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Sums of some multiplicative functions over a special set of integers

by

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1. Introduction. The sum of multiplicative functions and the value distribution of multiplicative functions are two central and important problems in analytic number theory. Two typical examples are

$$\sum_{n\le x}\mu(n) \quad \text{and} \quad \sum_{\varphi(n)\le x} 1,$$

where $\mu(n)$ is the Möbius function and $\varphi(n)$ the Euler function. The first one is equivalent to the prime number theorem and the second was studied by Erdős & Turán [5], Bateman [3] and Balazard & Tenenbaum [2]. Another interesting aspect is the study of the value distribution restricted to a certain set of integers, for example, the set of squarefree integers [10] and the set of integers free of large prime factors [11]. It then seems natural to investigate a general sum

$$F_g(x) := \sum_{g(n)\le x} f(n),$$

where $f(n)$ and $g(n)$ are two multiplicative functions. Problems of similar fashion have been considered by other authors ([1], [7], [9] and [10]). Generalizing a result of Abbott & Subbarao, Balasubramanian & Ramachandra [1] studied $\sum_{ng(n)\le x} 1$ where g is multiplicative, $g(p)$ equals a fixed positive constant for all primes p and $g(n) \gg n^{-1/16}$ for all integers $n \ge 1$. By using tools of complex analysis, they established an asymptotic formula for this sum and answered a question of Erdős whether

$$\sum_{n/\tau(n)\le 2x} 1 \sim 2 \sum_{n/\tau(n)\le x} 1,$$

where $\tau(n)$ is the divisor function. Using elementary methods, Smati [9]

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showed that there is a positive constant c such that

$$\sum_{g(n)\le x} 1 = A(g)x + O(x\exp\{-c\sqrt{\log x\log_2 x}\}), \tag{1.1}$$

where $A(g) := \prod_p (1-p^{-1})\sum_{\nu=0}^{\infty} g(p^\nu)^{-1}$, g is a multiplicative function and (i) $g(p^\nu)$ is a polynomial of degree ν in p with leading coefficient 1 and all other coefficients in $[-1, B]$; (ii) $g(n) \gg n/(\log_2 n)^H$ for all integers $n \ge 3$ ($B \ge -1$ and $H > 0$ are constants).

In this paper, we are concerned with the general sum $F_g(x)$ where both f and g are multiplicative and satisfy some conditions given below. Based on the Selberg–Delange method, we obtain an asymptotic formula for $F_g(x)$ with a very good error term. The error term is sharpest subject to the present techniques. Any improvement will lead to a better error term in the prime number theorem. In order to achieve this error term, we need to borrow the method of Balazard & Tenenbaum [2]. Our result includes a wide class of multiplicative functions. Moreover, we can derive some new results related to local densities.

Let κ, θ, θ', θ'', $\widetilde{\theta}$, α, α', η, ψ, C_1, C_2, C_3 be given constants such that

$$\begin{cases} |\kappa| < 1/\eta, \quad \theta > 0, \quad \widetilde{\theta} > \theta > \theta' > \theta'', \quad \alpha > 0, \quad \alpha' \ne 0, \\ \eta > 0, \quad \psi > 1, \quad C_1 \ge 0, \quad C_2 \ge 0, \quad C_3 > 0. \end{cases} \tag{1.2}$$

Suppose that $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to (0,\infty)$ are two multiplicative functions such that for all prime numbers p:

(1) $|f(p) - \kappa| \le C_1/p^\eta$;
(2) $g(p) = \alpha p^\theta$ or $g(p) = \alpha p^\theta + \alpha' p^{\theta'} + t(p)$,
where $|t^{(l)}(u)| \le (C_2 l + 1)^l u^{\theta''-l}$ $(l = 0, 1, \ldots)$;
(3) $\sum_{\nu=2}^{\infty} |f(p^\nu)|/g(p^\nu)^{1/\widetilde{\theta}} \le C_3/p^\psi$.

As usual let $\tau_\kappa(n)$ be the Piltz function, defined by

$$\zeta(s)^\kappa = \sum_{n=1}^{\infty} \tau_\kappa(n)/n^s,$$

where $\zeta(s)$ is the Riemann function. In particular $\tau_2(n) = \tau(n)$ is the usual divisor function. Define

$$\mathbf{1}(n) \equiv 1, \quad j(n) := n, \quad \sigma(n) := \sum_{d\mid n} d, \quad \Omega(n) := \sum_{p^\nu \| n} \nu, \quad \omega(n) := \sum_{p^\nu\| n} 1.$$

It is easy to verify that the following function pairs:

$$(\mu, j), \quad (\mathbf{1}, \varphi), \quad (\mu, \varphi), \quad (\mathbf{1}, j/\tau), \quad (\mu^2, \sigma), \quad (\tau_\kappa, j), \quad (z^\Omega, j), \quad (z^\omega, \varphi)$$

satisfy our assumptions on (f, g) for suitable parameters.

For the Selberg–Delange method, we introduce the associated Dirichlet series

$$\mathcal{F}_g(s) := \sum_{n=1}^{\infty} \frac{f(n)}{g(n)^s}, \qquad \widetilde{\mathcal{F}}_g(s) := \frac{\mathcal{F}_g(s)}{\zeta(\theta s)^{\kappa/\alpha^s}}.$$

Under our assumptions (1)–(3), it is not difficult to show that there is a positive constant ϱ_0 such that $\widetilde{\mathcal{F}}_g(s)$ is uniformly convergent on any compact set in the half-plane $\sigma \geq 1/\theta - 10\varrho_0$ (see Theorem 4(i)). Thus we can write, for $|s - 1/\theta| < 10\varrho_0$,

$$(1.3)\qquad s^{-1}\widetilde{\mathcal{F}}_g(s)(\zeta(\theta s)(\theta s - 1))^{\kappa/\alpha^s} = \sum_{l=0}^{\infty} a_l (s - 1/\theta)^l = \sum_{l=0}^{\infty} \frac{a_l}{\theta^l}(\theta s - 1)^l,$$

where the coefficient $a_l = a_l(f, g)$ is given by

$$(1.4)\qquad a_l := \frac{1}{2\pi i} \int_{|s-1/\theta|=\varrho_0} \frac{\widetilde{\mathcal{F}}_g(s)(\zeta(s)(s-1))^{\kappa/\alpha^s}}{s(s-1/\theta)^{l+1}}\, ds \ll \frac{1}{\varrho_0^l}.$$

In particular

$$(1.5)\qquad a_0 = \prod_p (1 - 1/p)^{\kappa/\alpha^{1/\theta}} \sum_{\nu=0}^{\infty} f(p^\nu)/g(p^\nu)^{1/\theta}.$$

Let

$$b_{m,n} := \sum_{n_1+\ldots+n_m=n} \frac{1}{(n_1+1)!\ldots(n_m+1)!} \qquad (m \geq 1),$$

$$b_{0,n} := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$$

As usual we use $\Gamma(s)$ to denote the Euler Γ-function and define

$$\frac{1}{\Gamma_i(a)} := \left[\frac{d^i}{dz^i}\left(\frac{1}{\Gamma(z)}\right)\right]_{z=a}.$$

Put $\mathcal{L}(x) := \exp\{(\log x)^{3/5}/(\log_2 x)^{1/5}\}$, where $\log_k$ is the k-fold iterated logarithm. Let c_i and ϱ_i be constants depending at most on θ, θ', θ'', $\widetilde{\theta}$, α, α', η, ψ, ψ_1, ψ_2, C_1, C_2, C_3, C_4 (see Theorem 2 below). The ϱ_i are small enough to satisfy the conditions made precise later. The constants implied in $\ll$ or O depend at most on these parameters. Finally we define $0^0 = 1$.

THEOREM 1. *Let f, g satisfy the assumptions* (1)–(3). *Then for any integer $J \geq 0$, we have*

$$(1.6)\qquad F_g(x) = \frac{x^{1/\theta}}{(\log x)^{1-\kappa/\alpha^{1/\theta}}} \left\{ \sum_{j=0}^{J} \frac{P_j(\log_2 x)}{(\log x)^j} + O(R_{J,\lambda}(x)) \right\},$$

where $P_j(t) := \sum_{l=0}^{j} \lambda_{j,l} t^l$ *and the coefficient* $\lambda_{j,l} = \lambda_{j,l}(f,g)$ *is given by*

$$\lambda_{j,l} := \frac{\theta^{-\kappa/\alpha^{1/\theta}}}{l!} \sum_{m=l}^{j} \sum_{k=l}^{m} \sum_{i=0}^{k-l} \lambda^*_{m,k,i}, \tag{1.7}$$

where

$$\lambda^*_{m,k,i} := \frac{(-\log\alpha)^m (\kappa/\alpha^{1/\theta})^k (-\log\theta)^{k-l-i} a_{j-m} b_{k,m-k}}{(k-l-i)!\,i!\,\Gamma_i(\kappa/\alpha^{1/\theta} - j)}.$$

The error term $R_{J,\lambda}(x)$ *is defined by*

$$R_{J,\lambda}(x) := \left((c_1 J + 1)\frac{\lambda \log_2 x + c_2}{\log x}\right)^{J+1} + \frac{1}{\mathcal{L}(x)^{c_3}} \tag{1.8}$$

with $\lambda := \varrho_0 \alpha^{-1/\theta} |\kappa \log\alpha| e^{\varrho_0 |\log\alpha|} < 1$. *Moreover,*

$$\lambda_{j,l} \ll \lambda^l (c_4 j + 1)^j / l!. \tag{1.9}$$

Remarks. (i) The dominating term in the sum (1.6) is given by

$$\frac{a_0}{\theta^{\kappa/\alpha^{1/\theta}} \Gamma(\kappa/\alpha^{1/\theta})}$$

if $a_0/\theta^{\kappa/\alpha^{1/\theta}} \Gamma(\kappa/\alpha^{1/\theta}) \neq 0$. This is true when $\sum_{\nu=0}^{\infty} f(p^\nu)/g(p^\nu)^{1/\theta} \neq 0$ for all prime numbers p (see (1.5)) and $\kappa/\alpha^{1/\theta} \neq 0, -1, -2, \ldots$

(ii) If $\alpha = 1$, then $\lambda = 0$ and $\lambda_{j,l} = 0$ for $l = 1, \ldots, j$. Thus the asymptotic formula (1.6) in Theorem 1 is simplified to

$$F_g(x) = \frac{x^{1/\theta}}{(\log x)^{1-\kappa}} \left\{ \sum_{j=0}^{J} \frac{\lambda_{j,0}}{(\log x)^j} + O(R_J(x)) \right\}, \tag{1.10}$$

where $\lambda_{j,0} = a_j/\theta^\kappa \Gamma(\kappa - j)$ and

$$R_J(x) := \left(\frac{J+1}{c_5 \log x}\right)^{J+1} + \frac{1}{\mathcal{L}(x)^{c_3}}. \tag{1.11}$$

(iii) If $\alpha = 1$ and $\kappa = J_0 \in \mathbb{Z}$, then $\lambda = 0$, $\lambda_{j,l} = 0$ for $l = 1, \ldots, j$ and $\lambda_{j,0} = 0$ for $j \geq J_0$. Taking $J = (\log x)^{3/5}/(\log_2 x)^{6/5}$ in (1.10), we obtain

$$F_g(x) = \frac{x^{1/\theta}}{(\log x)^{1-J_0}} \left\{ \sum_{j=0}^{J_0-1} \frac{\lambda_{j,0}}{(\log x)^j} + O(\mathcal{L}(x)^{-c_6}) \right\}. \tag{1.12}$$

In particular if $J_0 \leq 0$, then

$$F_g(x) \ll x^{1/\theta}/\mathcal{L}(x)^{c_7}. \tag{1.13}$$

The next result comes from particular cases of (1.12) and (1.13). The first assertion is due to Balazard & Tenenbaum ([2], théorème 1), the second one is equivalent to the prime number theorem and the third one is new.

COROLLARY 1. *There is a constant $c > 0$ such that*

(i) $\sum_{\varphi(n)\le x} 1 = \{\zeta(2)\zeta(3)/\zeta(6)\}x + O(x/\mathcal{L}(x)^c)$;
(ii) $\sum_{n\le x} \mu(n) \ll x/\mathcal{L}(x)^c$;
(iii) $\sum_{\varphi(n)\le x} \mu(n) \ll x/\mathcal{L}(x)^c$.

Clearly Theorem 1 also improves Smati's (1.1), Corollaries 1–4 of Scourfield in [7], contains Selberg's asymptotic formula for $\sum_{n\le x} \tau_\kappa(n)$ [8] and generalizes the main result of Balasubramanian & Ramachandra [1].

From Theorem 1, we can derive some local density results.

THEOREM 2. *Let f and g satisfy the assumptions* (1), (2) *and furthermore assume*

(3)′ $|f(p^\nu)| \le C_4 g(p^\nu)^{1/\widetilde{\theta}}(\psi_1/p)^{\psi_2\nu}$ *for all primes p and all integers $\nu \ge 2$, where $\widetilde{\theta} > \theta$, $\psi_1 > 1/2$ and $\psi_2 > 1/2$ are given constants.*

Let λ and $R_{J,\lambda}(x)$ be defined as in Theorem 1. *Then for any integer $J \ge 0$ and any $\varepsilon > 0$, we have, uniformly for $x \ge 3$ and $1 \le k \le ((2-\varepsilon)/\psi_1)^{\psi_2}\log_2 x$,*

$$\sum_{\substack{g(n)\le x\\ \Omega(n)=k}} f(n) = \frac{x^{1/\theta}}{\log x}\bigg\{\sum_{j=0}^{J}\frac{Q_{j,k}(\log_2 x)}{(\log x)^j} + O\bigg(\bigg(\frac{\log_2 x}{k}\bigg)^k \frac{e^{k|\kappa|\alpha^{-1/\theta}}}{\sqrt{|\varrho|k+1}}R_{J,\lambda}(x)\bigg)\bigg\},$$

where $Q_{j,k}(t) := \sum_{n=0}^{j+k-1}\lambda_{j,n,k}t^n$ and the coefficient $\lambda_{j,n,k}$ is given by

$$\lambda_{j,n,k} := \sum_{\substack{l+m=n\\ 0\le l\le j,\, 0\le m\le k}} \frac{(\kappa\alpha^{-1/\theta})^m}{2\pi i m!}\oint_{|z|=1}\frac{\lambda_{j,l}(fz^\Omega, g)}{z^{k+1-m}}\,dz$$

and $\lambda_{j,l}(fz^\Omega, g)$ is defined by (1.7).

THEOREM 3. *Let f, g, λ and $R_{J,\lambda}(x)$ be defined as in Theorem* 1. *Then for any $A > 0$ and any integer $J \ge 0$, we have, uniformly for $x \ge 3$ and $1 \le k \le A\log_2 x$,*

$$\sum_{\substack{g(n)\le x\\ \omega(n)=k}} f(n) = \frac{x^{1/\theta}}{\log x}\bigg\{\sum_{j=0}^{J}\frac{\widetilde{Q}_{j,k}(\log_2 x)}{(\log x)^j} + O_A\bigg(\bigg(\frac{\log_2 x}{k}\bigg)^k \frac{e^{k|\kappa|\alpha^{-1/\theta}}}{\sqrt{|\kappa|k+1}}R_{J,\lambda}(x)\bigg)\bigg\},$$

where $\widetilde{Q}_{j,k}(t) := \sum_{n=0}^{j+k-1} \widetilde{\lambda}_{j,n,k} t^n$ *and the coefficient* $\widetilde{\lambda}_{j,n,k}$ *is given by*

$$\widetilde{\lambda}_{j,n,k} := \sum_{\substack{l+m=n \\ 0\le l\le j,\, 0\le m\le k}} \frac{(\kappa\alpha^{-1/\theta})^m}{2\pi i m!} \oint_{|z|=1} \frac{\lambda_{j,l}(fz^{\omega}, g)}{z^{k+1-m}}\, dz$$

and $\lambda_{j,l}(fz^{\omega}, g)$ *is defined by* (1.7).

Clearly Theorems 2 and 3 contain Selberg's classical results [8] on

$$N_k(x) := |\{n \le x : \Omega(n) = k\}| \quad \text{and} \quad \pi_k(x) := |\{n \le x : \omega(n) = k\}|.$$

Here we state some consequences, which are new.

COROLLARY 2. *For any* $\delta > 0$, *we have*, *uniformly for* $J \ge 0$, $x \ge 3$ *and* $1 \le k \le (2-\delta)\log_2 x$,

$$\sum_{\substack{\varphi(n)\le x \\ \Omega(n)=k}} 1 = \frac{x}{\log x}\left\{ \sum_{j=0}^{J} \frac{W_{j,k}(\log_2 x)}{(\log x)^j} + O_\delta\left(\frac{(\log_2 x)^k}{k!} R_J(x) \right) \right\}, \tag{1.14}$$

where

$$W_{j,k}(t) := \sum_{n=0}^{k-1} \frac{\varsigma_j^{(k-1-n)}(0)}{n!(k-1-n)!} t^n, \qquad \varsigma_j(z) := \frac{a_j(z^{\Omega}, \varphi)}{z\Gamma(z-j)}$$

and $a_j(z^{\Omega}, \varphi)$ *is defined by* (1.4). *Moreover*, *under the same conditions*,

$$\sum_{\substack{\varphi(n)\le x \\ \Omega(n)=k}} 1 = \frac{x}{\log x} \cdot \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \varsigma_0\left(\frac{k-1}{\log_2 x}\right) + O_\delta\left(\frac{k}{(\log_2 x)^2}\right) \right\}, \tag{1.15}$$

where

$$\varsigma_0(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{pz}{(p-1)(p-z)}\right)\left(1 - \frac{1}{p}\right)^z.$$

COROLLARY 3. *For any* $A > 0$, *we have*, *uniformly for* $J \ge 0$, $x \ge 3$ *and* $1 \le k \le A\log_2 x$,

$$\sum_{\substack{\varphi(n)\le x \\ \omega(n)=k}} 1 = \frac{x}{\log x}\left\{ \sum_{j=0}^{J} \frac{\widetilde{W}_{j,k}(\log_2 x)}{(\log x)^j} + O_A\left(\frac{(\log_2 x)^k}{k!} R_J(x) \right) \right\}, \tag{1.16}$$

where

$$\widetilde{W}_{j,k}(t) := \sum_{n=0}^{k-1} \frac{\widetilde{\varsigma}_j^{(k-1-n)}(0)}{n!(k-1-n)!} t^n, \qquad \widetilde{\varsigma}_j(z) := \frac{a_j(z^{\omega}, \varphi)}{z\Gamma(z-j)}$$

and $a_j(z^\omega, \varphi)$ *is defined by* (1.4). *Moreover*, *under the same conditions*,

$$(1.17)\qquad \sum_{\substack{\varphi(n)\le x\\ \omega(n)=k}} 1 = \frac{x}{\log x}\cdot\frac{(\log_2 x)^{k-1}}{(k-1)!}\left\{\widetilde{\varsigma}_0\left(\frac{k-1}{\log_2 x}\right) + O_A\left(\frac{k}{(\log_2 x)^2}\right)\right\},$$

where

$$\widetilde{\varsigma}_0(z) = \frac{1}{\Gamma(z+1)}\prod_p\left(1+\frac{pz}{(p-1)^2}\right)\left(1-\frac{1}{p}\right)^z.$$

In what follows the letter s always denotes a complex number and we implicitly define the real numbers σ and τ by the relation $s = \sigma + i\tau$. We let $\beta(t) := (\log t)^{-2/3}(\log_2 t)^{-1/3}$ for $t \ge 3$ and $T := |\tau| + 3$. The next result is a generalization of Theorem 2 in [2], which plays a key role in the proof of Theorem 1.

Theorem 4. *Under the above conditions*, *there is a positive constant* ϱ_0 *such that*

(i) $\widetilde{\mathcal{F}}_g(s)$ *is uniformly convergent on any compact set in the half-plane* $\sigma \ge 1/\theta - 10\varrho_0$,

(ii) $|\widetilde{\mathcal{F}}_g(s)| \ll (\log^2 T \log_2 T)^{\frac{2}{3}|\kappa|\alpha^{-\sigma}}$ *for* $\sigma \ge 1/\theta - 10\varrho_0\beta(T)$.

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2. Some preparations. This section is devoted to proving some preliminary lemmas. The following result is a variant of Lemma 1 of [2].

Lemma 1. *Let* $0 < b_0 < 1 < b_1$, $\delta > 0$ *be fixed and* $\delta_0 := \min\{\sqrt{\delta}/4, 1/10\}$. *Let* $P \in \mathbb{Z}$, $Q \in \mathbb{N}$ *and* $D \ge Q^\delta$. *Suppose that* $h \in \mathcal{C}^\infty([P, P+Q], \mathbb{R})$ *and for all* $P \le u \le P+Q$ *and* $1 \le l \le \log D/(8\delta_0^2 \log Q) + 1$,

$$(2.1)\qquad D/(b_1 Q)^l \le |h^{(l)}(u)|/l! \le D/(b_0 Q)^l.$$

Then there exist three positive constants $A = A(\delta)$, $c = c(\delta)$ *and* $Q_0 = \max\{b_0^{-1/\delta_0^2}, b_1^{1/\delta_0}\}$ *such that*

$$\max_{Q_1\le Q}\Big|\sum_{P<n\le P+Q_1} e(h(n))\Big| \le AQ\exp\{-c(\log Q)^3/(\log D)^2\}\qquad (Q \ge Q_0),$$

where $e(t) = e^{2\pi i t}$ $(t \in \mathbb{R})$.

Proof. For $Q \geq Q_0$, we choose $K \in \mathbb{N}$ such that $Q^{8\delta_0^2(K-1)} \leq D < Q^{8\delta_0^2 K}$. Then

$$2 \leq \log D/(8\delta_0^2 \log Q) < K \leq \log D/(8\delta_0^2 \log Q) + 1,$$
$$D/(b_0 Q)^K \leq Q^{-(1-9\delta_0^2)K}/(b_0 Q^{\delta_0^2})^K \leq Q^{-(1-9\delta_0^2)K}.$$

Similarly for $\delta_0 K \leq l \leq 2\delta_0 K$, we have

$$D/(b_0 Q)^l \leq Q^{8\delta_0^2 K}/(b_0 Q)^l \leq Q^{-(1-9\delta_0)l}/(b_0 Q^{\delta_0})^l \leq Q^{-(1-9\delta_0)l},$$
$$D/(b_1 Q)^l \geq Q^{8\delta_0^2(K-1)}/(b_1 Q)^l \geq Q^{4\delta_0^2 K}/(b_1 Q)^l$$
$$\geq Q^{-(1-\delta_0)l}(Q^{\delta_0}/b_1)^l \geq Q^{-(1-\delta_0)l}.$$

Our result follows immediately by Lemma 0 in [2]. ∎

Let $\Lambda(n)$ be the von Mangoldt function and $g_0(u) := \alpha u^\theta + \alpha' u^{\theta'} + t(u)$. Define

$$S_M(\tau) := \sup_{M<N\leq 2M} \Big| \sum_{M<n\leq N} \Lambda(n) g_0(n)^{-i\tau} \Big|.$$

The next two lemmas are generalizations of Lemmas 2 and 3 in [2], which play a key role in the proof of Theorem 4.

Lemma 2. *There exists a positive constant ϱ_1 such that*

$$S_M(\tau) \ll M^{1-\varrho_1 \beta(T)} + MT^{-1} + M\mathcal{L}(M)^{-\varrho_1}$$

for $T := |\tau| + 3 \leq M^{\theta_1(1+\varrho_1\beta(M))}$, where $\theta_1 := \theta - \theta'$.

Proof. Define

$$V_\tau(z) := \sum_{n\leq z} \frac{\Lambda(n)}{n^{i\tau}}, \qquad w_\tau(z) := \frac{z^{i\theta\tau}}{g_0(z)^{i\tau} \log z}.$$

It is easy to see that

$$\frac{dw_\tau(z)}{dz} \ll \frac{z^{-\theta_1}T + 1}{z \log z}.$$

Thus we can deduce that for $M \leq N \leq 2M$,

$$\begin{aligned}\sum_{M<p\leq N} g_0(p)^{-i\tau} &= \sum_{M<n\leq N} w_\tau(n) \frac{\Lambda(n)}{n^{i\theta\tau}} + O(1) \\ &= \int_M^N w_\tau(z)\, dV_{\theta\tau}(z) + O(1) \\ &\ll \sup_{M\leq z\leq N} |V_{\theta\tau}(z)| \bigg(\frac{1}{\log M} + \int_M^N \frac{z^{-\theta_1}T + 1}{z \log z}\, dz \bigg) + 1 \\ &\ll \sup_{M\leq z\leq N} |V_{\theta\tau}(z)| (M^{-\theta_1}T + 1) + 1.\end{aligned}$$

According to (12) in [2], we have $V_\tau(z) \ll z^{1-c\beta(T)} + zT^{-1} + z\mathcal{L}(z)^{-c}$ for $z \geq 3$ and $\log T \leq (\log z)^{3/2}/(\log_2 z)^2$, where $c > 0$ is an absolute constant. Hence we deduce that for some suitable constant $c_8 > 0$,

$$\sum_{M<p\leq N} g_0(p)^{-i\tau} \ll (M^{-\theta_1}T+1)(M^{1-c_8\beta(T)} + M/T + M/\mathcal{L}(M)^{c_8})$$
$$\ll M^{1-c_8\beta(T)+\theta_1\varrho_1\beta(M)} + M/T + M^{1+\theta_1\varrho_1\beta(M)}/\mathcal{L}(M)^{c_8},$$

which implies the desired result provided $\varrho_1 > 0$ is suitably small. ■

LEMMA 3. *There exist two positive constants ϱ_2 and ϱ_3 such that*

$$S_M(\tau) \ll M\{e^{-\varrho_2(\log M)^3/(\log T)^2} + (M^{\theta_1}/T)^{1/2}(\log M)^{7/2}\}$$

for $3 \leq M^{\theta_1} \leq T := |\tau| + 3 \leq \exp\{\varrho_3(\log M)^2\}$, *where* $\theta_1 := \theta - \theta'$.

Proof. Clearly the assertion is trivial if $T \leq 10$. Thus we can suppose that $T > 10$ and $T \asymp |\tau|$. We define $\theta_2 := \theta - \theta''$, $\theta_0 := \min\{3\theta_1/2, \theta_2\}$ and

$$\delta := \min\left\{\frac{\theta_0-\theta_1}{2(3+\theta_0+\theta_1)}, \frac{1}{12(1+\theta_1)}\right\} > 0. \tag{2.2}$$

Applying Vaughan's identity ([4], (24.6)) with $U = V = M^{1/2-\delta}$, for $M \leq N \leq 2M$ we have

$$\sum_{n\leq N} \Lambda(n) g_0(n)^{-i\tau} \ll M^{1/2-\delta} + S_N'(\tau)\log M + \{S_N''(\tau)M\}^{1/2}(\log M)^3, \tag{2.3}$$

where

$$S_N'(\tau) := \sum_{n\leq M^{1-2\delta}} \max_w \Big|\sum_{w\leq r\leq N/n} g_0(nr)^{-i\tau}\Big|,$$

$$S_N''(\tau) := \max_{\substack{M^{1/2-\delta}\leq Q\leq N/M^{1/2-\delta}\\ M^{1/2-\delta}<j\leq N/Q}} \sum_{M^{1/2-\delta}<k\leq N/Q} \Big|\sum_{\substack{Q<m\leq 2Q\\ m\leq N/\max\{j,k\}}} \Big(\frac{g_0(km)}{g_0(jm)}\Big)^{i\tau}\Big|.$$

In order to bound the sum $S_N'(\tau)$, we first observe that

$$S_N'(\tau) \leq M^{1-\delta} + \sum_{n\leq M^{1-2\delta}} \max_{M^\delta\leq w\leq N/n} \sum_{\nu\geq 0} \Big|\sum_{\max\{w,N/(2^{\nu+1}n)\}\leq r\leq N/(2^\nu n)} g_0(nr)^{-i\tau}\Big|. \tag{2.4}$$

Thus it is sufficient to bound the sum $\sum_{U<r\leq U'} e(h(r))$, where $h(u) := -(\tau/(2\pi))\log g_0(nu)$, $M^\delta \leq U \leq \min\{N/n, M\}$ and $U + M^\delta \leq U' \leq 2U$. This will be done by applying Lemma 1. Let $v(u) := \alpha' u^{\theta'} + t(u)$. We expand

$h(u)$ in a series form

$$h(u) = -\frac{\tau}{2\pi}\left\{\log\alpha + \theta\log(nu) - \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu\alpha^\nu}\left(\frac{v(nu)}{(nu)^\theta}\right)^\nu\right\}. \tag{2.5}$$

By the assumption (2), we easily show that $|v^{(l)}(u)| \le (c_9 l + 1)^l u^{\theta'-l}$. Note that

$$\frac{d^l}{du^l}\phi(nu) = n^l\phi^{(l)}(nu), \qquad \frac{d^l}{du^l}\prod_{i=1}^{\nu}\phi_i(u) = \sum_{l_1+\ldots+l_\nu=l}\frac{l!}{l_1!\ldots l_\nu!}\prod_{i=1}^{\nu}\phi_i^{(l_i)}(u).$$

Then for $l \ge 1$ we have

$$\begin{aligned}
&\left|\frac{d^l}{du^l}\left(\frac{v(nu)}{(nu)^\theta}\right)^\nu\right| \\
&= n^l\left|\sum_{l_1+\ldots+l_{\nu+1}=l}\frac{(-1)^{l_{\nu+1}}l!(\theta\nu)\ldots(\theta\nu+l_{\nu+1}-1)}{l_1!\ldots l_{\nu+1}!(nu)^{\theta\nu+l_{\nu+1}}}\prod_{i=1}^{\nu}v^{(l_i)}(nu)\right| \\
&\le \frac{l!}{(nu)^{\theta_1\nu}u^l}\sum_{l_1+\ldots+l_{\nu+1}=l}\frac{(\theta\nu)\ldots(\theta\nu+l_{\nu+1}-1)}{l_1!\ldots l_{\nu+1}!}\prod_{i=1}^{\nu}(c_9 l_i+1)^{l_i} \\
&\le \frac{l!}{(nu)^{\theta_1\nu}u^l}\sum_{l_1+\ldots+l_{\nu+1}=l}\frac{(c_{10}l)^{l_1+\ldots+l_\nu}(\theta\nu+l)^{l_{\nu+1}}}{l_1!\ldots l_{\nu+1}!} \\
&\le (c_{11}\nu l)^l(nu)^{-\theta_1\nu}u^{-l}.
\end{aligned} \tag{2.6}$$

Hence if $1 \le l \le c_{12}\log U$ and $U \le u \le 2U$, we deduce that

$$\begin{aligned}
\frac{d^l}{du^l}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu\alpha^\nu}\left(\frac{v(nu)}{(nu)^\theta}\right)^\nu &\le \left(\frac{c_{11}l}{u}\right)^l\sum_{\nu=1}^{\infty}\frac{\nu^{l-1}}{(\alpha(nu)^{\theta_1})^\nu} \\
&\le \left(\frac{c_{13}l}{u}\right)^l\sum_{\nu=1}^{\infty}\frac{1}{(\alpha(nu)^{\theta_1})^{3\nu/4}} \\
&\ll \left(\frac{c_{14}l}{u}\right)^l\frac{1}{(nu)^{3\theta_1/4}}.
\end{aligned} \tag{2.7}$$

Inserting this into (2.5) and using the Stirling formula yield, for $1 \le l \le c_{12}\log U$ and $U \le u \le 2U$,

$$\frac{T}{(b_1U)^l} \le \frac{1}{l!}|h^{(l)}(u)| = \frac{\theta|\tau|}{2\pi l u^l} + O\left(\left(\frac{c_{15}}{u}\right)^l\frac{|\tau|}{u^{\theta_1/2}}\right) \le \frac{T}{(b_0U)^l},$$

where $b_i = b_i(\alpha,\alpha',\theta,\theta',\theta'',C)$ are constants satisfying $0 < b_0 < 1 < b_1$.

Under our assumptions on M, T, U, U', it is easy to see that the condition of Lemma 1 is satisfied with $\delta = \theta_1$, $D = T$, $P = U$, $Q = U' - U$, $b_0 = b_0$

and $b_1 = b_1$. Thus Lemma 1 implies

$$\sum_{U<r\le U'} e(h(r)) \ll Ue^{-c_{16}(\log M)^3/(\log T)^2}.$$

Inserting this estimate into (2.4), we obtain

$$S'_N(\tau) \ll M^{1-\delta} + Me^{-c_{16}(\log M)^3/(\log T)^2}\log M \ll Me^{-c_{17}(\log M)^3/(\log T)^2}. \tag{2.8}$$

In view of $S''_N(\tau)$, it suffices to deal with the sum

$$S''_N(\tau,j,Q) := \sum_{M^{1/2-\delta}<k\le N/Q} \Bigg| \sum_{\substack{Q<m\le 2Q\\ m\le N/\max\{j,k\}}} \left(\frac{g_0(km)}{g_0(jm)}\right)^{i\tau}\Bigg|,$$

where $M^{1/2-\delta} \le Q \le N/M^{1/2-\delta}$ and $M^{1/2-\delta} < j \le N/Q$.

The contribution of the sum over k such that $|j-k| \le M^{1/2-2\delta}$ is $\ll M^{1/2-2\delta}Q \ll M^{1-\delta}$ and the sum over k with $N/\max\{j,k\} < Q+Q^{1/2}$ is $\ll NQ^{-1/2} \ll M^{3/4+\delta/2} \ll M^{1-\delta}$. Therefore we have, for $N/\max\{j,k\} \ge Q+Q^{1/2}$,

$$S''_N(\tau,j,Q) \ll M^{1-\delta} + \sum_{\substack{M^{1/2-\delta}<k\le N/Q\\ |k-j|>M^{1/2-2\delta}}} \Bigg| \sum_{\substack{Q<m\le 2Q\\ m\le N/\max\{j,k\}}} \left(\frac{g_0(km)}{g_0(jm)}\right)^{i\tau}\Bigg|. \tag{2.9}$$

Define $h_{j,k}(u) := (\tau/(2\pi))\log(g_0(ku)/g_0(ju))$. Similarly to (2.5), we write

$$h_{j,k}(u) = \frac{\tau}{2\pi}\bigg\{\theta\log\frac{k}{j} + \frac{v(ku)}{\alpha(ku)^\theta} - \frac{v(ju)}{\alpha(ju)^\theta} - \sum_{\nu=2}^{\infty}\frac{(-1)^\nu}{\nu\alpha^\nu}\bigg[\left(\frac{v(ku)}{(ku)^\theta}\right)^\nu - \left(\frac{v(ju)}{(ju)^\theta}\right)^\nu\bigg]\bigg\}.$$

Recall that $v(u) = \alpha' u^{\theta'} + t(u)$. By (2.6) with $t(nu)$ in place of $v(nu)$ and $\nu = 1$, we have

$$\begin{aligned}\frac{d^l}{du^l}&\left(\frac{v(ku)}{(ku)^\theta} - \frac{v(ju)}{(ju)^\theta}\right)\\ &= \alpha'\prod_{i=1}^{l}(1-\theta_1-i)(k^{-\theta_1}-j^{-\theta_1})u^{-\theta_1-l} + \frac{d^l}{du^l}\left(\frac{t(ku)}{(ku)^\theta} - \frac{t(ju)}{(ju)^\theta}\right)\\ &= \alpha'\prod_{i=1}^{l}(1-\theta_1-i)(k^{-\theta_1}-j^{-\theta_1})u^{-\theta_1-l} + O\left(\frac{(c_{11}l)^l u^{-\theta_2-l}}{\min\{k,j\}^{\theta_2}}\right).\end{aligned}$$

Similarly to (2.7), we can deduce that for $1 \le l \le c_{18} \log Q$,

$$\begin{aligned}\frac{h_{j,k}^{(l)}(u)}{l!} &= \frac{\tau}{2\pi\alpha l!}\bigg\{\frac{d^l}{du^l}\bigg(\frac{v(ku)}{(ku)^\theta} - \frac{v(ju)}{(ju)^\theta}\bigg) \\ &\qquad + O((c_{19}l)^l \min\{k,j\}^{-3\theta_1/2}u^{-3\theta_1/2-l})\bigg\} \\ &= \frac{\tau}{2\pi\alpha l!}\bigg\{\alpha' \prod_{i=1}^{l}(1-\theta_1-i)(k^{-\theta_1} - j^{-\theta_1})u^{-\theta_1-l} \\ &\qquad + O((c_{19}l)^l \min\{k,j\}^{-3\theta_1/2}u^{-3\theta_1/2-l} \\ &\qquad + (c_{11}l)^l \min\{k,j\}^{-\theta_2}u^{-\theta_2-l})\bigg\} \\ &= \frac{\alpha'\tau}{2\pi\alpha}\bigg\{\prod_{i=1}^{l}\bigg(\frac{1-\theta_1}{i} - 1\bigg)(k^{-\theta_1} - j^{-\theta_1})u^{-\theta_1-l} \\ &\qquad + O(c_{20}^l \min\{k,j\}^{-\theta_0}u^{-\theta_0-l})\bigg\}.\end{aligned}$$

Since $M^{1/2-\delta} \le j,k \le M^{1/2+\delta}$, we have, in view of (2.2),

$$\begin{aligned}|k^{-\theta_1} - j^{-\theta_1}| &\ge M^{(1/2+\delta)(-\theta_1-1)}|k-j| \ge M^{(1/2+\delta)(-\theta_1-1)+1/2-2\delta} \\ &= M^{-(3+\theta_1)\delta-\theta_1/2} \ge M^{-(1/2-\delta)\theta_0} \ge \min\{k,j\}^{-\theta_0}.\end{aligned}$$

Thus for $1 \le l \le c_{18} \log Q$, we have

$$\frac{|h_{j,k}^{(l)}(u)|}{l!} = \frac{|\alpha'\tau(k^{-\theta_1} - j^{-\theta_1})|u^{-\theta_1-l}}{2\pi\alpha}\bigg\{\prod_{i=1}^{l}\bigg|\frac{1-\theta_1}{i} - 1\bigg| + O(c_{20}^l u^{-(\theta_0-\theta_1)})\bigg\}.$$

Since $\theta_0 > \theta_1$, there exists $b_i' = b_i'(\alpha, \alpha', \theta, \theta', \theta'', C)$ with $0 < b_0' < 1 < b_1'$ such that we have, for $1 \le l \le c_{18} \log Q$ and $Q \le u \le 2Q$,

$$\frac{T|k^{-\theta_1} - j^{-\theta_1}|Q^{-\theta_1}}{(b_1'Q)^l} \le \frac{|h_{j,k}^{(l)}(u)|}{l!} \le \frac{T|k^{-\theta_1} - j^{-\theta_1}|Q^{-\theta_1}}{(b_0'Q)^l}. \tag{2.10}$$

We then consider the following two cases according to the size of T. Let $\Theta := (1+2\delta)\theta_1 + 4\delta$.

Case 1: $M^\Theta \le T \le \exp\{\varrho_3(\log M)^2\}$. We appeal to Lemma 1. In view of (2.10), we take

$$D = T|k^{-\theta_1} - j^{-\theta_1}|Q^{-\theta_1}.$$

Since $\max\{j,k,Q\} \le M^{1/2+\delta}$ and $|j-k| \ge M^{1/2-2\delta}$, we have

$$\begin{aligned}D &\ge M^{(1+2\delta)\theta_1+4\delta-(1/2+\delta)(1+\theta_1)+1/2-2\delta}Q^{-\theta_1} = M^{\delta+\theta_1/2+\theta_1\delta}Q^{-\theta_1} \\ &\ge Q^{(\delta+\theta_1/2+\theta_1\delta)/(1/2+\delta)-\theta_1} = Q^{\delta/(1/2+\delta)} \ge Q^\delta,\end{aligned}$$

and

$$\log D/(8\delta_0^2 \log Q) + 1 \le \log T/(4\delta_0^2 \log Q) \le \varrho_3(\log M)^2/(4\delta_0^2 \log Q) \le c_{18} \log Q.$$

Lemma 1 yields

$$\sum_{\substack{Q<m\le 2Q \\ m\le N/\max\{j,k\}}} \left(\frac{g_0(km)}{g_0(jm)}\right)^{i\tau} \ll Q \exp\left\{-c_{20}\frac{\log^3 M}{\log^2 T}\right\}. \tag{2.11}$$

Case 2: $M^{\theta_1} \le T \le M^{\Theta}$. We apply van der Corput's classical result ([6], Theorem 2.9): if $h \in \mathcal{C}^2[Q, 2Q]$ satisfies $h^{(l)}(u) \asymp H/Q^l$ for $l = 1, 2$ and $Q \le u \le 2Q$, then

$$\sup_{Q\le Q_1\le 2Q} \Big|\sum_{Q<n\le Q_1} e(h(n))\Big| \ll H^{1/2}Q^{1/2} + H^{-1}Q.$$

The relation (2.10) shows that this result is applicable to $h = h_{j,k}$ with $H = T|k^{-\theta_1} - j^{-\theta_1}|Q^{-\theta_1}$. Since $\min\{j, k, Q\} \ge M^{1/2-\delta}$ and $\max\{j, k\} \le N/Q \le 2M/Q$, we have

$$\begin{cases} H \ge T \max\{j, k\}^{-\theta_1-1}|j-k|Q^{-\theta_1} \gg TQ|k-j|/M^{1+\theta_1}, \\ H \le 2M^{(1+2\delta)\theta_1+4\delta-2\theta_1(1/2-\delta)} = M^{4(1+\theta_1)\delta} \le M^{1/3}. \end{cases}$$

Thus we obtain

$$\sum_{\substack{Q<m\le 2Q \\ m\le N/\max\{j,k\}}} \left(\frac{g_0(km)}{g_0(jm)}\right)^{i\tau} \ll M^{1/6}Q^{1/2} + \frac{M^{1+\theta_1}}{T|k-j|}. \tag{2.12}$$

Now combining (2.11) and (2.12) with (2.9), we find

$$\begin{aligned} S''_N(\tau, j, Q) &\ll M^{1-\delta} + Me^{-c_{21}(\log M)^3/(\log T)^2} \\ &\quad + M^{23/24} + T^{-1}M^{1+\theta_1} \log M \\ &\ll Me^{-c_{21}(\log M)^3/(\log T)^2} + T^{-1}M^{1+\theta_1} \log M. \end{aligned} \tag{2.13}$$

Finally inserting (2.8) and (2.13) into (2.3), we get

$$\sum_{n\le N} \Lambda(n)g_0(n)^{-i\tau} \ll M\{e^{-c_{22}(\log M)^3/(\log T)^2} + (M^{\theta_1}/T)^{1/2}(\log M)^{7/2}\},$$

which implies the desired result. Our proof is complete. ■

The fourth lemma is an asymptotic formula on page 248 of [2]. Since the proof given there is quite sketchy, we present a detailed proof for convenience of the reader. The proof was provided by Balazard & Tenenbaum and we reproduce it here with their permission.

Lemma 4. *For any positive constants, given ϕ_1 and ϕ_2, there exists a positive constant $\varrho_0' = \varrho_0'(\phi_1, \phi_2)$ such that*

$$\sum_{n\le z} \frac{\Lambda(n)}{n^{1+i\tau}} = -\frac{\zeta'}{\zeta}(1+i\tau) + O(z^{-\varrho_0'\beta(|\tau|+3)}\log z)$$

for $e^{\phi_1/(\beta(|\tau|+3))} \le z \le (|\tau|+3)^{\phi_2}$.

Proof. Let $F(s) := -\zeta'(s)/\zeta(s)$. By the Perron formula ([12], Theorem II.2.3), we can write

$$U(z) := \sum_{n\le z} \frac{\Lambda(n)}{n^{1+i\tau}} \log\left(\frac{z}{n}\right) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} F(1+i\tau+w)\frac{z^w}{w^2}\,dw, \tag{2.14}$$

where $\xi := 1/\log z$. According to Vinogradov–Korobov's well known bound, we have

$$F(1+i\tau+w) \ll 1/(\beta(|\tau+\operatorname{Im} w|+3)), \qquad \operatorname{Re} w \ge -\varrho_0''\beta(|\tau+\operatorname{Im} w|+3), \tag{2.15}$$

where $\varrho_0'' > 0$ is a constant. We truncate the integral in (2.14) to $|\operatorname{Im} w| = \sqrt{z}$ with an error

$$\int_{\substack{\operatorname{Re} w=\xi\\ |\operatorname{Im} w|\ge\sqrt{z}}} F(1+i\tau+w)\frac{z^w}{w^2}\,dw \ll \frac{\log z}{\sqrt{z}},$$

where we have used (2.15) in the form

$$\begin{aligned} F(1+i\tau+w) &\ll \max\{1/(\beta(|\tau|+3)),\, 1/(\beta(|\operatorname{Im} w|+3))\}\\ &\ll \max\{\log z,\, \log(|\operatorname{Im} w|+3)\}\end{aligned}$$

for $\operatorname{Re} w = \xi$ and $e^{\phi_1/(\beta(|\tau|+3))} \le z$. We move the segment of integration from $[\xi - i\sqrt{z}, \xi + i\sqrt{z}\,]$ to $\operatorname{Re} w = -2\varrho_0'\beta(|\tau|+3)$, where $\varrho_0' \le \varrho_0''/10$ is a suitable positive constant depending on ϕ_1 and ϕ_2. The contribution of the vertical segment is

$$\ll \frac{z^{-2\varrho_0'\beta(|\tau|+3)}}{\beta(|\tau|+3)}\int_0^{\sqrt{z}} \frac{dt}{t^2+\beta(|\tau|+3)^2} \ll \frac{z^{-2\varrho_0'\beta(|\tau|+3)}}{\beta(|\tau|+3)^2} \ll z^{-2\varrho_0'\beta(|\tau|+3)}\log^2 z$$

and the contribution of the horizontal segments is

$$\ll \frac{1}{z\beta(|\tau|+3)}\left(\frac{1}{\log z} + \beta(|\tau|+3)\right) \ll \frac{1}{z},$$

where we have used $e^{\phi_1/(\beta(|\tau|+3))} \le z \le (|\tau|+3)^{\phi_2}$. Hence the residue theorem gives

$$U(z) = F(1+i\tau)\log z + F'(1+i\tau) + O(z^{-2\varrho_0'\beta(|\tau|+3)}\log^2 z).$$

From this, we deduce that for $\sqrt{z} \le y \le z^2$,

$$U(z+y)-U(z) = \log\left(1+\frac{y}{z}\right)\sum_{n\le z}\frac{\Lambda(n)}{n^{1+i\tau}} + \sum_{z<n\le z+y}\frac{\Lambda(n)}{n^{1+i\tau}}\log\left(\frac{z+y}{n}\right)$$
$$= \log(1+y/z)F(1+i\tau) + O(z^{-2\varrho_0'\beta(|\tau|+3)}\log^2 z).$$

Thus for $\sqrt{z} \le y \le z^2$ we obtain

$$\sum_{n\le z}\frac{\Lambda(n)}{n^{1+i\tau}} = F(1+i\tau) + O\left(\frac{z^{-2\varrho_0'\beta(|\tau|+3)}\log^2 z}{\log(1+y/z)} + \sum_{z<n\le z+y}\frac{\Lambda(n)}{n}\right). \tag{2.16}$$

It remains to estimate the last sum in (2.16). Defining

$$\psi(t) := \sum_{n\le t}\Lambda(n)$$

and using the Brun–Titchmarsh inequality ([12], Theorem I.4.9), we can deduce that for $\sqrt{z} \le y \le z^2$,

$$\sum_{z<n\le z+y}\frac{\Lambda(n)}{n} = \int_z^{z+y}\frac{d\psi(t)}{t} = \frac{\psi(z+y)}{z+y} - \frac{\psi(z)}{z} + \int_z^{z+y}\frac{\psi(t)}{t^2}\,dt$$
$$= \frac{\psi(z+y)-\psi(z)}{z+y} - \frac{\psi(z)y}{(z+y)z} + \int_z^{z+y}\frac{\psi(t)}{t^2}\,dt$$
$$\ll \frac{y}{z+y} + \log\left(1+\frac{y}{z}\right) \ll \log\left(1+\frac{y}{z}\right).$$

Take y such that $\log(1+y/z) = z^{-\varrho_0'\beta(|\tau|+3)}\log z$, i.e.

$$y = z(z^{z^{-\varrho_0'\beta(|\tau|+3)}} - 1).$$

It is easy to see that $\sqrt{z} \le z^{1-\varrho_0'\beta(|\tau|+3)}\log z \le y \le z^2$. This completes the proof. ∎

The last lemma is a variant of Hankel's formula ([12], Theorem II.5.2).

For $a \in \mathbb{R}$ and $r > 0$, we use $\mathcal{H}(a,r)$ to denote the Hankel contour surrounding the point $s = a$ with radius r, which is defined as the path formed from the circle $|s-a| = r$ excluding the point $s = a-r$, together with the half-line $(-\infty, a-r]$ traced twice, with respective arguments $+\pi$ and $-\pi$. For each $X > |a-r|+1$, let $\mathcal{H}_X(a,r)$ be the part of the Hankel contour $\mathcal{H}(a,r)$ situated in the half-plane $\sigma > -X$.

LEMMA 5. *For $X > 1$, $z \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, we have*

$$\frac{1}{2\pi i}\int_{\mathcal{H}_X(0,r)} s^{-z}e^s(\log s)^k\,ds = (-1)^k\frac{d^k}{dz^k}\left(\frac{1}{\Gamma(z)}\right) + E_{k,z}(X),$$

where

$$|E_{k,z}(X)| \le \frac{e^{\pi|\operatorname{Im} z|}}{2\pi} \int_X^\infty \sigma^{-\operatorname{Re} z} e^{-\sigma} (\log\sigma + \pi)^k \, d\sigma. \tag{2.17}$$

Proof. According to the Hankel formula ([12], Theorem II.5.2), we have

$$\frac{1}{2\pi i} \int_{\mathcal{H}(0,r)} s^{-z} e^s \, ds = \frac{1}{\Gamma(z)}.$$

Since the integral on the left-hand side is absolutely and uniformly convergent on any compact set in the z-plane, we can differentiate under the integral sign to obtain

$$\frac{1}{2\pi i} \int_{\mathcal{H}(0,r)} s^{-z} e^s (\log s)^k \, ds = (-1)^k \frac{d^k}{dz^k}\left(\frac{1}{\Gamma(z)}\right).$$

For $\sigma > 1$ and $s = \sigma e^{\pm i\pi}$, we have

$$|s^{-z} e^s (\log s)^k| \le \sigma^{-\operatorname{Re} z} e^{\pi|\operatorname{Im} z| - \sigma} (\log\sigma + \pi)^k.$$

This implies the desired estimate for $E_{k,z}(X)$. ■

3. Proof of Theorem 4. From the assumptions (1)–(3), we deduce that for $\sigma \ge 1/\theta - \min\{\eta/(2\theta),\, 1/\theta - 1/\widetilde{\theta},\, 1/(4\theta)\}$,

$$\begin{aligned}\widetilde{\mathcal{F}}_g(s) &= \prod_p \left\{1 + \frac{\kappa}{g(p)^s} + O\left(\frac{1}{p^{1+\eta/2}} + \frac{1}{p^\psi}\right)\right\}\left\{1 - \frac{\kappa}{(\alpha p^\theta)^s} + O\left(\frac{1}{p^{3/2}}\right)\right\} \\ &= \prod_p \left\{1 + \kappa\left(\frac{1}{g(p)^s} - \frac{1}{(\alpha p^\theta)^s}\right) + O\left(\frac{1}{p^{1+\eta/2}} + \frac{1}{p^\psi} + \frac{1}{p^{3/2}}\right)\right\},\end{aligned}$$

which implies, for $\sigma \ge 1/\theta - \min\{\eta/(2\theta),\, 1/\theta - 1/\widetilde{\theta},\, 1/(4\theta)\}$,

$$\widetilde{\mathcal{F}}_g(s) = \exp\left\{\kappa \sum_p (g(p)^{-s} - (\alpha p^\theta)^{-s}) + O(1)\right\}. \tag{3.1}$$

Noticing that $g(p) = \alpha p^\theta\{1 + O(p^{-\theta_1})\}$, we have

$$g(p)^{-s} - (\alpha p^\theta)^{-s} \ll |s| p^{-\theta\sigma - \theta_1} \ll |s| p^{-1-\theta_1/2}.$$

This proves the assertion (i) provided $\varrho_0 \le \frac{1}{10} \min\{\eta/(2\theta),\, 1/\theta - 1/\widetilde{\theta},\, 1/(4\theta)\}$.

In view of (3.1), in order to prove (ii), it suffices to show that for $\sigma \ge 1/\theta - 10\varrho_0\beta(T)$,

$$\Big|\sum_p (g(p)^{-s} - (\alpha p^\theta)^{-s})\Big| \le \alpha^{-\sigma}\left\{\frac{4}{3}\log_2 T + \frac{2}{3}\log_3 T + O(1)\right\}. \tag{3.2}$$

We only need to consider the case of $g(p) = \alpha p^\theta + \alpha' p^{\theta'} + t(p) = g_0(p)$.

Let $T_0 := e^{1/(10\theta\varrho_0\beta(T))}$ and $T_1 := T^{(1+20\theta\theta_1^{-1}\varrho_0\beta(T))/\theta_1}$. Then we have

$$\sum_{p>T_1} |g_0(p)^{-s} - (\alpha p^\theta)^{-s}| \ll |s| \sum_{p>T_1} p^{-(\theta\sigma+\theta_1)} \ll T \sum_{p>T_1} p^{-(1+\theta_1-10\theta\varrho_0\beta(T))} \ll 1$$

provided $\varrho_0 \le \theta_1/(20\theta\beta(3))$. Noticing that

$$|g_0(p)^{-s} - (\alpha p^\theta)^{-s}| \le 2(\alpha p^\theta\{1+O(p^{-\theta_1})\})^{-\sigma} \le 2\alpha^{-\sigma} p^{-1+10\theta\varrho_0\beta(T)}\{1+O(p^{-\theta_1})\},$$

we deduce

$$\Big|\sum_{p\le T_0} (g_0(p)^{-s} - (\alpha p^\theta)^{-s})\Big| \le 2\alpha^{-\sigma}\bigg\{\sum_{p\le T_0} \frac{1+O(\beta(T)\log p)}{p} + O(1)\bigg\} \le \alpha^{-\sigma}\bigg\{\frac{4}{3}\log_2 T + \frac{2}{3}\log_3 T + O(1)\bigg\}.$$

By using Lemma 4, we have

$$\begin{aligned}\sum_{T_0<p\le T_1} \frac{1}{(\alpha p^\theta)^s} &= \alpha^{-s}\sum_{T_0<n\le T_1} \frac{\Lambda(n)}{n^{\theta s}\log n} + O(1)\\ &= \alpha^{-s}\int_{T_0}^{T_1} \frac{z^{1-\theta\sigma}}{\log z}\, d\bigg(\sum_{n\le z}\frac{\Lambda(n)}{n^{1+i\theta\tau}}\bigg) + O(1)\\ &= \alpha^{-s}\int_{T_0}^{T_1} \frac{z^{1-\theta\sigma}}{\log z}\, dO(z^{-2c_{23}\beta(T)}\log z) + O(1)\\ &\ll \alpha^{-\sigma}\bigg\{1 + \bigg(|1-\theta\sigma| + \frac{1}{\log T_0}\bigg)\int_{T_0}^{T_1}\frac{dz}{z^{1+c_{23}\beta(T)}}\bigg\} \ll 1,\end{aligned}$$

provided $\varrho_0 \le c_{23}/(10\theta)$.

Our remaining task is to show

$$\sum_{T_0<p\le T_1} g_0(p)^{-s} \ll 1. \tag{3.3}$$

We divide this sum into two parts according as $T_0 < p \le T'$ or $T' < p \le T_1$, where $T' := T^{(1-\varrho_1^2\beta(T))/\theta_1}$ and ϱ_1 is the constant determined by Lemma 2. We use W_1 and W_2 to denote the corresponding contribution and apply Lemmas 3 and 2 to treat them.

Let $M_j := \min\{2^j T', T_1\}$ and $J \in \mathbb{N}$ such that $2^J T' \le T_1 < 2^{J+1}T'$. We have

$$\begin{aligned} W_2 &= \sum_{j=0}^{J} \int_{M_j}^{M_{j+1}} \frac{1}{g_0(z)^\sigma \log z}\, d\Big(\sum_{M_j<n\le z} \Lambda(n) g_0(n)^{-i\tau}\Big) + O(1) \\ &\ll \sum_{j=0}^{J} M_j^{-\theta\sigma} S_{M_j}(\tau) + 1. \end{aligned}$$

The choice of T' and the fact that $T^{1/(2\theta_1)} \le T' \le M_j \le T_1 \le T^{2/\theta_1}$ guarantee that Lemma 2 is applicable. Thus we deduce that, provided $\varrho_0 \le \varrho_1/(20\theta)$,

$$\begin{aligned} W_2 &\ll \sum_{j=0}^{J} M_j^{10\theta\varrho_0\beta(T)} (M_j^{-\varrho_1\beta(T)} + T^{-1} + \mathcal{L}(M_j)^{-\varrho_1}) + 1 \\ &\ll \sum_{j=0}^{J} (2^j T')^{-\varrho_1\beta(T)/2} + 1 \ll 1. \end{aligned}$$

Let $N_k := \min\{2^k T_0, T'\}$ and $K \in \mathbb{N}$ such that $2^K T_0 \le T' < 2^{K+1} T_0$. As before we have

$$W_1 \ll \sum_{k=0}^{K} N_k^{-\theta\sigma} S_{N_k}(\tau) + 1.$$

Assume that $\varrho_0 \le \varrho_3^{2/3}/(10\theta)$, where ϱ_3 is given in Lemma 3. Since

$$\begin{aligned} N_k^{\theta_1} &\le T'^{\theta_1} \le T \\ &\le \exp\{(10\theta\varrho_0 \log T_0)^{3/2}\} \\ &\le \exp\{\varrho_3 (\log T_0)^2\} \le \exp\{\varrho_3 (\log N_k)^2\}, \end{aligned}$$

Lemma 3 is applicable. Thus

$$\begin{aligned} W_1 &\ll \sum_{k=0}^{K} N_k^{10\theta\varrho_0\beta(T)} (e^{-\varrho_2 (\log N_k)^3/(\log T)^2} + (N_k^{\theta_1}/T)^{1/2} (\log N_k)^4) \\ &\ll (\log T)^4 \sum_{k=0}^{K} 2^{10\theta\varrho_0\beta(T)k} (e^{-\varrho_2(\log N_k)^3/(\log T)^2} + T^{-\varrho_1^2\beta(T)/2}). \end{aligned}$$

Noticing that

$$(\log N_k)^3/(\log T)^2 \ge k^3/(4(\log T)^2) + \log_2 T/(10\theta\varrho_0)^3$$

for $0 \le k \le K$, we deduce

$$W_1 \ll (\log T)^5\{(\log T)^{-\varrho_2/(10\theta\varrho_0)^3} \sup_{k\ge 0} 2^{10\theta\varrho_0\beta(T)k} e^{-\varrho_2 k^3/(4(\log T)^2)} + T^{(10\theta\varrho_0/\theta_1-\varrho_1^2/2)\beta(T)}\}$$
$$\ll (\log T)^{5-\varrho_2/(10\theta\varrho_0)^3} + T^{(10\theta\varrho_0/\theta_1-\varrho_1^2/2)\beta(T)}(\log T)^5 \ll 1,$$

provided $\varrho_0 \le \min\{(\varrho_2/10)^{1/3}/(10\theta), \theta_1\varrho_1^2/(40\theta)\}$. This completes the proof. ∎

4. Proof of Theorem 1. For simplicity, we introduce the notations:

$$z_j := \kappa/\alpha^{1/\theta} - j, \quad M(x) := x^{1/\theta}(\log x)^{\kappa/\alpha^{1/\theta}-1},$$
$$\frac{1}{\Gamma_i(a)} := \left[\frac{d^i}{dz^i}\left(\frac{1}{\Gamma(z)}\right)\right]_{z=a}.$$

Let $\xi = 1/\theta + 1/\log x$. By Theorem II.2.3 in [12], we can write

$$\int_0^x F_g(t)\,dt = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \mathcal{F}_g(s)\frac{x^{s+1}}{s(s+1)}\,ds. \tag{4.1}$$

Let $\varrho_4 < \varrho_0$ be a positive number small enough that

$$\begin{cases} |(\theta s-1)\log(\theta s-1)| \le \theta\varrho_0/2 \\ |\log(\theta s-1)| \ge 1 \end{cases} \qquad (|s-1/\theta| \le 10\varrho_4). \tag{4.2}$$

Let $U > 1$ be a parameter to be chosen later. The residue theorem allows us to deform the segment of integration $[\xi - iU, \xi + iU]$ into the following path symmetrically with respect to the real axis. Its upper part is made up of: the upper portion (above the real axis) of the truncated Hankel contour $\mathcal{H}_1 = \mathcal{H}_{4\varrho_4\beta(3)-1/\theta}(1/\theta, \varrho_4/\log x)$, the curve $\sigma = 1/\theta - 4\varrho_4\beta(\tau+3)$ for $0 \le \tau \le U$; and the horizontal segment $[1/\theta - 4\varrho_4\beta(U+3) + iU, \xi + iU]$.

From Theorem 4(ii) and the classic bound for $\zeta(s)$ ([12], Note of Chapter II.3), we have

$$\mathcal{F}_g(s) \ll (\log(|\tau|+3))^{c_{24}} \qquad (\sigma > 1/\theta - 10\varrho_0\beta(|\tau|+3)). \tag{4.3}$$

Using this estimate, we easily see that the contribution from the vertical half-lines $[\xi \pm iU, \xi \pm i\infty]$ and from the horizontal segments

$$[1/\theta - 4\varrho_4\beta(U+3) \pm iU, \xi \pm iU]$$

is $\ll x^{1+1/\theta}/\sqrt{U}$.

Finally the integral over the arcs $\sigma = 1/\theta - 4\varrho_4\beta(|\tau|+3)$ $(0 \le |\tau| \le U)$ is

$$\ll x^{1+1/\theta-4\varrho_4\beta(U+3)}\int_0^U \frac{(\log(\tau+3))^{c_{24}}}{(\tau+1)^2}\,d\tau \ll x^{1+1/\theta-4\varrho_4\beta(U+3)}.$$

Inserting these estimates into (4.1) and taking $U = \mathcal{L}(x)^{c_{25}}$, we can obtain

$$\int_0^x F_g(t)\,dt = \Psi(x) + O(x^{1+1/\theta}/\mathcal{L}(x)^{c_{26}}), \tag{4.4}$$

where

$$\Psi(x) = \frac{1}{2\pi i}\int_{\mathcal{H}_1} \mathcal{F}_g(s)\frac{x^{s+1}}{s(s+1)}\,ds.$$

Next we need to study the function $\Psi(x)$. Clearly it is an infinitely differentiable function of x on $\mathbb{R}^+$, and we have

$$\Psi'(x) = \frac{1}{2\pi i}\int_{\mathcal{H}_1} \mathcal{F}_g(s)\frac{x^s}{s}\,ds, \qquad \Psi''(x) = \frac{1}{2\pi i}\int_{\mathcal{H}_1} \mathcal{F}_g(s)x^{s-1}\,ds.$$

Now for $s \notin (-\infty, 1/\theta]$, we can write

$$\begin{aligned}(\theta s-1)^{-\kappa/\alpha^s+z_0} &= \exp\{z_0(1-\alpha^{-(\theta s-1)/\theta})\log(\theta s-1)\}\\ &= \sum_{m=0}^{\infty}\frac{(z_0(\theta s-1)\log(\theta s-1))^m}{m!}\left(\frac{1-\alpha^{-(\theta s-1)/\theta}}{\theta s-1}\right)^m.\end{aligned}$$

Note that for $m \ge 0$ we have

$$\left(\frac{1-\alpha^{-(\theta s-1)/\theta}}{\theta s-1}\right)^m = \left(\frac{\log\alpha}{\theta}\right)^m \sum_{n=0}^{\infty} b_{m,n}\left(-\frac{\log\alpha}{\theta}\right)^n(\theta s-1)^n,$$

where $b_{m,n}$ is defined as in Section 1. Obviously

$$b_{m,n} \le \sum_{n_1+\ldots+n_m=n}\frac{1}{n_1!\ldots n_m!} = \frac{m^n}{n!}. \tag{4.5}$$

It follows that

$$\begin{aligned}&(\theta s-1)^{-\kappa/\alpha^s+z_0}\\ &\quad= \sum_{m=0}^{\infty}\frac{(z_0\log(\theta s-1))^m}{m!}\sum_{n=0}^{\infty}(-1)^n b_{m,n}\left(\frac{\log\alpha}{\theta}\right)^{m+n}(\theta s-1)^{m+n}\\ &\quad= \sum_{m=0}^{\infty}\left\{\left(\frac{\log\alpha}{\theta}\right)^m\sum_{k=0}^{m}\frac{(-1)^{m-k}b_{k,m-k}}{k!}(z_0\log(\theta s-1))^k\right\}(\theta s-1)^m.\end{aligned}$$

Noticing that $s^{-1}\mathcal{F}_g(s) = s^{-1}\widetilde{\mathcal{F}}_g(s)\{\zeta(\theta s)(\theta s-1)\}^{\kappa/\alpha^s}(\theta s-1)^{-\kappa/\alpha^s}$ and using (1.3) and (1.4), we deduce that, for $s \notin (-\infty, 1/\theta]$ and $|s-1/\theta| < 10\varrho_0$,

$$\text{(4.6)} \quad \frac{\mathcal{F}_g(s)}{s}$$
$$= \sum_{j=0}^{\infty} \Big\{ \sum_{m=0}^{j} \frac{a_{j-m}(\log\alpha)^m}{\theta^j} \sum_{k=0}^{m} \frac{(-1)^{m-k} b_{k,m-k}}{k!} (z_0 \log(\theta s - 1))^k \Big\} (\theta s - 1)^{-z_j}$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{e_{j,k}}{\theta^j} (\theta s - 1)^{-z_j} (\log(\theta s - 1))^k,$$

where

$$\text{(4.7)} \qquad \begin{aligned} e_{j,k} &:= \frac{z_0^k}{k!} \sum_{m=k}^{j} (-1)^{m-k} (\log\alpha)^m a_{j-m} b_{k,m-k} \\ &\ll \frac{|z_0|^k}{k!} \sum_{m=k}^{j} \frac{|\log\alpha|^m}{\varrho_0^{j-m}} \cdot \frac{k^{m-k}}{(m-k)!} \\ &\ll \frac{(\varrho_0 |z_0 \log\alpha|)^k}{\varrho_0^j k!} \sum_{n=0}^{j-k} \frac{(\varrho_0 k |\log\alpha|)^n}{n!} \\ &\ll \varrho_0^{-j} \frac{(\varrho_0 |z_0 \log\alpha| e^{\varrho_0 |\log\alpha|})^k}{k!} \\ &\ll \varrho_0^{-j} \frac{\lambda^k}{k!}. \end{aligned}$$

For any integer $J \geq 0$, we split the double sum in (4.6) into two parts to obtain

$$\text{(4.8)} \qquad \Psi'(x) = M_J(x) + \Xi_J(x),$$

where

$$M_J(x) := \sum_{j=0}^{J} \sum_{k=0}^{j} \frac{e_{j,k}}{\theta^j} \cdot \frac{1}{2\pi i} \int_{\mathcal{H}_1} x^s (\theta s - 1)^{-z_j} (\log(\theta s - 1))^k \, ds,$$
$$\Xi_J(x) := \frac{1}{2\pi i} \int_{\mathcal{H}_1} x^s \sum_{j=J+1}^{\infty} \sum_{k=0}^{j} \frac{e_{j,k}}{\theta^j} (\theta s - 1)^{-z_j} (\log(\theta s - 1))^k \, ds.$$

Observing that $\mathcal{H}_1$ is contained in the disc $|s - 1/\theta| \leq 10\varrho_4$, the inequalities (4.2) and (4.7) allow us to deduce that the double sum in $\Xi_J(x)$ is, for $s \in \mathcal{H}_1$,

$$\ll |\theta s - 1|^{-\operatorname{Re} z_0} \sum_{j>J} \sum_{k \leq j} \frac{|\lambda \log(\theta s - 1)|^k}{k!} \cdot \frac{|\theta s - 1|^j}{(\theta\varrho_0)^j}$$

$$\ll |\theta s-1|^{-\operatorname{Re} z_0} \sum_{j>J} (1+|\lambda \log(\theta s-1)|^j) \frac{|\theta s-1|^j}{(\theta\varrho_0)^j}$$

$$\ll |\theta s-1|^{-\operatorname{Re} z_0} \left(\frac{|\theta s-1|^{J+1}}{(\theta\varrho_0)^{J+1}} + \frac{|\lambda(\theta s-1)\log(\theta s-1)|^{J+1}}{(\theta\varrho_0)^{J+1}} \right).$$

Noticing that $|\log(\theta s-1)| \le \log_2 x + c_{27}$ $(s\in\mathcal{H}_1)$, we find that for $s\in\mathcal{H}_1$,

$$\sum_{j=J+1}^{\infty} \sum_{k=0}^{j} \frac{e_{j,k}}{\theta^j} (\theta s-1)^{-z_j} (\log(\theta s-1))^k$$
$$\ll |\theta s-1|^{-\operatorname{Re} z_0} \left(\frac{\lambda \log_2 x + c_{28}}{\theta \varrho_0} |\theta s-1| \right)^{J+1}.$$

From this we deduce that

$$\Xi_J(x) \ll (\lambda \log_2 x + c_{28})^{J+1} \Xi_J^*(x),$$

where

$$\Xi_J^*(x) := \frac{1}{(\theta\varrho_0)^{J+1}} \int\limits_{1/\theta-4\varrho_4\beta(3)}^{1/\theta-r} x^{\sigma} |1-\theta\sigma|^{J+1-\operatorname{Re} z_0}\, d\sigma + \frac{x^{1/\theta+r}}{r^{\operatorname{Re} z_0 - J-2}}$$
$$\ll \frac{|M(x)|}{(\theta\varrho_0 \log x)^{J+1}} \Big\{ \int\limits_{\varrho_4}^{4\varrho_4\beta(3)\log x} t^{J+1-\operatorname{Re} z_0} e^{-t}\, dt + 1 \Big\}$$
$$\ll \frac{|M(x)|}{(\theta\varrho_0 \log x)^{J+1}} \{ \Gamma(J+2-\operatorname{Re} z_0) + 1 \}$$
$$\ll |M(x)| \left(\frac{c_{29} J + 1}{\log x} \right)^{J+1}.$$

Thus

$$\Xi_J(x) \ll |M(x)| \left((c_{29}J+1) \frac{\lambda \log_2 x + c_{28}}{\log x} \right)^{J+1}. \tag{4.9}$$

Using the change of variable $w=(s-1/\theta)\log x$ and Lemma 5, we have, with the notation $\mathcal{H}_0 := \mathcal{H}_X(0,\varrho_4)$ and $X := 4\varrho_4\beta(3)\log x$,

$$M_J(x) = M(x) \sum_{j=0}^{J} \frac{\theta^{-z_0}}{(\log x)^j} \sum_{k=0}^{j} \frac{e_{j,k}}{2\pi i} \int\limits_{\mathcal{H}_0} \frac{e^w \{\log w + \log(\theta/\log x)\}^k}{w^{z_j}}\, dw$$
$$= M(x) \sum_{j=0}^{J} \frac{\theta^{-z_0}}{(\log x)^j} \sum_{k=0}^{j} e_{j,k} \sum_{i=0}^{k} \binom{k}{i} \left(\log \frac{\theta}{\log x} \right)^{k-i}$$
$$\times \left\{ \frac{(-1)^i}{\Gamma_i(z_j)} + E_{k,z_j}(X) \right\}.$$

The contribution of $(-1)^i/\Gamma_i(z_j)$ to $M_J(x)$ is

$$M(x)\sum_{j=0}^{J}\frac{\theta^{-z_0}}{(\log x)^j}\sum_{k=0}^{j}e_{j,k}\sum_{i=0}^{k}\binom{k}{i}\left(\log\frac{\theta}{\log x}\right)^{k-i}\frac{(-1)^i}{\Gamma_i(z_j)} = M(x)\sum_{j=0}^{J}\frac{P_j(\log_2 x)}{(\log x)^j},$$

where

$$\begin{aligned}
P_j(t) &:= \frac{1}{\theta^{z_0}}\sum_{k=0}^{j}e_{j,k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(\log\theta-t)^{k-i}}{\Gamma_i(z_j)} \\
&= \frac{1}{\theta^{z_0}}\sum_{k=0}^{j}e_{j,k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\sum_{l=0}^{k-i}\binom{k-i}{l}\frac{(\log\theta)^{k-i-l}}{\Gamma_i(z_j)}(-t)^l \\
&= \frac{1}{\theta^{z_0}}\sum_{k=0}^{j}e_{j,k}\sum_{l=0}^{k}\sum_{i=0}^{k-l}(-1)^i\binom{k}{i}\binom{k-i}{l}\frac{(\log\theta)^{k-i-l}}{\Gamma_i(z_j)}(-t)^l \\
&= \frac{1}{\theta^{z_0}}\sum_{l=0}^{j}\sum_{k=l}^{j}e_{j,k}\sum_{i=0}^{k-l}(-1)^i\binom{k}{i}\binom{k-i}{l}\frac{(\log\theta)^{k-i-l}}{\Gamma_i(z_j)}(-t)^l \\
&= \sum_{l=0}^{j}\lambda_{j,l}t^l
\end{aligned}$$

and from (4.7),

$$\begin{aligned}
\lambda_{j,l} &:= \frac{(-1)^l}{\theta^{z_0}}\sum_{k=l}^{j}e_{j,k}\sum_{i=0}^{k-l}(-1)^i\binom{k}{i}\binom{k-i}{l}\frac{(\log\theta)^{k-l-i}}{\Gamma_i(z_j)} \\
&= \frac{\theta^{-z_0}}{l!}\sum_{k=l}^{j}(\kappa\alpha^{-1/\theta})^k \\
&\qquad\times\sum_{m=k}^{j}(-\log\alpha)^m a_{j-m}b_{k,m-k}\sum_{i=0}^{k-l}\frac{(-\log\theta)^{k-l-i}}{(k-l-i)!i!\Gamma_i(z_j)} \\
&= \frac{\theta^{-\kappa/\alpha^{1/\theta}}}{l!}\sum_{m=l}^{j}\sum_{k=l}^{m}\sum_{i=0}^{k-l}\lambda^*_{m,k,i},
\end{aligned}$$

where

$$\lambda^*_{m,k,i} := \frac{(-\log\alpha)^m(\kappa\alpha^{-1/\theta})^k(-\log\theta)^{k-l-i}a_{j-m}b_{k,m-k}}{(k-l-i)!i!\Gamma_i(\kappa/\alpha^{1/\theta}-j)}.$$

The contribution of $E_{k,z_j}(X)$ to $M_J(x)$ is, via (2.17),

$$\begin{aligned}
&\ll |M(x)| \sum_{j=0}^{J} \sum_{k=0}^{j} \frac{|e_{j,k}|}{(\log x)^j} \sum_{i=0}^{k} \binom{k}{i} \left(\log \frac{\theta}{\log x} \right)^{k-i} \\
&\quad \times \int_X^\infty \sigma^{|z_j|} (\log \sigma + \pi)^i e^{-\sigma}\, d\sigma \\
&\ll |M(x)| \sum_{j=0}^{J} \frac{1}{(\varrho_0 \log x)^j} \sum_{k=0}^{j} \frac{\lambda^k}{k!} \int_X^\infty (2\log_2 x + 2\log \sigma)^k \sigma^{|z_j|} e^{-\sigma}\, d\sigma \\
&\ll |M(x)| \sum_{j=0}^{J} \frac{1}{(\varrho_0 \log x)^j} \int_X^\infty e^{2\lambda \log_2 x + 2\lambda \log \sigma} \sigma^{|z_j|} e^{-\sigma}\, d\sigma \\
&\ll |M(x)| \sum_{j=0}^{J} \frac{(\log x)^{2\lambda}}{(\varrho_0 \log x)^j} \int_X^\infty \sigma^{|z_j|+2\lambda} e^{-\sigma}\, d\sigma.
\end{aligned}$$

Since

$$\begin{aligned}
\int_X^\infty \sigma^{|z_j|+2\lambda} e^{-\sigma}\, d\sigma &\ll e^{-X/2} \int_X^\infty \sigma^{|z_j|+2\lambda} e^{-\sigma/2}\, d\sigma \\
&\ll e^{-X/2} 2^j \int_{X/2}^\infty \sigma^{|z_j|+2\lambda} e^{-\sigma}\, d\sigma \\
&\ll e^{-X/2} 2^j \Gamma(j + [|z_0| + 2\lambda] + 2) \\
&\ll e^{-X/2} (|2z_0| + 4\lambda + 4)^j (j+1)!,
\end{aligned}$$

the contribution of $E_{k,z_j}(X)$ to $M_J(x)$ is

$$\begin{aligned}
&\ll |M(x)| e^{-X/4} \sum_{j=0}^{J} \left(\frac{|2z_0| + 4\lambda + 4}{\varrho_0 \log x} \right)^j (j+1)! \\
&\ll |M(x)| e^{-X/4} \sum_{j=0}^{J} \left(\frac{8}{X} \right)^j (j+1)! \\
&\ll |M(x)| e^{-X/4} \left(\frac{8}{X} \right)^J \sum_{j=0}^{J} \left(\frac{X}{8} \right)^{J-j} \frac{(J+1)!}{(J-j)!} \\
&\ll |M(x)| e^{-X/8} \left(\frac{8}{X} \right)^J (J+1)! \ll |M(x)| \left(\frac{c_{29} J + 1}{\log x} \right)^{J+1}
\end{aligned}$$

provided $\varrho_4 \le \varrho_0 / (\beta(3)(|z_0| + 2\lambda + 2))$. Combining these estimates yields

$$M_J(x) = M(x) \left\{ \sum_{j=0}^{J} \frac{P_j(\log_2 x)}{(\log x)^j} + O\left(\left(\frac{c_{29} J + 1}{\log x} \right)^{J+1} \right) \right\}. \tag{4.10}$$

Inserting (4.9) and (4.10) into (4.8), we obtain

$$\text{(4.11)}\quad \Psi'(x) = M(x)\bigg\{\sum_{j=0}^{J}\frac{P_j(\log_2 x)}{(\log x)^j} + O\bigg(\bigg((c_{29}J+1)\frac{\lambda\log_2 x + c_{28}}{\log x}\bigg)^{J+1}\bigg)\bigg\}.$$

Obviously, we have

$$\Psi''(x) = \frac{1}{2\pi i}\int_{\mathcal{H}_1}\mathcal{F}_g(s)x^{s-1}\,ds \ll x^{1/\theta-1}(\log x)^{|\kappa|/\alpha^{1/\theta}}.$$

Assume that $f(n)\ge 0$ for all integers n. Then, with (4.4), we have

$$\begin{aligned}\text{(4.12)}\quad hF_g(x) &\le \int_x^{x+h} F_g(t)\,dt\\ &= \Psi(x+h)-\Psi(x)+O(x^{1+1/\theta}\mathcal{L}(x)^{-c_{26}})\\ &= h\Psi'(x)+h^2\int_0^1(1-t)\Psi''(x+th)\,dt\\ &\quad + O(x^{1+1/\theta}\mathcal{L}(x)^{-c_{26}})\\ &= h\Psi'(x)+O(h^2x^{1/\theta-1}(\log x)^{|\kappa|/\alpha^{1/\theta}}+x^{1+1/\theta}\mathcal{L}(x)^{-c_{26}}).\end{aligned}$$

Taking $h = x\mathcal{L}(x)^{-c_{30}}$, we obtain

$$F_g(x)\le\Psi'(x)+O(x^{1/\theta}\mathcal{L}(x)^{-c_3}).$$

Similarly, we can prove

$$F_g(x)\ge\frac{1}{h}\int_{x-h}^{x}F_g(t)\,dt\ge\Psi'(x)+O(x^{1/\theta}\mathcal{L}(x)^{-c_3}).$$

This proves the desired asymptotic formula in the case of $f(n)\ge 0$.

For the general case, write $|F_g|(x)=\sum_{g(n)\le x}|f(n)|$. We first check that the conditions in Theorem 1 are satisfied by $|F_g|$. Since

$$\bigg|F_g(x)-\frac{1}{h}\int_x^{x+h}F_g(t)\,dt\bigg|\le\frac{1}{h}\int_x^{x+h}|F_g|(t)\,dt-|F_g|(x),$$

the desired asymptotic formula then follows from (4.11) and (4.12).

Finally we prove (1.9). Noticing that for $0\le\alpha\le\beta$,

$$\begin{aligned}\int_1^\infty e^{-\sigma}\sigma^\beta(\log\sigma)^\alpha\,d\sigma &\ll (\log(\alpha+2))^\alpha\int_1^\infty e^{-\sigma/2}\sigma^\beta\,d\sigma\\ &\ll (\log(\alpha+2))^\alpha(c_{31}\beta+1)^\beta,\end{aligned}$$

we deduce

$$\begin{aligned}\frac{1}{|\Gamma_n(z_j)|} &= \left|\frac{1}{2\pi i}\int_{\mathcal{H}(0,1)} s^{-z_j} e^s (\log s)^n\, ds\right| \\ &\ll \int_1^\infty \sigma^{|z_0|+j} e^{-\sigma}(\log\sigma+\pi)^n\, d\sigma + \pi^n \\ &\ll (\log(n+2))^n (c_{32}j+1)^j.\end{aligned}$$

Now from this and (4.7), we have

$$\begin{aligned}|\lambda_{j,l}| &\ll \sum_{k=l}^{j} \varrho_0^{-j}\frac{\lambda^k}{k!}\sum_{i=0}^{k-l}\binom{k}{i}\binom{k-i}{l}(\log\theta)^{k-l-i}(\log(i+2))^i(c_{32}j+1)^j \\ &\ll \frac{(c_{32}j+1)^j}{l!}\sum_{k=l}^{j}\frac{\lambda^k}{(k-l)!}\sum_{i=0}^{k-l}\binom{k-l}{i}(\log\theta)^{k-l-i}(\log(i+2))^i \\ &\ll \frac{\lambda^l(c_{32}j+1)^j}{l!}\sum_{k=l}^{j}\frac{\lambda^{k-l}}{(k-l)!}(\log\theta+\log(k-l+2))^{k-l} \\ &\ll \frac{\lambda^l(c_{32}j+1)^j(j+2)^\lambda}{l!} \ll \frac{\lambda^l(c_4 j+1)^j}{l!}.\end{aligned}$$

This completes the proof. ■

5. Proofs of Theorems 2 and 3. We only prove Theorem 2. The other one can be treated completely in the same way.

It is easy to show that the assumption $(3)'$ and $|z| \le ((2-\varepsilon)/\psi_1)^{\psi_2}$ imply that $f(n)z^{\Omega(n)}$, $g(n)$ satisfy the assumption (3) with $\psi = 2\psi_2 > 1$. Thus Theorem 1 allows us to deduce that, uniformly for $|z| \le ((2-\varepsilon)/\psi_1)^{\psi_2}$,

$$(5.1)\qquad \sum_{g(n)\le x} f(n)z^{\Omega(n)} = \frac{x^{1/\theta}}{(\log x)^{1-\kappa z/\alpha^{1/\theta}}}\left\{\sum_{j=0}^{J}\frac{P_j(\log_2 x)}{(\log x)^j} + O(R_{J,\lambda}(x))\right\},$$

where $P_j(t) := \sum_{l=0}^{j}\lambda_{j,l}(z)t^l$ and the coefficient $\lambda_{j,l}(z) = \lambda_{j,l}(fz^{\Omega}, g)$ is given by

$$\lambda_{j,l}(z) := \frac{\theta^{-\kappa z/\alpha^{1/\theta}}}{l!}\sum_{m=l}^{j}\sum_{k=l}^{m}\sum_{i=0}^{k-l}\lambda^*_{m,k,i}(z),$$

where

$$\lambda^*_{m,k,i}(z) := \frac{(-\log\alpha)^m(\kappa z\alpha^{-1/\theta})^k(-\log\theta)^{k-l-i}a_{j-m}(fz^{\Omega}, g)b_{k,m-k}}{\Gamma_i(\kappa z/\alpha^{1/\theta}-j)(k-l-i)!i!}.$$

Obviously $\lambda_{j,l}(z)$ is analytic in the disc $|z| < ((2-\varepsilon)/\psi_1)^{\psi_2}$ and we can write, in this disc and for any $0 < r < ((2-\varepsilon)/\psi_1)^{\psi_2}$,

$$\lambda_{j,l}(z) = \sum_{n=0}^{\infty} \chi_{j,l}(n) z^n, \qquad \chi_{j,l}(n) = \frac{1}{2\pi i} \int_{|z|=r} \frac{\lambda_{j,l}(z)}{z^{n+1}}\, dz.$$

Thus

$$\begin{aligned}(\log x)^{\kappa z/\alpha^{1/\theta}} &\lambda_{j,l}(z)(\log_2 x)^l \\ &= (\log_2 x)^l \sum_{m=0}^{\infty} \frac{(\kappa \alpha^{-1/\theta} \log_2 x)^m}{m!} \sum_{n=0}^{\infty} \chi_{j,l}(n) z^{m+n} \\ &= \sum_{k=0}^{\infty} z^k \sum_{m=0}^{k} \frac{\chi_{j,l}(k-m)(\kappa\alpha^{-1/\theta})^m}{m!} (\log_2 x)^{l+m}.\end{aligned}$$

Notice that

$$\sum_{\substack{g(n)\le x \\ \Omega(n)=k}} f(n) = \frac{1}{2\pi i} \int_{|z|=r} \sum_{g(n)\le x} f(n) z^{\Omega(n)} \frac{dz}{z^{k+1}},$$

hence the contribution of the main term in (5.1) is

$$\begin{aligned}\frac{x^{1/\theta}}{\log x} \sum_{j=0}^{J} \frac{1}{(\log x)^j} \sum_{l=0}^{j} \sum_{m=0}^{k} & \frac{\chi_{j,l}(k-m)(\kappa\alpha^{-1/\theta})^m}{m!} (\log_2 x)^{l+m} \\ &= \frac{x^{1/\theta}}{\log x} \sum_{j=0}^{J} \frac{Q_{j,k}(\log_2 x)}{(\log x)^j}.\end{aligned}$$

It remains to estimate the error term. Taking $r = k/\log_2 x$ and writing $\kappa = |\kappa| e^{i\phi}$ show that this is

$$\begin{aligned}&\ll_A \frac{x^{1/\theta} R_{J,\lambda}(x)}{\log x} \oint_{|z|=r} \frac{(\log x)^{\mathrm{Re}(\kappa z/\alpha^{1/\theta})}}{|z|^{k+1}} |dz| \\ &\ll_A \frac{x^{1/\theta} R_{J,\lambda}(x)}{\log x} \left(\frac{\log_2 x}{k}\right)^k \int_0^{2\pi} e^{k|\kappa|\alpha^{-1/\theta}\cos(\phi+\vartheta)}\, d\vartheta.\end{aligned}$$

Denote the last integral by I. Then

$$\begin{aligned}I &\le 4 \int_0^{\pi/2} e^{k|\kappa|\alpha^{-1/\theta}\cos\vartheta}\, d\vartheta = 4 \int_0^1 \frac{e^{k|\kappa|\alpha^{-1/\theta} t}}{\sqrt{1-t^2}}\, dt \\ &\le 4 e^{k|\kappa|\alpha^{-1/\theta}} \int_0^1 \frac{e^{-k|\kappa|\alpha^{-1/\theta}(1-t)}}{\sqrt{1-t}}\, dt \ll \frac{e^{k|\kappa|\alpha^{-1/\theta}}}{\sqrt{|\kappa| k + 1}}.\end{aligned}$$

This completes the proof. ■

6. Proofs of Corollaries 2 and 3. As before, we only prove Corollary 2 and the other one will follow by the same argument.

We choose $\varepsilon = \varepsilon(\delta) > 0$ so small that $\{(2-\varepsilon)/(1/2+10\varepsilon)\}^{1/2+\varepsilon} > 2-\delta$. It is easy to verify that $f(n) \equiv 1$ and $\varphi(n)$ satisfy the assumptions (1)–(3)$'$ with $\psi_1 = 1/2+10\varepsilon$ and $\psi_2 = 1/2+\varepsilon$. Thus Theorem 2 is applicable. Next we show that the main term can take the simple form as stated. In view of Remarks (ii), we have $\lambda_{j,l}(z) = 0$ for $l = 1, \ldots, j$ and $\lambda_{j,0}(z) = a_j(z^{\Omega}, \varphi)/\Gamma(z-j)$. Observing that $\lambda_{j,0}(0) = 0$, we see that $\varsigma_j(z) := a_j(z^{\Omega}, \varphi)/(z\Gamma(z-j))$ is analytic in the disc $|z| \le 2-\delta$. Thus Theorem 2 implies (1.14).

Now we prove (1.15). If $k = 1$, the formula (1.15) can be obtained directly from Theorem 2 with $J = 0$. Next suppose $k \ge 2$. Similarly to (5.1) with $J = 0$, we have, uniformly for $|z| \le 2-\delta$,

$$\sum_{\varphi(n)\le x} z^{\Omega(n)} = \frac{x}{(\log x)^{1-z}}\left\{z\varsigma_0(z) + O\left(\frac{1}{\log x}\right)\right\}.$$

Dividing both sides by $2\pi i z^{k+1}$ and integrating over $|z| = r$ yields, for any $r \le 2-\delta$,

$$\sum_{\substack{\varphi(n)\le x\\ \Omega(n)=k}} 1 = \frac{x}{\log x}\cdot\frac{1}{2\pi i}\int_{|z|=r}\frac{\varsigma_0(z)(\log x)^z}{z^k}\,dz + O_\delta(R), \tag{6.1}$$

where

$$R := \frac{x}{(\log x)^2}\int_{|z|=r}\frac{(\log x)^{\operatorname{Re} z}}{|z|^{k+1}}\,|dz|.$$

To evaluate the principal term in (6.1), we write

$$\varsigma_0(z) = \varsigma_0(r) + (z-r)\varsigma_0'(r) + (z-r)^2\int_0^1 (1-t)\varsigma_0''(r+t(z-r))\,dt.$$

Taking $r := (k-1)/\log_2 x$, the Cauchy formula gives

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{(z-r)(\log x)^z}{z^k}\,dz = \frac{(\log_2 x)^{k-2}}{(k-2)!} - r\frac{(\log_2 x)^{k-1}}{(k-1)!} = 0.$$

Therefore it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=r}\frac{\varsigma_0(z)(\log x)^z}{z^k}\,dz &= \frac{1}{2\pi i}\int_{|z|=r}\frac{\varsigma_0(r)(\log x)^z}{z^k}\,dz + O_\delta(R') \\ &= \frac{(\log_2 x)^{k-1}}{(k-1)!}\varsigma_0\left(\frac{k-1}{\log_2 x}\right) + O_\delta(R'),\end{aligned} \tag{6.2}$$

where

$$R' := \int_{|z|=r} \left|(z-r)^2 \int_0^1 (1-t)\varsigma_0''(r+t(z-r))\,dt\right| (\log x)^{\operatorname{Re} z}|z|^{-k}|dz|.$$

Since $|r+t(z-r)| \le (1-t)r + tr = r$ $(0 \le t \le 1, |z| = r)$, we have $\varsigma_0''(r+t(z-r)) \ll_\delta 1$ and

$$(6.3)\quad R' \ll_\delta \int_0^{2\pi} |e^{i\vartheta}-1|^2 r^{3-k} e^{r\log_2 x \cos\vartheta}\,d\vartheta$$
$$\ll_\delta r^{3-k} \int_0^{2\pi} e^{(k-1)\cos\vartheta}(1-\cos\vartheta)\,d\vartheta$$
$$\ll_\delta r^{3-k}\Big(\int_0^1 e^{(k-1)t}\sqrt{1-t}\,dt + 2\pi\Big) \ll_\delta r^{3-k}e^{k-1}(k-1)^{-3/2}$$
$$\ll_\delta \frac{(\log_2 x)^{k-1}}{(k-1)!}\cdot\frac{k-1}{(\log_2 x)^2}.$$

Similarly, we have

$$(6.4)\quad R \ll_\delta \frac{x}{(\log x)^2} e^{k-1} r^{-k}(k-1)^{-1/2} \ll_\delta \frac{x}{\log x}\cdot\frac{(\log_2 x)^{k-1}}{(k-1)!}\cdot\frac{k-1}{(\log_2 x)^2}.$$

The estimate (1.15) now follows from (6.2)–(6.4). This completes the proof. ■

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