate by means of Lemmas 4.3 and 4.5 of [14]. Let $T_0 = 4\pi N/(RQ) = 4\pi PL/R$. Since

$$\left|\frac{d}{du}\left(\frac{t}{2\pi}\log u + a_j\lambda u\right)\right| = \left|\frac{t}{2\pi u} + a_j\lambda\right| \ge \frac{|t|}{4\pi N_j}$$

for $|t| > T_0$, and

$$\left|\frac{d^2}{du^2}\left(\frac{t}{2\pi}\log u + a_j\lambda u\right)\right| = \left|-\frac{t}{2\pi u^2}\right| \ge \frac{|t|}{2\pi N_j^2},$$

we have

$$\int_{M_j}^{N_j} u^{-1/2} e\left(\frac{t}{2\pi}\log u + a_j\lambda u\right) du \ll \begin{cases} \sqrt{N_j}/\sqrt{|t|+1} & \text{for } |t| \le T_0, \\ \sqrt{N_j}/|t| & \text{for } T_0 < |t| \le T. \end{cases} \tag{5.4}$$

Therefore,

$$\max_{|\lambda|\le 1/(rQ)} |\widehat{W}_j(\chi,\lambda)| \ll \sqrt{N_j}\sum_{\mathbf{D}}\left\{\int_{|t|\le T_0}\left|F_{\mathbf{D}}\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{\sqrt{|t|+1}} + \int_{T_0<|t|\le N_j}\left|F_{\mathbf{D}}\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|}\right\} + \frac{NL^{12}}{RQ}. \tag{5.5}$$

Thus,

$$\begin{aligned} &\sum_{\substack{r\sim R\\ d\mid r}}\sideset{}{^*}\sum_{\chi \bmod r}\max_{|\lambda|\le 1/(rQ)} |\widehat{W}_j(\chi,\lambda)| \\ &\quad\ll \sqrt{N_j}\,L\sum_{\mathbf{D}}\left\{\max_{Y\le T_0}\frac{1}{\sqrt{Y}}\sum_{\substack{r\sim R\\ d\mid r}}\sideset{}{^*}\sum_{\chi \bmod r}\int_{|t|\sim Y}\left|F_{\mathbf{D}}\left(\frac{1}{2}+it,\chi\right)\right| dt \right.\\ &\qquad\left. + \max_{T_0<Z\le N_j}\frac{1}{Z}\sum_{\substack{r\sim R\\ d\mid r}}\sideset{}{^*}\sum_{\chi \bmod r}\int_{|t|\sim Z}\left|F_{\mathbf{D}}\left(\frac{1}{2}+it,\chi\right)\right| dt\right\} + \frac{NR}{Qd}\,L^{12}. \end{aligned} \tag{5.6}$$

Applying now Lemma 3.4 with $X = N_j$, the above is

$$\ll \left(\frac{R^2}{d}\sqrt{T_0 N_j} + \frac{R}{\sqrt{d}}\,N_j^{4/5} + N_j\right)L^c. \tag{5.7}$$

Notice that for any function $H(r)$,

$$\sum_{r\sim R}\frac{\sqrt{(\varpi,r)}}{r}\,H(r) \le \frac{1}{R}\sum_{\substack{d\mid\varpi\\ d\ll R}}\sqrt{d}\sum_{\substack{r\sim R\\ d\mid r}}|H(r)|. \tag{5.8}$$

Hence by (5.7) and the definition of $J_j(R)$ in (3.4),

$$J_j(R) \ll \frac{1}{R} \sum_{\substack{d|\varpi \\ d \ll R}} \left(\sqrt{\frac{T_0 N_j}{d}}\, R^2 + R N_j^{4/5} + \sqrt{d}\, N_j \right) L^c$$
$$\ll \tau(\varpi)(\sqrt{RPN_j} + N_j^{4/5} + R^{-1/2} N_j) L^c.$$

In view of the assumption that $R \ll P \le N_j^{2/5}$, this yields the bound in Lemma 3.3(i).

To prove the alternative bound for $J_j(R)$ in Lemma 3.3(ii), we note trivially that

$$r^{-1}\sqrt{(\varpi, r)} \le r^{-1/2}. \tag{5.9}$$

Hence by (5.7) and definition (3.4),

$$J_j(R) \ll R^{-1/2}\{R^2 \sqrt{T_0 N_j} + R N_j^{4/5} + N_j\} L^c$$
$$\ll \{R\sqrt{PN_j} + \sqrt{R}\, N_j^{4/5} + R^{-1/2} N_j\} L^c.$$

This yields the desired bound in Lemma 3.3(ii) provided (in addition to the condition $R \ll N^{1/10}$) that R is greater than a sufficiently large power of L.

It remains, therefore, to consider the situation when $R \ll L^{c_4}$ for any large constant c_4. In this case we shall obtain the bound

$$J_j(R) \ll \frac{N}{|a_j|} \exp\{-cL^{1/5}\}, \tag{5.10}$$

which is good enough.

We begin with the explicit formula (see [4, pp. 109 and 120])

$$\sum_{m \le u} \Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma| \le T} \frac{u^\varrho}{\varrho} + O\left\{ \left(\frac{u}{T} + 1 \right) \log^2(ruT) \right\}, \tag{5.11}$$

where $\varrho = \beta + i\gamma$ is a typical non-trivial zero of the function $L(s, \chi)$ and T is a parameter satisfying $2 \le T \le u$. Taking $T = M_j$ in (5.11) and then inserting it into $\widehat{W}_j(\chi, \lambda)$, we get

$$\widehat{W}_j(\chi, \lambda) = \int_{M_j}^{N_j} e(a_j u \lambda)\, d\Big\{ \sum_{n \le u} (\Lambda(m)\chi(m) - \delta_\chi) \Big\} \tag{5.12}$$
$$= - \sum_{|\gamma| \le T} \int_{M_j}^{N_j} u^{\varrho - 1} e(a_j u \lambda)\, du + O\{(1 + |\lambda| N) L^2\}.$$

The last integral is bounded in the same way as in (5.4) and we have

$$\begin{aligned}\int_{M_j}^{N_j} u^{\varrho-1}e(a_ju\lambda)\,du &= \int_{M_j}^{N_j} u^{\beta-1}e\biggl(\frac{\gamma}{2\pi}\log u + a_j\lambda u\biggr)\,du\\ &\ll \begin{cases} N_j^{\beta}/\sqrt{|\gamma|+1} & \text{if } |\gamma|\le T_0,\\ N_j^{\beta}/|\gamma| & \text{if } T_0<|\gamma|\le T,\end{cases}\end{aligned}$$

where, as before, $T_0 = 4\pi N/(RQ)$. Applying this to (5.12) we have

$$\begin{aligned}J_j(R) &\ll \sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\max_{|\lambda|\le 1/(rQ)}|\widehat{W}_j(\chi,\lambda)| \\ &\ll \sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\sum_{|\gamma|\le T_0}\frac{N_j^{\beta}}{\sqrt{|\gamma|+1}} \\ &\quad + \sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\sum_{T_0<|\gamma|\le N_j}\frac{N_j^{\beta}}{|\gamma|} + RPL^2 \\ &=: \sum_{r\sim R}H_1 + \sum_{r\sim R}H_2 + RPL^2,\end{aligned}\tag{5.13}$$

say. The last term above is clearly acceptable.

Vinogradov's zero-free region (see Satz VIII.6.2 in Prachar [13]) states that for any χ mod r, there exists a constant $c_5>0$ such that $L(\sigma+it,\chi)\ne 0$ in the region

$$\sigma \ge 1 - \frac{c_5}{\log r + \log^{4/5}(|t|+2)}$$

except for the possible Siegel zero. However, for $r\ll L^{c_4}$ the Siegel zero does not exist. It follows that $L(s,\chi)$ is zero-free for $\sigma\ge 1-\eta(\tau)$ and $|t|\le\tau$, where $\eta(\tau) = c_5/(2\log^{4/5}\tau)$. Consequently, the inner sum in H_2 is

$$\ll N_j^{1-\eta(N_j)}\sum_{|\gamma|\le N_j}\frac{1}{|\gamma|} \ll N_j\exp\biggl\{-\frac{c_5}{3}\log^{1/5}N_j\biggr\},$$

and

$$H_2 \ll N_j\exp\biggl\{-\frac{c_5}{4}\log^{1/5}N\biggr\}.$$

Recall that $R\ll L^{c_4}$, so the contribution of H_2 to $J_j(R)$ is acceptable.

Finally, we bound the remaining sum involving H_1, by using the zero-density theorem that

$$\sum_{\chi \bmod r} N(\chi,\tau) \ll (r\tau)^{3(1-\sigma)/(2-\sigma)}(\log r\tau)^9,$$

where $N(\chi,\tau)$ denotes the number of zeros $\varrho = \beta + i\gamma$ of $L(s,\chi)$ with $\sigma \le \beta \le 1$, $|\gamma| \le \tau$. Then

$$\begin{aligned} H_1 &\ll L \max_{Z \le T_0} \frac{1}{\sqrt{Z}} \sum_{\chi \bmod r} \sum_{|\gamma| \sim Z} N_j^\beta \\ &\ll L^{10} \max_{Z \le T_0} \frac{1}{\sqrt{Z}} \int_{1/2}^{1-\eta(Z)} N_j^\sigma (rZ)^{3(1-\sigma)/(2-\sigma)}\, d\sigma, \end{aligned}$$

by Stieltjes integration. The exponent of Z here is

$$\phi(\sigma) = \frac{3(1-\sigma)}{2-\sigma} - \frac{1}{2},$$

which is positive when $\sigma < 4/5$ and is negative when $\sigma > 4/5$. Thus, by dividing the above integral at the point $4/5$, we have

$$H_1 \ll L^c \int_{1/2}^{4/5} N_j^\sigma T_0^{\phi(\sigma)}\, d\sigma + L^c \int_{4/5}^{1-\eta(T_0)} N_j^\sigma\, d\sigma. \tag{5.14}$$

The second term here, by definition of T_0, is

$$\ll L^c N_j^{1-\eta(T_0)} \ll N_j \exp\left\{-\frac{c_5}{10}\log^{1/5} N_j\right\},$$

which is good enough. For the first integral in (5.14), since $T_0 \ll P \le N_j^{2/5}$, it is

$$\ll \int_{1/2}^{4/5} N_j^\sigma P^{\phi(\sigma)}\, d\sigma \le \int_{1/2}^{4/5} N_j^{\phi_1(\sigma)}\, d\sigma, \tag{5.15}$$

where

$$\phi_1(\sigma) = \sigma - \frac{1}{5} + \frac{6(1-\sigma)}{5(2-\sigma)}.$$

The maximum value of $\phi_1(\sigma)$ for $\sigma \in [1/2, 4/5]$ is $\phi_1(4/5) = 4/5$. This leads to the bound $N_j^{4/5}$ for the integral in (5.15). In view of (5.13) and the assumption that $R \ll L^{c_4}$, the desired bound (5.10) follows. This finishes the proof of Lemma 3.3. ∎

We now come to prove the bounds for K_j in Lemma 3.2.

Proof of Lemma 3.2. First, by Gallagher's lemma (see [5, Lemma 1]), we have

$$(5.16)\qquad \int\limits_{|\lambda|\le 1/(rQ)} |\widehat{W}_j(\chi,\lambda)|^2\,d\lambda \ll \left(\frac{1}{rQ}\right)^2 \int\limits_{-\infty}^{\infty} \Big| \sum_{\substack{v<|a_j|m\le v+rQ\\ M<|a_j|m\le N}} (\varLambda(m)\chi(m)-\delta_\chi)\Big|^2 dv$$
$$\ll \left(\frac{1}{rQ}\right)^2 \int\limits_{M-rQ}^{N} \Big| \sum_{\substack{v<|a_j|m\le v+rQ\\ M<|a_j|m\le N}} (\varLambda(m)\chi(m)-\delta_\chi)\Big|^2 dv.$$

Thus, in view of the definition (3.5),

$$(5.17)\qquad K_j(g,R) \ll \sqrt{N}\sum_{r\sim R} \frac{\sqrt{(\varpi,[g,r])}}{[g,r]}\,\frac{1}{rQ} \times \sideset{}{^*}\sum_{\chi \bmod r} \max_{I_r}\Big|\sum_{m\in I_r}(\varLambda(m)\chi(m)-\delta_\chi)\Big|,$$

where the maximum is over all intervals I_r lying in $[M_j,N_j]$ of length $\ll rQ|a_j|^{-1}$. For the sum $\sum_{m\in I_r}$ in (5.17), we apply Heath-Brown's identity in the same way as for J_j in (5.2)–(5.3). If we write $I_r=(Y,X]$ where $M_j\le Y<X\le N_j$, then as in (5.3),

$$\sum_{m\in I_r}\varLambda(m)\chi(m) = \sum_{\mathbf{D}}\sigma(u;\mathbf{D}) = \frac{1}{2\pi}\sum_{\mathbf{D}}\int\limits_{|t|\le N_j} F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\frac{X^{1/2+it}-Y^{1/2+it}}{1/2+it}\,dt+O(L^{12}).$$

The factor $(X^{1/2+it}-Y^{1/2+it})(1/2+it)^{-1}$ inside the integral is clearly $\ll\sqrt{N_j}/|t|$ for $|t|>T_0$ and is

$$\ll\Big|\int\limits_{Y}^{X}u^{-1/2+it}\,du\Big| \ll \frac{X-Y}{\sqrt{N_j}} \ll \frac{rQ}{\sqrt{N_j}|a_j|}$$

for $|t|\le T_0$. Here $T_0=N/(RQ)$ is the same as before. Therefore,

$$\max_{I_r}\Big|\sum_{m\in I_r}(\varLambda(m)\chi(m)-\delta_\chi)\Big| \ll \sum_{\mathbf{D}}\frac{rQ}{\sqrt{|a_j|N}}\int\limits_{|t|\le T_0}\Big|F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\Big|\,dt + \sum_{\mathbf{D}}\sqrt{N_j}\int\limits_{T_0<|t|\le N_j}\Big|F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\Big|\,\frac{dt}{|t|}+L^{12}.$$

Similarly to (5.9) and applying Lemma 3.4, we have

$$\begin{aligned}(5.18)\qquad &\sum_{\substack{r\sim R\\ d|r}}\ \sideset{}{^*}\sum_{\chi \bmod r}\max_{I_r}\Big|\sum_{m\in I_r}(\Lambda(m)\chi(m)-\delta_\chi)\Big|\\ &\ll \sum_{\mathbf{D}}\frac{RQ}{\sqrt{|a_j|N}}\sum_{\substack{r\sim R\\ d|r}}\ \sideset{}{^*}\sum_{\chi \bmod r}\int_{|t|\le T_0}\Big|F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\Big|\,dt\\ &\quad+\sum_{\mathbf{D}}\sqrt{N_j}\,L\max_{T_0<Z\le N_j}\frac{1}{Z}\sum_{\substack{r\sim R\\ d|r}}\ \sideset{}{^*}\sum_{\chi \bmod r}\int_{|t|\sim Z}\Big|F_{\mathbf{D}}\Big(\frac12+it,\chi\Big)\Big|\,dt+\frac{R^2}{d}\,L^{12}\\ &\ll L^{11}\sqrt{N_j}\bigg(\frac{R^2}{d}+\frac{R}{\sqrt{dT_0}}\,N_j^{3/10}+\frac{\sqrt{N_j}}{T_0}\bigg).\end{aligned}$$

To prove Lemma 3.2(i), we observe that

$$\begin{aligned}(5.19)\qquad \frac{\sqrt{(\varpi,[g,r])}}{[g,r]}&=\frac{(g,r)}{gr}\sqrt{\Big(\varpi,g\,\frac{r}{(g,r)}\Big)}\\ &\le\frac{\sqrt{(\varpi,g)}}{g}\,\frac{(g,r)}{r}\sqrt{\Big(\varpi,\frac{r}{(g,r)}\Big)}\end{aligned}$$

and hence, for any function $H(r)$,

$$(5.20)\qquad \sum_{r\sim R}\frac{\sqrt{(\varpi,[g,r])}}{[g,r]}\,|H(r)|\ll\frac{\sqrt{(\varpi,g)}}{gR}\sum_{\substack{h|g\\ k|\varpi}}h\sqrt{k}\sum_{\substack{r\sim R\\ hk|r}}|H(r)|.$$

Using this and (5.18) we deduce from (5.17) that

$$\begin{aligned}(5.21)\qquad &K_j(g,R)\\ &\ll\frac{\sqrt{(\varpi,g)}}{g}\,L^{11}\,\frac{\sqrt{NN_j}}{R^2Q}\sum_{\substack{h|g,\,k|\varpi\\ hk\le 2R}}h\sqrt{k}\bigg(\frac{R^2}{hk}+\frac{R}{\sqrt{hkT_0}}\,N_j^{3/10}+\frac{\sqrt{N_j}}{T_0}\bigg)\\ &\ll\frac{\sqrt{(\varpi,g)}}{g}\,L^c\tau(g)\tau(\varpi)\,\frac{P}{R^2\sqrt{|a_j|}}\bigg(R^2+\frac{R^{3/2}}{\sqrt{T_0}}\,N_j^{3/10}+R\,\frac{\sqrt{N_j}}{T_0}\bigg)\\ &\ll\frac{\sqrt{(\varpi,g)}}{g}\,L^c\tau(g)\tau(\varpi)\bigg(\frac{P}{\sqrt{|a_j|}}+\frac{\sqrt{P}}{\sqrt{|a_j|}}\,N_j^{3/10}+\frac{\sqrt{N}}{|a_j|}\bigg)\\ &\ll\frac{\sqrt{(\varpi,g)}}{g}\,L^c\tau(g)\tau(\varpi)\,\frac{\sqrt{N}}{|a_j|},\end{aligned}$$

on recalling that $T_0 = N/(RQ)$ and $P = (N/A)^{2/5} \le N_j^{2/5}$. This proves the bound in Lemma 3.2(i).

To prove the bound in (ii), it is sufficient to use, instead of (5.20), the cruder inequality

$$\begin{aligned}(5.22)\quad \sum_{r\sim R}\frac{\sqrt{(\varpi,[g,r])}}{[g,r]}|H(r)| &\le \frac{\sqrt{(\varpi,g)}}{g}\sum_{r\sim R}\frac{(g,r)}{r}\sqrt{\left(\varpi,\frac{r}{(g,r)}\right)}|H(r)|\\ &= \frac{\sqrt{(\varpi,g)}}{g}\sum_{r\sim R}\frac{\sqrt{(g,r)}\,|H(r)|}{\sqrt{r}}\\ &\ll \frac{\sqrt{(\varpi,g)}}{g\sqrt{R}}\sum_{\substack{d\mid g\\ d\le 2R}}\sqrt{d}\sum_{\substack{r\sim R\\ d\mid r}}|H(r)|\end{aligned}$$

for any function $H(r)$. So, parallel to the deduction of (5.21), we have

$$\begin{aligned}(5.23)\quad & K_j(g,R)\\ &\ll \frac{\sqrt{(\varpi,g)}}{g}L^{11}\sum_{\substack{d\mid g\\ d\le 2R}}\sqrt{d}\,\frac{\sqrt{N_jN}}{R^{3/2}Q}\left(\frac{R^2}{d}+\frac{R}{\sqrt{T_0 d}}N_j^{3/10}+\frac{\sqrt{N_j}}{T_0}\right)\\ &\ll \frac{\sqrt{(\varpi,g)}}{g}L^c\tau(g)\,\frac{\sqrt{N_jN}}{R^{3/2}Q}\left(R^2+\frac{R}{\sqrt{T_0}}N_j^{3/10}+\frac{\sqrt{RN_j}}{T_0}\right)\\ &\ll \frac{\sqrt{(\varpi,g)}}{g}L^c\tau(g)\left(\frac{P\sqrt{R}}{\sqrt{|a_j|}}+\frac{\sqrt{P}}{\sqrt{|a_j|}}N_j^{3/10}+\sqrt{\frac{N_j}{|a_j|}}\right).\end{aligned}$$

Since $P \le N_j^{2/5}$ and $R \ll N_j^{1/10}$, the above yields the bound in Lemma 3.2(ii). This completes our proof of Lemma 3.2. ∎

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