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Note on the fractional parts of $\lambda\theta^n$

by

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Let S be the set of *Pisot numbers* (also known as *Pisot–Vijayaraghavan numbers*), that is, real algebraic integers greater than 1 whose other conjugates have modulus strictly less than 1. Then S is a closed set (Salem [10]) and the smallest element of S is the positive zero of $x^3 - x - 1$ (Siegel [11]).

For real numbers $\lambda > 0$ and $\theta > 1$ we put $\lambda\theta^n = a_n + \varepsilon_n$ where $a_n \in \mathbb{Z}, -1/2 < \varepsilon_n \le 1/2$ for $n \ge 0$. The interesting relations between arithmetical properties of λ, θ and the asymptotic behaviour of ε_n have been studied by several researchers as exceptional cases of uniform distribution. In 1912 Thue [12] showed that, if ε_n decays faster than an exponential rate, then θ must be algebraic. Moreover Hardy [5] showed that $\theta \in S$ and λ must belong to the field $\mathbb{Q}(\theta)$ if θ is algebraic and ε_n tends to 0 as $n \to \infty$.

In 1938 Pisot [7] proved that the condition

$$\sum_{n=0}^{\infty} \varepsilon_n^2 < \infty$$

implies that $\theta \in S$ and $\lambda \in \mathbb{Q}(\theta)$. In particular, $\sum_{n=0}^{\infty} \varepsilon_n^2 = \infty$ if θ is transcendental. Later Pisot and Salem [9] asked whether there exists a transcendental number θ with the property that ε_n tends to 0 as $n \to \infty$ for some $\lambda > 0$, which is still an open and major question.

In 1946 Pisot [8] also proved that $\theta \in S \cup T$ and $\lambda \in \mathbb{Q}(\theta)$ if the condition

$$\sup_{n \ge 0} |\varepsilon_n| \le \frac{1}{2e\theta(\theta+1)(1+\log\lambda)}$$

is fulfilled when $\lambda \ge 1$, where T is the set of *Salem numbers*, that is, real algebraic integers greater than 1 whose other conjugates have modulus at most 1, one at least having a modulus equal to 1. It is known that a Salem number must be a unit of even degree at least equal to 4. The above ℓ^∞-

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condition was weakened by Cantor [2] (see also [3, Corollaire 5.2] or [1, Theorem 5.1.3]) to

$$\sup_{n\geq 0}|\varepsilon_n| \leq \frac{1}{e(\theta+1)^2(2+\sqrt{\log\lambda})}.$$

Pisot's ℓ^2-condition on $\{\varepsilon_n\}$ mentioned above can be replaced by an inequality of the type $|\varepsilon_n| \leq c/\sqrt{n}$ for all sufficiently large n with some constant $c > 0$. This type of inequality was first discussed by Gelfond [4] in a more general situation and his result was generalized by Környei [6]. An effective inequality was given by Decomps-Guilloux and Grandet-Hugot [3, Théorème 3.1]; they showed that, if

$$|\varepsilon_n| \leq \frac{1}{2\sqrt{2}\,(\theta+1)^2\sqrt{n}}$$

for $n \geq n_0$, then $\theta \in S$ and $\lambda \in \mathbb{Q}(\theta)$. See also [1, Theorem 5.4.4]. This can now be improved as follows:

THEOREM. *If*

$$\limsup_{n\to\infty}\sqrt{n}\,|\varepsilon_n| < \frac{c}{\theta^2}$$

with

$$c = \exp\left(-\frac{1}{2}\int_0^1 \log\log\left(1+\frac{1}{x}\right)dx\right) = 0.90258454\ldots,$$

then $\theta \in S$ *and* $\lambda \in \mathbb{Q}(\theta)$.

Proof. There exist an integer N and $\alpha \in (0, c)$ satisfying $|\varepsilon_n| < \alpha\theta^{-2}n^{-1/2}$ for all $n \geq N$. Let D_n be the Kronecker determinant for $\{a_n\}$, that is,

$$D_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix}.$$

We have immediately

$$(-1)^{n+1}D_n = \begin{vmatrix} \varepsilon_0-\lambda & \varepsilon_1-\lambda\theta & \cdots & \varepsilon_n-\lambda\theta^n \\ \varepsilon_1-\lambda\theta & \varepsilon_2-\lambda\theta^2 & \cdots & \varepsilon_{n+1}-\lambda\theta^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_n-\lambda\theta^n & \varepsilon_{n+1}-\lambda\theta^{n+1} & \cdots & \varepsilon_{2n}-\lambda\theta^{2n} \end{vmatrix}$$

$$= \begin{vmatrix} \varepsilon_0 & \cdots & \varepsilon_n \\ \varepsilon_1 & \cdots & \varepsilon_{n+1} \\ \vdots & \ddots & \vdots \\ \varepsilon_n & \cdots & \varepsilon_{2n} \end{vmatrix} - \lambda \sum_{i=0}^{n} (-\theta)^i \begin{vmatrix} 1 & \varepsilon_0 & \cdots & \varepsilon_{i-1} & \varepsilon_{i+1} & \cdots & \varepsilon_n \\ \theta & \varepsilon_1 & \cdots & \varepsilon_i & \varepsilon_{i+2} & \cdots & \varepsilon_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \theta^n & \varepsilon_n & \cdots & \varepsilon_{i+n-1} & \varepsilon_{i+n+1} & \cdots & \varepsilon_{2n} \end{vmatrix}.$$

For brevity we denote the determinants on the right-hand side by $A_n, B_n^{(i)}$ respectively. It then follows from Hadamard's inequality that

$$\begin{aligned} |B_n^{(i)}|^2 &\le \Big(\sum_{i=0}^{n} \theta^{2i} \Big) \prod_{\substack{j=0 \\ j\ne i}}^{n} \sum_{k=0}^{n} \varepsilon_{j+k}^2 \\ &< \frac{2\theta^{2n+2}}{\theta^2-1} \left(\frac{\alpha}{\theta^2}\right)^{2(n-N+1)} (n+1)^N \prod_{j=N}^{n} (S_{j+n} - S_{j-1}) \end{aligned}$$

for all $n \ge N$, since $|\varepsilon_m| < 1$ and $S_{j+n} - S_{j-1} > 1/2$, where

$$S_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}.$$

Now we put

$$S_m = \log m + \gamma + \eta_m/m$$

where γ is Euler's constant. Using the formula

$$1 - \gamma = \int_1^\infty \frac{\{t\}}{t^2}\, dt,$$

where $\{t\}$ denotes the fractional part of t, we obtain

$$0 < \eta_m = m(S_m - \log m - \gamma) = m \int_m^\infty \frac{\{t\}}{t^2}\, dt < m \int_m^\infty \frac{dt}{t^2} = 1.$$

Therefore

$$\begin{aligned} \log(S_{j+n} - S_{j-1}) &= \log\left(\log\left(1 + \frac{n}{j}\right) + \frac{\eta_{j+n}}{j+n} + \frac{1-\eta_j}{j} \right) \\ &= \log\log\left(1 + \frac{n}{j}\right) + \frac{\omega_{j,n}}{j}, \end{aligned}$$

where

$$0 < \omega_{j,n} = j \log\left(1 + \frac{\eta_{j+n}/(j+n) + (1-\eta_j)/j}{\log(1+n/j)}\right) < j \log\left(1 + \frac{2}{j\log 2}\right) < \frac{2}{\log 2}$$

for any $N \le j \le n$. We thus have

$$\frac{2}{n} \log |B_n^{(i)}| < 2\log\left(\frac{\alpha}{\theta}\right) + \frac{1}{n} \sum_{j=N}^{n} \log\log\left(1 + \frac{n}{j}\right) + O\left(\frac{\log n}{n}\right) \quad (n \to \infty).$$

The second term on the right-hand side is almost the Riemann sum for the improper integral $\int_0^1 \log\log(1+1/x)\,dx$ and it can be easily verified that the Riemann sum actually converges to this integral. Since the above estimate is uniform in i,

$$\max_{0\le i\le n} |B_n^{(i)}| < \left(\frac{\alpha}{c\,\theta}\right)^n e^{o(n)} \qquad (n\to\infty)$$

and similarly

$$|A_n| < \left(\frac{\alpha}{c\,\theta^2}\right)^n e^{o(n)} \qquad (n\to\infty).$$

Hence we get

$$\begin{aligned} |D_n| &< |A_n| + \lambda\,\frac{\theta^{n+1}}{\theta-1} \max_{0\le i\le n} |B_n^{(i)}| \\ &< \left(1+\frac{\lambda\theta}{\theta-1}\right)\left(\frac{\alpha}{c}\right)^n e^{o(n)} \to 0 \qquad (n\to\infty). \end{aligned}$$

It thus follows that $D_n = 0$ for all sufficiently large n, since $D_n \in \mathbb{Z}$. Finally, the conclusion of the theorem will be obtained by the same method as in Pisot [7]. This completes the proof.

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References

[1] M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse and J.-P. Schreiber, *Pisot and Salem Numbers*, Birkhäuser, Basel, 1992.

[2] D. G. Cantor, *On power series with only finitely many coefficients* (mod 1): *Solution of a problem of Pisot and Salem*, Acta Arith. 34 (1977), 43–55.

[3] A. Decomps-Guilloux et M. Grandet-Hugot, *Nouvelles caractérisations des nombres de Pisot et de Salem*, ibid. 50 (1988), 155–170.

[4] A. O. Gelfond, *On fractional parts of linear combinations of polynomials and exponential functions*, Rec. Math. [Mat. Sb.] 9 (51) (1941), 721–726 (in Russian).

[5] G. H. Hardy, *A problem of Diophantine approximation*, J. Indian Math. Soc. 11 (1919), 162–166.

[6] I. Környei, *On a theorem of Pisot*, Publ. Math. Debrecen 34 (1987), 169–179.

[7] Ch. Pisot, *La répartition modulo* 1 *et les nombres algébriques*, Ann. Scuola Norm. Sup. Pisa (2) 7 (1938), 205–248.

[8] —, *Répartition* (mod 1) *des puissances successives des nombres réels*, Comment. Math. Helv. 19 (1946), 153–160.

[9] Ch. Pisot and R. Salem, *Distribution modulo* 1 *of the powers of real numbers larger than* 1, Compositio Math. 16 (1964), 164–168.

[10] R. Salem, *A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan*, Duke Math. J. 11 (1944), 103–108.

[11] C. L. Siegel, *Algebraic integers whose conjugates lie in the unit circle*, ibid., 597–602.

[12] A. Thue, *Über eine Eigenschaft, die keine transcendente Grösse haben kann*, Kra. Vidensk. Selsk. Skrifter. I. Mat. Nat. Kl. 20 (1912), 1–15; also in: Selected Mathematical Papers of Axel Thue, with an introduction by C. L. Siegel; ed. by T. Nagell *et al.*, Universitetforlaget, Oslo, 1977, 479–491.

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