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Zero spacing distributions for differenced L-functions

by

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1. Introduction. This paper establishes results on the vertical spacing distribution of zeros of certain deformations of Dirichlet L-functions formed using averaging and differencing operators. Study of these zero spacings is motivated in part by the GUE conjecture for vertical zero spacings of L-functions, which we now recall.

There is a great deal of evidence suggesting that the normalized spacings between the nontrivial zeros of the Riemann zeta function have a "random" character described by the eigenvalue statistics of a random Hermitian matrix whose size $N \to \infty$. The resulting statistics are the large N limit of normalized eigenvalue spacings for random Hermitian matrices drawn from the GUE distribution ("Gaussian unitary ensemble"). This limiting distribution is identical to the large N limit of normalized eigenvalue spacings for random unitary matrices drawn from the CUE distribution ("circular unitary ensemble"), i.e. eigenvalues of matrices drawn from $U(N)$ using Haar measure. (The GUE and CUE spacing distributions are not the same for finite N.) More precisely, one compares the normalized spacings of k consecutive zeros with the limiting joint probability distribution of the normalized spacings of k adjacent eigenvalues of random hermitian $N \times N$ matrices, as $N \to \infty$. The relation of zeta zeros with random matrix theory was first suggested by work of H. L. Montgomery [22] which concerned the pair correlation of zeros of the zeta function. Montgomery's results showed (conditionally on the Riemann hypothesis) that there must be some randomness in the spacings of zeros, and were consistent with the prediction of the GUE distribution. A. M. Odlyzko [24] made extensive numerical computations with zeta zeros, now up to height $T = 10^{22}$, which show an extremly impressive fit of zeta zero spacings with predictions of the GUE distribution.

The GUE distribution of zero spacings is now thought to hold for all automorphic L-functions, specifically for principal L-functions attached to

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GL(n) (cf. Katz and Sarnak [13]). Further evidence for this was given in Rudnick and Sarnak [25], conditionally on a suitable generalized Riemann hypothesis. They showed that the evaluation of consecutive zero gaps against certain test functions (of limited compact support) agrees with the GUE predictions. There is also supporting numerical evidence for certain principal L-functions attached to GL(2).

To describe the GUE conjecture, we define the *normalized (vertical) zero spacing* δ_n of the nth zero $\varrho_n = \beta_n + i\gamma_n$ to be

$$\delta_n := (\gamma_{n+1} - \gamma_n)\,\frac{1}{2\pi}\log\frac{|\gamma_n|}{2\pi}.$$

The GUE conjecture prescribes the limiting distribution of any finite set of consecutive normalized zero spacings $(\delta_n, \delta_{n+1}, \ldots, \delta_{n+k})$, with k fixed and $n \to \infty$. For simplicity we will consider the case of consecutive zeros ($k = 1$).

GUE CONJECTURE FOR CONSECUTIVE ZEROS. *The vertical spacings of consecutive normalized zeros $\{\delta_n : 1 \le n \le T\}$ of the Riemann zeta function have a limiting distribution as $T \to \infty$ given by $p(0,u)du$, the limiting distribution for consecutive normalized spacings of eigenvalues for Gaussian random $N \times N$ Hermitian matrices as $N \to \infty$.*

The density $p(0,u)$ is a continuous density supported on the half-line $u \ge 0$ with $p(0,0) = 0$ and which is positive for all $u > 0$. More generally there is a density $p(k-1,u)$ giving the distribution of the spacing to the kth consecutive normalized zero, which is positive for all $u > 0$ (see Mehta and des Cloizeaux [21], Mehta [20], and Odlyzko [24]).

At present there is no structural reason known that would explain why the normalized zeta zeros might obey the GUE distribution. However it is known that the GUE n-level spacing distribution is completely specified by its moments, and some moments of the GUE distribution can be related to (conjectural) distributions of prime k-tuples over various ranges (see Goldston and Montgomery [11] and Montgomery and Soundararajan [23]). Recent work of Conrey and Gamburd [6] shows unconditionally that "pseudo-moments" of partial sums of the zeta function on the critical line exhibit behavior predicted by the GUE distribution.

To gain insight into the origin and stability of the GUE property it seems useful to study the effect of operators on functions that preserve the property of having all zeros on a line. One can then study the effect of such operators on the distribution of local zero spacings. Here we consider certain sum and difference operators constructed using the translation operator $\mathbf{T}_h f(x) := f(x+h)$, such as $\mathbf{A}_h(f)(x) := \frac{1}{2}\left(f(x+h) + f(x-h)\right)$ and $\mathbf{B}_h(f)(x) := \frac{1}{2i}\left(f(x+h) - f(x-h)\right)$. We apply these operators to the Riemann ξ-function, and later to (completed) Dirichlet L-functions.

We obtain families of functions by viewing h as a real parameter. Applied to the Riemann ξ-function $\xi(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, we obtain the family of "averaged" functions $A_h(s) = \frac{1}{2}\left(\xi(s+h) + \xi(s-h)\right)$, where h is a real parameter, and another family, consisting of "differenced" functions: $B_h(s) = -\frac{1}{2i}\left(\xi(s+h) - \xi(s-h)\right)$. In fact, one can naturally insert a second parameter $0 \le \theta < 2\pi$ and consider

$$A_{h,\theta}(s) := \frac{1}{2}\left(\cos\theta(\xi(s+h) + \xi(s-h)) + i\sin\theta(\xi(s+h) - \xi(s-h))\right).$$

There is an analogous extended family $B_{h,\theta}(s)$ associated to $B_h(s)$.

In §2 we show unconditionally that for fixed $|h| \ge 1/2$ the zeros of $A_h(s)$ and $B_h(s)$ lie on the critical line $\Re(s) = 1/2$ and are simple zeros, and they interlace. Assuming the Riemann hypothesis, we show that the same result holds for all nonzero h.

In §3 we show that for all real h the functions in this family have the same asymptotic density of zeros as the zeta function. The results together permit a definition of normalized zero spacings for the functions in this family.

In §4 we establish the main result of the paper (Theorem 4.1). We show that for nonzero h the functions $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$ have a limiting normalized zero spacing distribution for any fixed number k of consecutive zeros, which corresponds to a delta measure with equal spacings of size 1. This is proved unconditionally when $|h| \ge 1/2$, and proved conditionally on the Riemann hypothesis for the remaining range $0 < |h| < 1/2$. These results assert that for $|h| \ne 0$ the normalized zero spacings are completely regular. We conclude that these differencing operations destroy the GUE property. Of course for $h = 0$ the GUE distribution is expected to hold.

In §5 we observe that these results also hold for zeros of (completed) Dirichlet L-functions $\xi(s, \chi)$ for primitive characters χ, with little change in the proofs. The proof method should also extend to various automorphic L-functions for $\mathrm{GL}(N)$, for $N \ge 2$.

In §6 we interpret the results of §2 in terms of the de Branges theory of Hilbert spaces of entire functions. This was another motivation for this work, and the notations in this paper are compatible with de Branges's theory, under the change of variable $s = 1/2 - iz$. The interpretation is that these sum and differencing operators applied to L-functions produce structure functions of de Branges Hilbert spaces of entire functions when $|h| \ge 1/2$, and, conditionally on the Riemann hypothesis, for all nonzero h. For example, Lemma 2.1 asserts that $E_h(1/2 - iz) := \xi(1/2 + h - iz)$ is a de Branges structure function, under these hypotheses. The particular case with $h = 1/2$ gives the structure function $E(z) = \xi(1 - iz)$ which was discussed in 1986 by de Branges [3]. We describe some features of the de Branges theory, and observe that it provides a "spectral" interpretation of the zeros $1/2 + i\gamma$ of the functions $A_{h,\theta}(s)$ (resp. $B_{h,\theta}(s)$). It produces for each θ an unbounded self-adjoint operator on a Hilbert space having the negative imaginary parts of the zeros (that is, $-\gamma_n$) as its spectrum.

In §7 we make concluding remarks and suggest some directions for further work. In particular there likely exist random matrix analogues of these results.

We make some comments regarding the proofs. In Theorem 2.1 we establish simplicity of zeros, all lying on the critical line, using a function-theoretic result of de Branges [1] which applies generally to all de Branges structure functions. For completeness we include a self-contained proof of this result (Lemma 2.2). The results of §3 are standard. The main result of §4 is proved by viewing the zeros of $E_{h,\theta}(s) = A_h(s) - iB_h(s)$ as perturbations of zeros of the real part of the archimedean factor contribution on the critical line. These zeros are regularly spaced, and under the given hypotheses the perturbations are shown to be sufficiently small not to disturb these spacings significantly.

Concerning prior work, Xian-Jin Li informs me that the results in §2 and §5 were known to de Branges in the late 1980's. These include Lemma 2.1, Theorem 2.1 and their extensions to Dirichlet L-functions stated in §5. De Branges reportedly covered some of these results in a lecture course on Hilbert spaces of entire functions given at Purdue in 1987-1988. In this regard, the present paper serves to make proofs of these results available. Recently Haseo Ki [14] obtained results analogous to those in §2 for averagings of the meromorphic function $\widehat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = 2\xi(s)/s(s-1)$, e.g. for $h \geq 1/2$ all zeros of $\widetilde{A}_h(s) = \frac{1}{2}(\widehat{\zeta}(s+h) + \widehat{\zeta}(s-h))$ lie on the critical line. Ki's methods should also apply to the function $\xi(s)$.

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2. Differenced Riemann ξ-function. The Riemann ξ-function $\xi(s)$ is given by

$$\xi(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It is real on the real axis. More importantly here, it is real on the critical line $\Re(s) = 1/2$, and satisfies the functional equation $\xi(s) = \xi(1-s)$. Its zeros are confined to the (open) critical strip $0 < \Re(s) < 1$, and as an entire function of order 1 it has the Hadamard factorization

$$\xi(s) = e^{A(\chi_0)+B(\chi_0)s} \prod_{\varrho} \left(1 - \frac{s}{\varrho}\right) e^{s/\varrho}.$$

The spacing of its zeros is asymptotically regular, and this permits one to show that the convergence factors $e^{s/\varrho}$ can be factored out into the lead term, if the product is taken in a suitable order, for example considering partial products over all terms with $|\varrho| < T$ and taking a limit as $T \to \infty$. In that case one obtains a modified Hadamard product representation

$$\xi(s) = e^{A^*(\chi_0)+B^*(\chi_0)s} \prod_{\varrho,*} \left(1 - \frac{s}{\varrho}\right), \tag{2.1}$$

where the asterisk in the product indicates that its terms are to be combined in a suitable order.

For the ξ function it suffices to group all zeros in pairs $(\varrho, 1-\varrho)$ if on the critical line, or in quadruples $(\varrho, 1-\varrho, \overline{\varrho}, 1-\overline{\varrho})$ otherwise, to get absolute convergence of the modified Hadamard product (2.1). We have

$$\xi(s) = \lim_{T\to\infty} e^{A^*(\chi_0)+B^*(\chi_0)s} \prod_{|\varrho|<T} \left(1 - \frac{s}{\varrho}\right).$$

It is known that

$$A(\chi_0) = A^*(\chi_0) = \log \xi(0) = -\log 2$$

and $B(\chi_0)$ is real with

$$B(\chi_0) = -\frac{1}{2}\gamma - 1 + \frac{1}{2}\log 4\pi \approx -0.023,$$

where γ is Euler's constant (see Davenport [9, p. 83]). In addition $B(\chi_0) = -\sum_{\varrho} 1/\varrho$ where the sum takes the zeros in complex conjugate pairs, and from this one deduces that

$$B^*(\chi_0) = 0. \tag{2.2}$$

For the purposes here, what is important is that $\Re(B^*(\chi_0)) \geq 0$ (see Lemma 2.1).

In this section we consider for a real parameter h the functions

$$A_h(s) := \frac{1}{2}\left(\xi(s+h) + \xi(s-h)\right),$$
$$B_h(s) := -\frac{1}{2i}\left(\xi(s+h) - \xi(s-h)\right).$$

We have

$$\xi(s-h) = \overline{\xi(s+h)},$$

which follows from the reflection symmetry of $\xi(s)$ around the line $\Re(s) = 1/2$, i.e.

$$\xi(1/2 - x + iy) = \overline{\xi(1/2 + x - iy)}.$$

It follows that $A_h(s)$ and $B_h(s)$ are both real on the critical line and satisfy there

$$A_h(1/2+it) = \Re(\xi(1/2+h+it)), \qquad B_h(1/2+it) = -\Im(\xi(1/2+h+it)).$$

In particular $A_0(s) = \xi(s)$. It is also immediate that $A_{-h}(s) = A_h(s)$ and $B_{-h}(s) = -B_h(s)$, so that without loss of generality we need only consider $h \geq 0$. The fact that $\xi(s)$ is real on the real axis gives rise to the extra symmetry $A_h(\overline{s}) = \overline{A_h(s)}$, and similarly for $B_h(s)$.

Lemma 2.1.

(1) *If* $h \geq 1/2$, *then*

$$|\xi(h+s)| > |\xi(h+1-\overline{s})| \qquad \text{for } \Re(s) > 1/2. \tag{2.3}$$

(2) *Assuming the Riemann hypothesis*, *the inequality* (2.3) *holds for each* $h > 0$.

Remark. The inequality (2.3) can be stated alternatively (and apparently more elegantly) as

$$|\xi(h+s)| > |\xi(h+(1-s))|,$$

making use of the reflection symmetry of $\xi(s)$ around the real axis, i.e. $\xi(\overline{s}) = \overline{\xi(s)}$. However the restated form does not generalize to Dirichlet L-functions, which generally do not have this symmetry. In the form stated the lemma generalizes to Dirichlet L-functions, with the relevant property being that the (completed) L-functions have constant phase on the critical line.

Proof of Lemma 2.1. We show that the inequality holds term by term for each factor in the modified Hadamard product (2.1) for $\xi(s)$. For the exponential factor, let $s = \sigma + it$, and we have

$$|e^{B^*(\chi_0)(h+s)}| = e^{\Re(B^*(\chi_0))(h+\sigma)} e^{-\Im(B^*(\chi_0))t},$$
$$|e^{B^*(\chi_0)(h+1-\overline{s})}| = e^{\Re(B^*(\chi_0))(h+1-\sigma)} e^{-\Im(B^*(\chi_0))t}.$$

The condition $\Re(B^*(\chi_0)) \geq 0$ is sufficient to imply that for $\sigma > 1/2$,

$$|e^{B^*(\chi_0)(h+s)}| \geq |e^{B^*(\chi_0)(h+1-\overline{s})}|,$$

as desired. In fact $B^*(\chi_0) = 0$ by (2.2).

Now we consider the product factors for each zero separately. Let $\varrho = \beta + i\gamma$ be a zero of $\xi(s)$, so that $0 < \beta < 1$, and under the Riemann hypothesis $\beta = 1/2$. Now set $s = \sigma + it$, where we will suppose $\sigma > 1/2$, and we will show under the stated hypotheses that

$$\left|1 - \frac{h+s}{\varrho}\right| > \left|1 - \frac{h+1-\overline{s}}{\varrho}\right|. \tag{2.4}$$

Multiplying by $|\varrho|$ gives the equivalent inequality

$$|\beta - h - \sigma + i(\gamma - t)| > |\beta - h - 1 + \sigma + i(\gamma - t)|,$$

and this in turn is equivalent to the inequality

$$|\beta - h - \sigma| > |\beta - h - (1 - \sigma)|. \tag{2.5}$$

(1) Suppose $h \ge 1/2$, so that $\beta - h - 1/2 < 0$. Then one has

$$|\beta - h - \sigma| = |(\beta - h - 1/2) - (\sigma - 1/2)| = |\beta - h - 1/2| + |\sigma - 1/2|,$$

while

$$|\beta - h - 1 + \sigma| = |(\beta - h - 1/2) + (\sigma - 1/2)| < |\beta - h - 1/2| + |\sigma - 1/2|.$$

The application of the triangle inequality is strict because both terms in it are nonzero and have opposite signs. This establishes (2.5), hence (2.4), in this case.

(2) Now assume RH, so that $\beta = 1/2$, and suppose $h > 0$. Now $\beta - \sigma = 1/2 - \sigma < 0$, and then

$$|\beta - h - \sigma| = |(\beta - \sigma) - h| = |\beta - \sigma| + |h|,$$

while

$$|\beta - h - (1 - \sigma)| = |(\sigma - 1/2) - h| < |\beta - \sigma| + |h|.$$

The application of the triangle inequality in the last line is strict because both terms in it are nonzero and of opposite sign. This establishes (2.5), hence (2.4), in this case. ∎

The next lemma, due to de Branges [1, Lemma 5], formulates a basic property underlying the de Branges theory of Hilbert spaces of entire functions. The de Branges theory is formulated in terms of a variable z with $s = 1/2 - iz$ and considers functions satisfying $|E(z)| > |E^\sharp(z)|$ when $\Im(z) > 0$, where $E^\sharp(z) := \overline{E(\overline{z})}$ is an involution acting on entire functions. Here we re-express de Branges's result in terms of the s-variable, and the involution becomes $E^\sharp(s) = \overline{E(1 - \overline{s})}$.

Lemma 2.2. *Let $E(s)$ be an entire function that satisfies*

$$|E(s)| > |E(1 - \overline{s})| \quad \text{when } \Re(s) > 1/2. \tag{2.6}$$

Write $E(s) = A(s) - iB(s)$ with

$$A(s) = \frac{1}{2}\,(E(s) + \overline{E(1 - \overline{s})}), \qquad B(s) = -\frac{1}{2i}\,(E(s) - \overline{E(1 - \overline{s})}),$$

so that $A(s)$ and $B(s)$ are real-valued on the critical line $\Re(s) = 1/2$. Then $A(s)$ and $B(s)$ have all their zeros lying on the critical line $\Re(s) = 1/2$, and these zeros interlace.

Remarks. (1) The interlacing property allows $A(s)$ and $B(s)$ to have multiple zeros. By "interlacing of zeros" we mean there is a numbering of zeros of the two functions, $\{\varrho_n(A) = 1/2 + \gamma_n(A) : n \in \mathbb{Z}\}$ and $\{\varrho_n(B) = 1/2 + \gamma_n(B) : n \in \mathbb{Z}\}$, with imaginary parts in increasing order, counting zeros with multiplicity, such that

$$\gamma_n(A) \leq \gamma_n(B) \leq \gamma_{n+1}(A)$$

holds for all allowed n.

(2) There are no growth restrictions on the maximum modulus of functions $E(s)$ in Lemma 2.2. It can be shown that there exist entire functions $E(s)$ of fast growth, for example of infinite order, satisfying the condition (2.6).

Proof of Lemma 2.2. The definition of $A(s)$ and $B(s)$ shows that for real α they have the symmetry

$$A(1/2 - \alpha + it) = \overline{A(1/2 + \alpha + it)}, \tag{2.7}$$

and similarly $B(1/2 - \alpha + it) = \overline{B(1/2 + \alpha + it)}$. Consequently, $A(s)$ and $B(s)$ are real-valued on the critical line $\Re(s) = 1/2$.

To see that $A(s)$ has all its zeros on the critical line, we observe that for $\Re(s) > 1/2$, (2.6) gives $|A(s)| = |E(s) + \overline{E(1-\overline{s})}| \geq |E(s)| - |E(1-\overline{s})| > 0$. If $\Re(s) < 1/2$ then $\Re(1-\overline{s}) > 1/2$ and (2.6) gives

$$|A(s)| = |E(s) + \overline{E(1-\overline{s})}| \geq |E(1-\overline{s})| - |E(1-\overline{1-\overline{s}})| > 0.$$

This establishes that all zeros of $A(s)$ lie on the critical line, and the proof that $B(s)$ has the same property is similar.

To see that the zeros of $A(s)$ interlace with those of $B(s)$ on the critical line (counting multiplicities), we observe that the property (2.6) implies for real $\alpha > 0$ that

$$\lim_{\alpha \to 0^+} \frac{|E(1/2 + \alpha + it)|^2 - |E(1/2 - \alpha + it)|^2}{2\alpha} \geq 0.$$

On inserting $E(s)$ and its conjugate $\overline{E(s)}$ into this and using the symmetry (2.7) for $A(s)$ and $B(s)$ a calculation yields, for real $\alpha > 0$, on letting $s_\alpha = 1/2 + \alpha + it$,

$$\begin{aligned}
&\frac{|E(1/2 + \alpha + it)|^2 - |E(1/2 - \alpha + it)|^2}{2\alpha} \\
&\qquad = i\left(\frac{A(s_\alpha)\overline{B(s_\alpha)} - \overline{A(s_\alpha)}B(s_\alpha)}{\alpha}\right) \\
&\qquad = -iA(s_\alpha)\left(\frac{B(s_\alpha) - \overline{B(s_\alpha)}}{\alpha}\right) + iB(s_\alpha)\left(\frac{A(s_\alpha) - \overline{A(s_\alpha)}}{\alpha}\right).
\end{aligned}$$

Letting $\alpha \to 0^+$ we deduce that for $s = 1/2 + it$,

$$\left(i\frac{d}{ds}A(s)\right)B(s) - \left(i\frac{d}{ds}B(s)\right)A(s) \geq 0. \tag{2.8}$$

The left hand side of this inequality is real-analytic in t so it can vanish only at isolated points, provided it is not identically zero, and this would give the interlacing property. It remains to establish that it is not identically zero.

We can rephrase the result (2.8) in terms of a continuous real-valued "phase function" $\phi(t)$ defined by

$$E(s) := |E(1/2+it)|e^{i\phi(t)}, \tag{2.9}$$

where we choose the unique allowed value $0 \le \phi(0) < 2\pi$, and then extend it to all t by continuity. Now (2.6) implies that $E(s)$ has no zeros in $\Re(s) > 1/2$ so we can define a single-valued version of $\log E(s)$ there, such that $\phi(t) = \Im(\log E(1/2+it))$. Using $A(s) = |E(s)|\cos\phi(t)$ and $B(s) = -|E(s)|\sin\phi(t)$ and $\frac{d}{ds} = i\frac{d}{dt}$ on the line $s = 1/2+it$ one finds that the inequality (2.8) simplifies to

$$\left|E\left(\frac{1}{2}+it\right)\right|^2 \frac{d}{dt}\,\phi(t) \ge 0. \tag{2.10}$$

To show the left side is not identically zero, it suffices to show that $\phi(t)$ is nonconstant. We argue by contradiction. If it were constant, say $E(s) = |E(s)|e^{i\alpha}$, then the function $\widetilde{E}(s) = E(s)e^{-i\alpha}$ would be real-valued on the real axis, and it continues to satisfy (2.6). However the reflection principle would then give $\widetilde{E}(\overline{s}) = \overline{\widetilde{E}(s)}$, which contradicts (2.6). We conclude that the real-analytic function $\phi(t)$ is nonconstant, so $\frac{d}{dt}\phi(t)$ has isolated zeros, and (2.10) implies that the function $\phi(t)$ is a strictly increasing function of t. This certifies that the zeros of $A(t)$ and $B(t)$ interlace. ∎

If a function $E(s) = A(s) - iB(s)$ satisfies the hypothesis of Lemma 2.2 then so does the function $E_\theta(s) = e^{i\theta}E(s)$, when $0 \le \theta < 2\pi$. It has the decomposition $E_\theta(s) = A_\theta(s) - iB_\theta(s)$, where

$$A_\theta(s) = (\cos\theta)A(s) + (\sin\theta)B(s), \qquad B_\theta(s) = -(\sin\theta)A(s) + (\cos\theta)B(s).$$

In particular, $B_\theta(s) = A_{\theta+\pi/2}(s)$.

We now specialize to the case of the shifted ξ-function $E_h(s) = \xi(s+h)$, for which $E_h(s) = A_h(s) - iB_h(s)$ with

$$\begin{aligned} A_h(s) &:= \frac{1}{2}\left(\xi(s+h) + \overline{\xi(1-\overline{s}+h)}\right) = \frac{1}{2}\left(\xi(s+h) + \xi(s-h)\right), \\ B_h(s) &:= -\frac{1}{2i}\left(\xi(s+h) - \overline{\xi(1-\overline{s}+h)}\right) = -\frac{1}{2i}\left(\xi(s+h) - \xi(s-h)\right). \end{aligned}$$

Here we used $\xi(\overline{s}) = \overline{\xi(s)}$ and the functional equation $\xi(s) = \xi(1-s)$. We consider

$$E_{h,\theta}(s) = e^{i\theta}E_h(s) = e^{i\theta}\xi(s+h),$$

and obtain

$$
\begin{aligned}
A_{h,\theta}(s) &:= (\cos\theta)A_h(s) + (\sin\theta)B_h(s)\\
&= \frac{1}{2}(\cos\theta)(\xi(s+h)+\xi(s-h)) - \frac{1}{2i}(\sin\theta)(\xi(s+h)-\xi(s-h)),\\
B_{h,\theta}(s) &:= (-\sin\theta)A_h(s) + (\cos\theta)B_h(s)\\
&= -\frac{1}{2}(\sin\theta)(\xi(s+h)+\xi(s-h)) - \frac{1}{2i}(\cos\theta)(\xi(s+h)-\xi(s-h)).
\end{aligned}
$$

For $\theta = 0$ we recover $A_{h,0}(s) = A_h(s)$ and $B_{h,0}(s) = B_h(s)$.

THEOREM 2.1.

(1) *For $|h| \ge 1/2$ and any $0 \le \theta < 2\pi$, the entire functions $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$ have all their zeros on the critical line $\Re(s) = 1/2$. These zeros are all simple zeros, and they interlace.*

(2) *Assuming the Riemann hypothesis, for $0 < |h| < 1/2$ and any $0 \le \theta < 2\pi$ the functions $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$ have all their zeros on the critical line $\Re(s) = 1/2$. These zeros are all simple zeros, and they interlace.*

Proof. Lemma 2.1 shows that under the stated assumptions the function $E(s) := E_h(s) = \xi(s+h)$ satisfies the hypothesis (2.6) of Lemma 2.2, with $A(s) = A_h(s)$ and $B(s) = B_h(s)$. The same holds more generally for $E_{h,\theta}(s) = e^{i\theta}E_h(s)$. Applying Lemma 2.2 we conclude in both cases (1) and (2) that $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$ have all their zeros on the critical line $\Re(s) = 1/2$, and these zeros interlace.

To conclude that these zeros are all simple zeros, we argue by contradiction. Suppose that one of $A_{h,\theta}(s)$ or $B_{h,\theta}(s)$ had a multiple zero at a point s_0. Then by the interlacing property they necessarily have a common zero at s_0. Since this point is on the critical line, we infer that the real and imaginary parts of $e^{i\theta}\xi(s_0+h)$ both vanish, whence $\varrho_0 = s_0 + h$ is a zero of $\xi(s)$. In case (1), $\Re(\varrho_0) = \Re(s_0) + h = 1/2 + h \ge 1$, a contradiction. In case (2), assuming RH, $h > 0$ gives $\varrho_0 = 1/2 + h > 1/2$, a contradiction. ∎

3. Global asymptotics of zeros. We determine the asymptotic zero density of functions $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$. The proof method is standard, using the argument principle. Let $N(T,F)$ count the number of zeros of $F(s)$ having $|\Im(s)| \le T$.

THEOREM 3.1.

(1) *For $|h| \ge 1/2$ and any $0 \le \theta < 2\pi$, for all $|T| \ge 2$ we have*

$$N(T, A_{h,\theta}) = \frac{1}{\pi}T\log T - \frac{1}{\pi}(\log(2\pi)+1)T + O(\log T), \tag{3.1}$$

where the implied constant in the O-notation is independent of h and θ. A similar formula holds for $N(T, B_{h,\theta})$.

(2) *Assuming the Riemann hypothesis, the formula* (3.1) *for* $N(T, A_{h,\theta})$ *is valid for all nonzero* h. *A similar formula holds for* $N(T, B_{h,\theta})$ *for all nonzero* h.

Proof. We study $\xi(s+h)$ in a half-plane $\Re(s+h) > \sigma_0$ in which it has no zeros, where $\sigma_0 = 1$ unconditionally and $\sigma_0 = 1/2$ if the Riemann hypothesis is assumed. Under these conditions we can choose a single-valued branch of $\log \xi(s)$ in this half-plane, and we choose the branch which is real-valued on the real axis. We set

$$\xi(1/2 + h + it) = |\xi(1/2 + h + it)| e^{i\varphi_h(t)},$$

in which

$$\varphi_h(t) = \arg(\xi(1/2 + h + it)) = \Im(\log \xi(1/2 + h + it)).$$

The function $E_h(s) = \xi(s+h)$ satisfies the conditions of Lemma 2.2 and its proof shows that $\varphi_h(t)$ is a strictly increasing function of t. By Theorem 2.1 the zeros of $A_h(s)$ and $B_h(s)$ are simple and lie on the critical line. We write the zeros of $A_{h,\theta}(s)$ as $\varrho_n(A_{h,\theta}) = 1/2 + i\gamma_n(A_{h,\theta})$, enumerated in order as

$$\cdots < \gamma_{-1}(A_{h,\theta}) < 0 \le \gamma_0(A_{h,\theta}) < \gamma_1(A_{h,\theta}) < \cdots,$$

and we denote the zeros of $B_{h,\theta}(s)$ similarly, as $\varrho_n(B_{h,\theta}) = 1/2 + i\gamma_n(B_{h,\theta})$. In terms of the argument $\varphi_h(t)$, these zeros comprise all the solutions to

$$\varphi_h(\gamma_n(A_{h,\theta})) \equiv \frac{\pi}{2} + \theta \pmod{\pi} \quad \text{and} \quad \varphi_h(\gamma_n(B_{h,\theta})) \equiv \theta \pmod{\pi}.$$

In particular $\varphi_h(\gamma_n(A_{h,\theta})) = n\pi + O(1)$ and similarly for $\varphi_h(\gamma_n(B_{h,\theta}))$.

Because all zeros are on the critical line and the argument $\varphi_h(t)$ is strictly increasing there, to estimate $N_h(T)$ it suffices to bound the change in argument. By definition we have

$$\arg(\xi(s)) = \arg(s(s-1)) + \arg(\pi^{-s/2}\Gamma(s/2)) + \arg(\zeta(s)). \tag{3.2}$$

The contribution to the argument mainly comes from the Gamma function factor, which we deal with first.

We let $\widetilde{\varphi}_h(t)$ denote the argument of

$$G_h(t) := \pi^{-(1/4+h/2+it/2)}\Gamma\left(\frac{1}{4} + \frac{h}{2} + \frac{it}{2}\right),$$

normalized by the condition that it be zero on the positive real axis. It can be checked that $\widetilde{\varphi}_h(t)$ is a strictly increasing function of t. Stirling's formula is valid in any sector $-\pi + \delta < \arg(s) < \pi - \delta$ with $\delta > 0$, and asserts that

$$\log \Gamma(s) = (s - 1/2)\log s - s + \tfrac{1}{2}\log 2\pi + O(1/|s|), \tag{3.3}$$

where the O-constants depend on $\delta > 0$. We choose $\delta = \pi/2$, so can use the formula on the half-plane $\Re(s) > 0$. Letting $s = 1/2 + h + it$, and using $\arg(\pi^{-s/2}) = -\big(\tfrac{1}{2}\log 2\pi\big)t + O(1)$ we obtain, for $t \ge 0$,

$$\widetilde{\varphi}_h(t) = \frac{t}{2}\log\frac{t}{2} - \frac{t}{2}(\log 2\pi + 1) + \frac{\pi}{2}\left(-\frac{1}{4} + \frac{h}{2}\right) + O\left(\frac{1}{|t|+1}\right), \tag{3.4}$$

and we have $\widetilde{\varphi}_h(-t) = -\widetilde{\varphi}_h(t)$.

We estimate the remaining terms on the right side of (3.2). The first term on the right for $t \geq 1$ satisfies

$$\arg(s(s-1)) = \pi + O\left(\frac{1}{|t|+1}\right). \tag{3.5}$$

On taking $s = 1/2 + h + it$, we deduce that

$$\varphi_h(t) = \widetilde{\varphi}_h(t) + \pi + \arg(\zeta(1/2 + h + it)) + O\left(\frac{1}{|t|+1}\right).$$

It is a standard estimate that in a half-plane $\Re(s) > 1/2 + h$ containing no zeros

$$\arg(\zeta(1/2 + h + it)) = O(\log|t|)$$

(see Titchmarsh [26, Theorem 9.4] or Davenport [9, Chap. 15]). (It is based on bounding $\Re((\zeta'/\zeta)(s))$.) Substituting these estimates in (3.2) together with (3.4) yields, for $|t| \geq 2$,

$$\varphi_h(t) = \frac{t}{2}\log\frac{t}{2} - \frac{t}{2}(\log 2\pi + 1) + O(\log|t|).$$

Using

$$\left| N_h(T) - \frac{1}{\pi}\left|(\varphi_h(T) - \varphi_h(-T))\right| \right| \leq 2,$$

the bounds in (1) and (2) follow. ∎

We remark that for $0 < h < 1/2$ the asymptotics of zeros of the real part (resp. imaginary part) of $\xi(s)$ can be established unconditionally, with a worse error term. Haseo Ki [14] showed analogous results for the meromorphic function $\widehat{\zeta}(s) = 2\xi(s)/s(s-1)$; an earlier result of Levinson [17] gave the lower bound. Let $\widetilde{N}_h(T)$ count the number of zeros of the averaged function

$$\widetilde{F}_h(s) := \tfrac{1}{2}\left(\widehat{\zeta}(s+h) + \widehat{\zeta}(s-h)\right)$$

having imaginary part between 0 and T. Then Ki [14, p. 290] shows that for $h > 0$,

$$\widetilde{N}_h(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\widetilde{R}_h(T)\log T), \tag{3.6}$$

in which $\widetilde{R}_h(T)$ counts the number of zeros of $\zeta(s)$ in $\Re(s) > 1/2 + h$ with imaginary part between 0 and T. It is known that $\widetilde{R}_h(T) = O(T^{3(1-2h)/(3-2h)}(\log T)^5)$ for $0 < h < 1/2$ (Titchmarsh [26, Theorem 9.19(B)]). Ki's methods apply to give similar results for the function $\xi(s)$ as well, which unconditionally imply (3.1) replaced with the larger remainder term $O(\widetilde{R}_h(T)\log T)$.

4. Local spacing statistics of zeros. In general we can define local spacing statistics of zeros of an entire function $G(z)$ having all its zeros on the real axis, as follows. Let $\{\gamma_i : i \in \mathbb{Z}\}$ enumerate the zeros in increasing order, with γ_0 denoting the zero nearest the origin, and let $N(T)$ denote the number of such zeros in $-T \le x \le T$. We consider the set of differences of consecutive zeros

$$\Sigma(T) := \left\{(\gamma_{n+1} - \gamma_n)\,\frac{N(T)}{T} : |\gamma_n| \le T\right\},$$

normalized to have average spacing 1 over this interval. We put the uniform probability distribution to $\Sigma(T)$, i.e. assign equal weights to each observation. We say that the function $G(z)$ has *limiting local spacing statistics* if as $T \to \infty$ the distributions $\Sigma(T)$ weakly converge to a limiting probability distribution Σ on the real line. Here weak convergence means that for each finite interval $[A, B]$ with $0 \le A < B$ the fraction of the distribution $\Sigma(T)$ that falls in that interval converges as $T \to \infty$ to the fraction that Σ assigns that interval. More generally we consider distributions of k consecutive zero spacings $\Sigma_k(T)$ which sample $(\gamma_{n+1} - \gamma_n, \dots, \gamma_{n+k} - \gamma_{n+k-1})N(T)/T$.

We can apply this definition of local spacing distributions to the functions here by using the linear change of variable $s = 1/2 - iz$, which moves the critical line to the real axis. The definition above, which works for an arbitary set of zeros, differs slightly from the normalization of zero spacings used in Odlyzko [24]. Recall that he used the *normalized consecutive zero spacing*

$$\delta_n := (\gamma_{n+1} - \gamma_n)\,\frac{1}{2\pi}\log\frac{|\gamma_n|}{2\pi}.$$

Using the asymptotics

$$N_h(T) = \frac{1}{\pi}\,T\log\frac{T}{2\pi e} + O(\log T)$$

given by Theorem 3.1, we have

$$\gamma_n = 2\pi\,\frac{n}{\log n} + O\left(\frac{n}{(\log n)^2}\right) \quad \text{as } n \to \infty,$$

and this yields

$$\delta_n = (\gamma_{n+1} - \gamma_n)\,\frac{N(T)}{T} + O\left(\frac{1}{\log\log T}\right),$$

valid over the range $T \le n \le N(T)$. This bound is sufficient to imply that the set

$$\Sigma^*(T) := \{\delta_n : |\gamma_n| < T\},$$

assigned the uniform probability distribution on its $N(T)$ elements, will have the same limiting distribution as that of $\Sigma(T)$ as $T \to \infty$. This holds in the sense that, if either $\Sigma(T)$ or $\Sigma^*(T)$ has a limiting distribution as $T \to \infty$, then so does the other, and they agree.

The main observation of this paper is the following.

Theorem 4.1. *Let $k \geq 1$ be given.*

(1) *For $|h| \geq 1/2$ and $0 \leq \theta < 2\pi$ the functions $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$ both have a limiting joint distribution as $T \to \infty$ of sets of k normalized consecutive zero spacings $(\delta_n, \delta_{n+1}, \ldots, \delta_{n+k-1})$ in $[-T, T]$. This distribution of normalized zero spacings consists of a delta function located at $(x_1, \ldots, x_k) = (1, \ldots, 1)$, the "trivial" distribution.*

(2) *Assuming the Riemann hypothesis, the same results also hold for $0 < |h| < 1/2$ and $0 \leq \theta < 2\pi$, with the limiting distribution of normalized zero spacings existing and being the "trivial" distribution.*

The idea of the proof is that the zeros for $\xi_h(1/2 + it)$ can be viewed as a perturbation of "zeros" associated to the argument of the Γ-factor term

$$G_h(t) = \pi^{-(1/4+h/2+it/2)}\Gamma\left(\frac{1}{4} + \frac{h}{2} + \frac{it}{2}\right).$$

The "zeros" are values $\arg(G_h(t)) \equiv \alpha \pmod{\pi}$ for a fixed value of α. The spacing of these points is extremely regular, as determined by Stirling's formula. Under the given hypotheses the perturbation coming from the zeta function contribution is sufficiently "small" that the spacings of the perturbed zeros remain the same in the main term of their asymptotics. The argument breaks down at $h = 0$, as it must, assuming the GUE conjecture is valid.

As a preliminary, we collect needed estimates on $(\zeta'/\zeta)(s)$. Let

$$R_h(T) := \sup_{T \leq t \leq T+1} \left|\frac{\zeta'}{\zeta}(1/2 + h + it)\right|.$$

Lemma 4.1. *The function $R_h(T)$ satisfies the following bounds:*

(1) *For $h > 1/2$,*
$$R_h(T) = O(1),$$
where the O-constant depends on h.

(2) *For $h = 1/2$, and $|T| \geq 2$,*
$$R_h(T) = O\left(\frac{\log T}{\log\log T}\right).$$

(3) *Assuming the Riemann hypothesis, for $0 < h \leq 1/2$,*
$$R_h(T) = O((\log T)^{1-2h}),$$
where the O-constant depends on h.

Proof. (1) follows from logarithmic differentiation of the Euler product for $\zeta(s)$, noting that $h > 1/2$ is the region of absolute convergence. (2) is shown in Titchmarsh [26, 5.17.4], in which the condition $|T| \geq 2$ is present to avoid the pole at $s = 1$. (3) is shown in Titchmarsh [26, 14.5.1]. ∎

Proof of Theorem 4.1. Recall that

$$\begin{aligned}\varphi_h(t) &= \arg(\xi(1/2+h+it)) = \Im(\log \xi(1/2+h+it)),\\ \widetilde{\varphi}_h(t) &= \arg(G_h(t)) = \arg\left(\pi^{-(1/4+h/2+it/2)}\Gamma\left(\frac{1}{4}+\frac{h}{2}+\frac{it}{2}\right)\right).\end{aligned}$$

Under the hypotheses of the theorem (either (1) or (2)) both these functions are strictly increasing functions of t. The proof of Theorem 3.1 gives

$$\varphi_h(t) = \widetilde{\varphi}_h(t) + \pi + \arg(\zeta(1/2+h+it)) + O\left(\frac{1}{|t|+1}\right).$$

From the formula (3.4) for the argument of the Gamma factor we deduce, for $0 \le a \le 1$ and $t \ge 0$, that

$$\begin{aligned}&\widetilde{\varphi}_h(t+a) - \widetilde{\varphi}_h(t) \tag{4.1}\\ &= \frac{a}{2}\log\frac{t+a}{2} + \frac{t}{2}\log\left(1+\frac{a}{2t}\right) + \frac{a}{2}\log\frac{\pi}{e} + O\left(\frac{1}{|t|+1}\right)\\ &= \frac{a}{2}\log t + \frac{a}{2}\left(\frac{1}{2} - \log\frac{\pi}{2e}\right) + O\left(\frac{a^2+1}{|t|+1}\right) = \frac{a}{2}\log t + O\left(a + \frac{1}{|t|+1}\right).\end{aligned}$$

We now define $\widetilde{\gamma}_n(G_h)$ by

$$\widetilde{\varphi}_h(\widetilde{\gamma}_n(G_h)) = \pi/2 + n\pi.$$

From the regular variation of this argument we can deduce for $t \ge 1$ that

$$\widetilde{\gamma}_{n+1}(G_h) - \widetilde{\gamma}_n(G_h) = \frac{2\pi}{\log t} + O\left(\frac{1}{(\log t)^2}\right), \tag{4.2}$$

although we will not directly use this in what follows.

The proof of Theorem 3.1 deduces from (3.2) and (3.5) that

$$\varphi_h(t) = \widetilde{\varphi}_h(t) + \pi + \arg(\zeta(1/2+h+it)) + O\left(\frac{1}{|t|+1}\right).$$

We compare these quantities at t and $t+a$, with $0 \le a \le 1$, $t \ge 1$, to obtain

$$\begin{aligned}\varphi_h(t+a) - \varphi_h(t) &= (\widetilde{\varphi}_h(t+a) - \widetilde{\varphi}_h(t))\\ &\quad + \Delta\arg(\zeta(1/2+h+iu))\Big|_{u=t}^{u=t+a} + O\left(\frac{1}{|t|+1}\right).\end{aligned}$$

The change in argument of $\zeta(s)$ is estimated by

$$\begin{aligned}\Delta\arg\left(\zeta\left(\frac{1}{2}+h+iu\right)\right)\Big|_{u=t}^{u=t+a} &\le \left|\int_t^{t+a} \Re\left(\frac{\zeta'}{\zeta}(1/2+h+iu)\right)du\right|\\ &\le \int_t^{t+a}\left|\frac{\zeta'}{\zeta}(1/2+h+iu)\right|du = O(aR_h(t)).\end{aligned}$$

We bound the term $R_h(t)$ using Lemma 4.1. Using (4.1) we find for $0 < a \le 1$ that

$$\varphi_h(t+a) - \varphi_h(t) = \frac{a}{2}\log t + \widetilde{R}_h(t) + O\left(\frac{1}{|t|+1}\right), \tag{4.3}$$

in which the remainder term $\widetilde{R}_h(t)$ satisfies

$$\widetilde{R}_h(t) = \begin{cases} O(a) & \text{if } |h| > 1/2, \\ O\left(\dfrac{a\log t}{\log\log t}\right) & \text{if } |h| = 1/2, \\ O(a(\log t)^{1-2|h|}) & \text{if RH holds and } 0 < |h| \le 1/2. \end{cases}$$

The global spacing in Theorem 3.1 gives

$$\gamma_n(A_{h,\theta}) = 2\pi\,\frac{n}{\log n} + O\left(\frac{n}{(\log n)^2}\right).$$

We can now invert (4.3) to infer that, for $t = \gamma_n(A_{h,\theta})/\log\gamma_n(A_{h,\theta})$,

$$\gamma_{n+1}(A_{h,\theta}) - \gamma_n(A_{h,\theta}) = \frac{2\pi}{\log n} + O\left(\mathcal{R}_h\left(\frac{n}{\log n}\right)\right),$$

in which the remainder term $\mathcal{R}_h(t)$ satisfies

$$\mathcal{R}_h(t) = \begin{cases} O\left(\dfrac{1}{(\log t)^2}\right) & \text{if } |h| > 1/2, \\ O\left(\dfrac{1}{\log\log t}\right) & \text{if } |h| = 1/2, \\ O\left(\dfrac{1}{(\log t)^{1+2|h|}}\right) & \text{if RH holds and } 0 < |h| \le 1/2. \end{cases}$$

Now we know that

$$\gamma_n(A_{h,\theta}) = \frac{1}{2\pi}\,\frac{n}{\log n} + O\left(\frac{n}{(\log n)^2}\right),$$

so we conclude that the normalized zero spacings

$$\delta_n(A_{h,\theta}) := (\gamma_{n+1}(A_{h,\theta}) - \gamma_n(A_{h,\theta}))\,\frac{1}{2\pi}\log\frac{\gamma_n(A_{h,\theta})}{2\pi}$$

satisfy

$$\delta_n(A_{h,\theta}) = 1 + O\left((\log n)\mathcal{R}_h\left(\frac{n}{\log n}\right)\right).$$

The bounds on $\mathcal{R}_h(t)$ give in all cases

$$\delta_n(A_h) = 1 + O\left(\frac{1}{\log\log n}\right),$$

under the stated hypotheses. This then gives the result for k consecutive spacings for any fixed value of $k \ge 1$. The same results applies to $\delta_n(B_{h,\theta})$ by an identical argument. ∎

5. Differenced L-functions. The results of this paper extend without essential change to all GL(1) L-functions over $\mathbb{Q}$. These are the completed Dirichlet L-functions $\xi(s,\chi)$, associated to a primitive character χ of conductor N, given by

$$\xi(s,\chi) := \left(\frac{\pi}{N}\right)^{-(s+k)/2} \Gamma\left(\frac{s+k}{2}\right) L(\chi, s),$$

where $k = 0$ or 1 according as $\chi(-1) = \pm 1$, and $L(s,\chi)$ denotes the usual Dirichlet L-function. These functions satisfy the functional equation

$$\xi(s,\chi) = \xi(1-s,\overline{\chi}),$$

and are real-valued on the critical line. In this section we formulate the analogous results, omitting detailed proofs.

Lemma 5.1.

(1) *For $|h| \geq 1/2$, the function $E_h(s,\chi) := \xi(s+h,\chi)$ satisfies*

$$|E_h(s,\chi)| > |E_h(1-\overline{s},\chi)| \quad \text{for } \Re(s) > 1/2. \tag{5.1}$$

(2) *Assuming the Riemann hypothesis holds for $\xi(s,\chi)$, the inequality* (5.1) *is valid for all nonzero h.*

Proof. The proofs are similar to those for Lemma 2.1. We need to know that $\xi(s,\chi)$ has a modified Hadamard product of the form

$$\xi(s,\chi) = e^{A^*(\chi)+B^*(\chi)s} \prod_{\varrho,*} \left(1 - \frac{s}{\varrho}\right), \tag{5.2}$$

where ϱ runs over the zeros of $L(\chi, s)$ inside the open critical strip $0 < \Re(s) < 1$, and the product is interpreted as the limit as $T \to \infty$ over all zeros with $|\varrho| \leq T$, with

$$B^*(\chi) := B(\chi) + \lim_{T\to\infty} \sum_{|\varrho|<T} \frac{1}{\varrho}.$$

This can be derived from the Hadamard product formula, using the standard asymptotic formula for zeros up to height T, as given in Davenport [9, Chap. 16], to show that the zeros inside a box of side 1 at heights T and $-T$ respectively combine to give a convergent sum. In addition we have

$$\Re(B^*(\chi)) = 0, \tag{5.3}$$

which follows from Davenport [9, p. 83], who shows that

$$\Re(B(\chi)) = -\frac{1}{2}\sum_{\varrho}\left(\frac{1}{\varrho} + \frac{1}{\overline{\varrho}}\right) = -\sum_{\varrho} \Re\left(\frac{1}{\varrho}\right).$$

In particular $\Re(B^*(\chi)) \geq 0$, and the rest of the proof follows that of Lemma 2.1.

We now set

$$E_{h,\theta}(s,\chi) := e^{i\theta}\xi(s+h,\chi) = A_{h,\theta}(s,\chi) - iB_{h,\theta}(s,\chi)$$

with $A_{h,\theta}(s,\chi) - iB_{h,\theta}(s,\chi)$ given as in Lemma 2.2.

THEOREM 5.1. *Let χ be a primitive Dirichlet character that is nonprincipal, i.e. $\chi \neq \chi_0$.*

(1) *For $|h| \geq 1/2$ and any $0 \leq \theta < 2\pi$, the entire functions $A_{h,\theta}(s,\chi)$ and $B_{h,\theta}(s,\chi)$ have all their zeros on the critical line $\Re(s) = 1/2$. These zeros are all simple zeros, and they interlace.*
(2) *Assuming the Riemann hypothesis for $L(s,\chi)$, for $0 < |h| < 1/2$ and any $0 \leq \theta < 2\pi$ the functions $A_{h,\theta}(s,\chi)$ and $B_{h,\theta}(s,\chi)$ have all their zeros on the critical line $\Re(s) = 1/2$. These zeros are all simple zeros, and they interlace.*

Proof. These results are established by essentially the same proof as that of Theorem 2.1, using Lemma 5.1 in place of Lemma 2.1. ∎

Next we observe that an analogue of Theorem 3.1 holds, with a similar proof, following Davenport [9, Chap. 16]. It asserts that, for $|T| \geq 2$,

$$N(T, A_{h,\theta}(s,\chi)) = \frac{1}{\pi} T \log T - \frac{1}{\pi}\left(\log\left(\frac{2\pi}{N}\right) + 1\right)T + O(\log T + \log N).$$

It is valid unconditionally for $|h| \geq 1/2$ and, under the Riemann hypothesis for $L(s,\chi)$, for all nonzero h. Finally, using this asymptotic formula, one can establish by a proof similar to that of Theorem 4.1 the following result.

THEOREM 5.2. *Let χ be a primitive Dirichlet character which is nonprincipal, and let $k \geq 1$ be given.*

(1) *For $|h| \geq 1/2$ and $0 \leq \theta < 2\pi$ the functions $A_{h,\theta}(s,\chi)$ and $B_{h,\theta}(s,\chi)$ both have limiting distributions of their k consecutive normalized zero spacings. This distribution of normalized zero spacings consists of a delta function located at $x = 1$, the "trivial" distribution.*
(2) *Assuming the Riemann hypothesis for $L(s,\chi)$, the same results also hold for $0 < |h| < 1/2$ and $0 \leq \theta < 2\pi$, with the limiting distribution of k consecutive normalized zero spacings existing and being the "trivial" distribution.*

Proof. This is established in a manner similar to Theorem 4.1. The key ingredient is an analogue of the three parts of Lemma 4.1, which cover $|h| > 1/2$, $|h| = 1/2$ and $0 < |h| < 1/2$, respectively. Here the analogue of part (1), that $R_h(T) = O(1)$, follows from the absolute convergence of the Dirichlet series for $-(L'/L)(s,\chi)$ in $\Re(s) > 1$. The analogue of part (3), comprising the estimate

$$\left|\frac{L'}{L}(s,\chi)\right| = O((\log|T|)^{2-2\sigma})$$

valid for $\Re(s) = \sigma > 1/2$, with $|T| \to \infty$, follows from a result of Iwaniec and Kowalski [12, Theorem 5.17], on noting that the Ramanujan–Petersson conjecture assumed in that theorem holds for Dirichlet L-functions. The analogue of part (2), that

$$\left|\frac{L'}{L}(1+iT,\chi)\right| = O\left(\frac{\log|T|}{\log\log|T|}\right) \quad \text{for } |T| \geq 3,$$

is deducible from [12, Prop. 5.16], taking the test function $\phi(y) = 1 - y$ for $0 \leq y \leq 1$ and $\phi(y) = 0$ for $y \geq 1$, and using the standard zero-free region excluding zeros when $\sigma > 1 - c/\log|T|$ ([12, Theorem 5.10]) to bound the contribution over zeros. ∎

6. Hilbert spaces of entire functions. We now consider the functions $A_{h,\theta}(s)$ and $B_{h,\theta}(s)$ from the viewpoint of the de Branges theory of Hilbert spaces of entire functions ([2]).

We recall some fundamental facts about the de Branges theory of Hilbert spaces of entire functions, following [16], which gives a translation into operator-theoretic language of results stated in de Branges's book [2]. A *de Branges structure function* $E(z)$ is any entire function having the property that

$$|E(z)| > |\overline{E(\overline{z})}| \quad \text{for } \Im(z) > 0. \tag{6.1}$$

Associated to any entire function $E(z)$, whether it satisfies (6.1) or not, is a unique decomposition

$$E(z) = A(z) - iB(z), \tag{6.2}$$

in which $A(z), B(z)$ are entire functions that are real on the real axis. We define

$$E^{\#}(z) := \overline{E(\overline{z})},$$

which is itself an entire function; then $A(z) = \frac{1}{2}(E(z) + E^{\#}(z))$, $B(z) = -\frac{1}{2i}(E(z) - E^{\#}(z))$. The usefulness of (6.1) is that it implies that the functions $A(z)$ and $B(z)$ have only real zeros, and that these zeros interlace. This was shown by de Branges [1, Lemma 5], and is the content of Lemma 2.2 above, taking $s = 1/2 - iz$.

We assign to a structure function $E(z)$ a Hilbert space $\mathcal{H}(E(z))$ of entire functions, by a formulation given below, whose Hermitian scalar product for admissible functions $f(z), g(z)$ takes the form

$$\langle f(z), g(z)\rangle_{\mathcal{H}(E)} := \int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{|E(x)|^2}\, dx. \tag{6.3}$$

The functions belonging to $\mathcal{H}(E(z))$ are exactly those entire functions which have a finite norm $\|f\|_E^2 := \langle f(z), f(z)\rangle_{\mathcal{H}(E)}$ and whose growth with respect to $E(z)$ is controlled in the upper half-plane $\mathbb{C}^+ := \{z : \Im(z) > 0\}$, and in the lower half-plane $\mathbb{C}^- := \{z : \Im(z) < 0\}$, as follows. We require that the two functions $f(z)/E(z)$ and $\overline{f(\overline{z})}/E(z)$ each be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function $h(z)$ is of *bounded type* if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$, and it is of nonpositive mean type if it grows no faster than $e^{\varepsilon y}$ as $y \to \infty$ on the imaginary axis $\{iy : y > 0\}$, for each $\varepsilon > 0$. De Branges gives several different characterizations of the functions belonging to $\mathcal{H}(E)$ (e.g. [2, Theorem 20]) and shows in particular that this Hilbert space is never trivial; it is at least one-dimensional.

Also associated to the space $\mathcal{H}(E)$ is a (generally unbounded) multiplication operator $(M_z, \mathcal{D}_z)$ defined on the domain of all functions $f(z) \in \mathcal{H}(E)$ such that $zf(z) \in \mathcal{H}(E)$. The closure of $\mathcal{D}_z$ is either all of $\mathcal{H}(E)$ or a subspace of codimension one in $\mathcal{H}(E)$; we call the former case the "dense" case. In the "dense" case the operator M_z is a closed symmetric operator, with deficiency indices $(1,1)$ in von Neumann's sense. In particular it possesses self-adjoint extensions, the complete set of which forms a one-parameter family identifiable with $\mathbf{U}(1) = \{e^{i\theta} : 0 \le \theta < 2\pi\}$. The structure function $E(z)$ can be viewed as singling out one such self-adjoint extension, associated to the function $A(z)$, as follows. As noted above, $A(z)$ only has real zeros. This self-adjoint extension has pure discrete simple spectrum, given by the zeros of $A(z)$ (counted *without multiplicity*), with associated eigenfunction

$$f_\varrho(z) := \frac{A(z)}{z-\varrho} \in \mathcal{H}(E), \tag{6.4}$$

and requiring that

$$M_z(f_\varrho)(z) = \varrho f_\varrho(z) \tag{6.5}$$

hold for all ϱ. The domain of this self-adjoint extension is $\mathcal{D}_z(A) := \mathcal{D}_z \oplus \mathbb{C}[f_\varrho(z)]$ where we have adjoined the eigenfunction for any single zero; this domain is well defined since

$$f_\varrho(z) - f_{\varrho'}(z) = (\varrho' - \varrho)\,\frac{A(z)}{(z-\varrho)(z-\varrho')} \in \mathcal{D}_z.$$

For self-adjoint operators with discrete spectrum a standard fact is that the eigenfunctions $f_\varrho(z)$ for different zeros ϱ are orthogonal in the Hilbert space $\mathcal{H}(E)$. The complete family of self-adjoint extensions of the operator M_z is given by the spaces $\mathcal{H}(E_\theta)$ where $E_\theta(z) := e^{i\theta}E(z)$ for $0 \le \theta < 2\pi$. The Hilbert spaces $\mathcal{H}(E_\theta)$ are identical (same functions, same scalar product), but the variation in structure functions picks out different self-adjoint extensions, using the decomposition $E_\theta(z) = A_\theta(z) - iB_\theta(z)$. Note that $B(z) = A_\theta(z)$ with $\theta = \pi/2$. The de Branges theory thus gives a spectral interpretation of the zeros of $A(z)$ (counted without multiplicity) as the spectrum of the self-adjoint operator $(M_z, \mathcal{D}_z(A))$ in $\mathcal{H}(E(z))$.

The de Branges theory can be regarded as giving a normal form for a particular kind of non-self-adjoint operator $(M_z, \mathcal{D}_z)$ with deficiency indices $(1,1)$, much as the spectral theorem for (unbounded) self-adjoint operators gives a normal form as a multiplication operator, acting on a Hilbert space given with a spectral measure, and with a specified domain. The structure function $E(z)$ then encodes the data analogous to the spectral measure. The major content of the de Branges theory is a Fourier-like transform (which might be called the *de Branges transform*) that converts this multiplication operator into a 2×2 matrix integral operator of a certain kind acting on a different Hilbert space, much as the Fourier transform takes the multiplication operator to a differential operator. We do not address that aspect of the de Branges theory here.

The results of §2 can be reformulated in terms of the de Branges theory as follows.

LEMMA 6.1.

(i) *For $h \ge 1/2$ the function*

$$E_h(z) := \xi(1/2 + h - iz) \tag{6.6}$$

is a de Branges structure function, i.e. $|E_h(z)| > |E_h(\overline{z})|$ when $\Im(z) > 0$.

(ii) *If the Riemann hypothesis holds, then for all $h \neq 0$ the function $E_h(z)$ is a de Branges structure function.*

Proof. This is a restatement of Lemma 2.1. ∎

Under the hypotheses of Lemma 6.1 it follows that $E_{h,\theta}(z) := e^{i\theta}E_h(z)$ is a de Branges structure function, whose associated decomposition (6.2) is

$$E_{h,\theta}(z) = \widetilde{A}_{h,\theta}(z) - i\widetilde{B}_{h,\theta}(z),$$

in which $\widetilde{A}_{h,\theta}(z) := A_{h,\theta}(1/2 - iz)$ and $\widetilde{B}_{h,\theta}(z) := B_{h,\theta}(1/2 - iz)$. The de Branges theory now applies to give a "spectral" interpretation of the zeros of $A_{h,\theta}(s)$, or, equivalently, of the zeros of

$$f_{h,\theta}(t) := \Re(e^{i\theta}\xi(1/2 + h + it)),$$

as the eigenvalues of the self-adjoint operator $(M_z, \mathcal{D}_z(A_{h,\theta}))$ acting in the de Branges Hilbert space $\mathcal{H}(E_{h,\theta}(z))$. Theorem 2.1 established that the zeros of $A_{h,\theta}(s)$ are simple zeros for $|h| \geq 1/2$, and, assuming the Riemann hypothesis, for all nonzero h.

The particular de Branges space with structure function $E(z) = \xi(1-iz)$ was considered by de Branges [3] as a possible approach to the Riemann hypothesis. This structure function is included in the family

$$E_{h,\theta}(z) = A_{h,\theta}(1/2 - iz) - iB_{h,\theta}(1/2 - iz)$$

of Theorem 2.1, on taking $h = 1/2$, $\theta = 0$. Theorem 2.1 thus supplies a proof that $\xi(1 - iz)$ is a structure function. In [3] and [4] de Branges proved general theorems giving sufficient conditions on the inner product of a Hilbert space of entire functions $\mathcal{H}(E(z))$ which imply that the associated (normalized) structure function $E(z)$ necessarily has all its zeros on the line $\Im(z) = -1/2$. If any of these theorems applied to the de Branges space $\mathcal{H}(\xi(1 - iz))$, the Riemann hypothesis would follow. However, Conrey and Li [7] recently showed that the hypotheses of these theorems are not satisfied for the de Branges Hilbert spaces $\mathcal{H}(E(z))$ with $E(z) = \xi_\chi(1 - iz)$ with χ either the trivial character χ_0 or the real character χ_{-4}. There do exist de Branges spaces satisfying the inner product conditions of [3], presented in Li [19], see also [18].

In [16] we formulate another connection between the Riemann hypothesis for Dirichlet L-functions and certain Hilbert spaces of entire functions. This connection is conditional, in that the associated de Branges space exists if and only if the Riemann hypothesis holds for the corresponding L-function.

7. Concluding remarks. (1) This paper studied a two-parameter deformation $A_{h,\theta}(s)$ of the Riemann ξ-function using the parameters (h, θ). It showed that under the RH this deformation preserves a "Riemann hypothesis" condition, and proved this holds unconditionally when $|h| \geq 1/2$, for all θ. The deformations (h, θ) preserve a "functional equation", namely

$$A_{h,\theta}(s) = \overline{A_{h,\theta}(1 - \overline{s})},$$

which encodes the property that $A_{h,\theta}(s)$ is real on the critical line, embodying the reflection principle. The deformations also preserve a second "functional equation"

$$B_{h,\theta}(s) = \overline{B_{h,\theta}(1 - \overline{s})}.$$

However we do not know of any analogue of an Euler product (or Hecke operator factorization) that is preserved under such deformations.

(2) The arguments of §5 extend with little change to appropriate automorphic L-functions over $\mathrm{GL}(N)$ (principal L-functions over $\mathrm{GL}(N)$ over

the rational field $\mathbb{Q}$) for all $N \geq 2$, under suitable extra hypotheses indicated below. Given an L-function, one constructs an analogue of the ξ-function, adjusted to be real on the critical line, as is done in [15, §2]. One has global zero density estimates that the number of zeros to height T grows like $(N/\pi)T \log T$ (see Iwaniec and Kowalski [12, Theorem 5.8], or Lagarias [15, Theorem 2.1]). One can prove an unconditional result for $|h| > 1/2$, since the analogue of Lemma 4.1(1) holds: the Dirichlet series of $L(s,\pi)$ converges absolutely for $\Re(s) > 1$, and this implies an $O(1)$ bound for the analogue of $R_h(T,\pi)$ where the O-constant depends on h. The convergence of the Dirichlet series is formulated in Lagarias [15, Theorem 2.1], a result that goes back to work of Jacquet and Shalika. A conditional result can also be proved, valid for all nonzero h, assuming the truth of both the Riemann hypothesis and the Ramanujan–Petersson conjecture for the given automorphic L-function attached to an irreducible, cuspidal unitary representation of $\mathrm{GL}(N)$ over the adeles. (The Ramanujan–Petersson conjecture states that the local parameters $|\alpha_i(p)|$ are of absolute value at most one; see [12, p. 95].) Under these hypotheses a suitable analogue of Lemma 4.1(3) holds in view of Theorem 5.17 of Iwaniec and Kowalski [12].

(3) Theorem 4.1 implies that nontrivial zero spacing statistics (GUE-like statistics) for these functions are associated precisely with the critical line, i.e. $h = 0$. It has been asserted that the Riemann hypothesis, if true, is "just barely true". One precise formulation of this has to do with the de Bruijn–Newman constant (see Csordas, Smith and Varga [8]). Since $\xi(s)$ presumably satisfies GUE, we would not expect $\xi(s)$ itself to be produced by averaging shifts of another function with all its zeros on the critical line. We therefore ask: Is it true that any entire function $G(s)$ such that for some $h > 0$,

$$\xi(s) = \tfrac{1}{2}\left(G(s+h) + G(s-h)\right)$$

necessarily has the property that not all its zeros lie on the critical line $\Re(s) = 1/2$?

(4) The result of Theorem 4.1 also illustrates a heuristic principle that "averaging" and "differencing" operations smooth the spacings of zeros. The operator

$$\mathbf{A}_h(f)(s) := \tfrac{1}{2}\left(f(s+h) + f(s-h)\right)$$

is a convolution-type operator which convolves with a discrete probability measure (with masses of weight $1/2$ at the points $\pm h$). Such convolution operators may be expected to smooth zeros under some conditions on the operator, as studied in Cardon and Nielsen [5], for functions in the Laguerre–Pólya class.

In the case $f(s) = \xi(s)$ a heuristic goes as follows. The zeros of a shifted function $\xi_h(s)$ "feel the effect" of zeros of the unshifted function within distance $O(1)$. Assuming the Riemann hypothesis, there is a regular asymptotics for zeros over a range $[T, T+1]$ of this length for the ξ-function, namely

$$N(T+1) - N(T) = \frac{1}{2\pi}\log T + O\left(\frac{\log T}{\log\log T}\right)$$

(see Titchmarsh [26, Theorem 14.13]). The heuristic is that the effect of averaging is to make the zeros repel each other. There are definite restrictions needed to formulate in a general theorem, for example bounds on the growth rate (entire function of order less than 2) and also on the location of zeros of the function $f(s)$. D. Cardon (private communication) has examples showing that this heuristic cannot be valid without further hypotheses, e.g. that the density of zeros to height T grows faster than linear in T.

(5) One might also study the distribution of normalized zeros of the derivative $\xi'(s)$ of the ξ-function. Assuming the Riemann hypothesis, all the zeros of $\xi'(s)$ lie on the critical line. If the zeros of $\xi(s)$ satisfy the GUE conjecture, it seems reasonable to expect that the normalized spacings of zeros of $\xi'(s)$ also have a limiting distribution. This distribution cannot be the "trivial" distribution concentrated at equal normalized spacings of size 1, because the zeros of $\xi'(s)$ interlace with those of $\xi(s)$, and the GUE distribution predicts a positive probability of two consecutive normalized spacings each of which is at most $1/2 - \delta$, for fixed positive δ, so the associated normalized zero spacing of $\xi'(s)$ is with positive probability at most $1 - 2\delta$. However, this distribution is certainly not the GUE distribution. The distribution of normalized zero spacings of $\xi'(s)$ should be a new distribution, whose form is expected to be predictable using random matrix theory analogues. It should be more concentrated near the unit spacing than the GUE distribution, by analogy with results of Farmer and Rhodes [10].

(6) It seems plausible that there will be a random matrix theory analogue of Theorem 4.1. This would concern the behavior of the roots of differenced characteristic polynomials of random unitary matrices drawn from $U(N)$, followed by letting $N \to \infty$. One may also expect there to be a random matrix theory analogue of the distribution of the zeros of $\xi'(s)$. We hope to address these questions elsewhere.

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