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Linear independence of certain Lambert and allied series

by

PETER BUNDSCHUH (Köln) and KEIJO VÄÄNÄNEN (Oulu)

1. Introduction. In our recent work [3] we considered linear independence of q-analogues of certain classical constants connected with Lambert series

$$F(\underline{a}) := \sum_{n=1}^{\infty} \frac{a_n}{q^n - 1},$$

where q is a rational integer or an integer in an imaginary quadratic number field satisfying $|q| > 1$, and $\underline{a} := (a_n)$ is a non-zero periodic sequence of period length ≤ 2 with all a_n in the same field as above. Our results gave quantitative refinements of some results of Tachiya [7], e.g., a linear independence measure $2(\pi^2+4)/(\pi^2-8) = 14.83694\ldots$ was obtained for $1, F(\underline{a})$ and $F(\underline{b})$ with linearly independent sequences $\underline{a}$ and $\underline{b}$, giving the same measure for $1, L_q(1), L_q(-1)$, the values of the q-logarithm L_q defined below. Simultaneously and independently, the same measure was obtained by Zudilin [8] using another method based on Padé approximations of the second kind.

In the present paper, our purpose is to extend the results of [3] to more general algebraic numbers q and to

$$f(\underline{a}, \alpha) := \sum_{n=1}^{\infty} \frac{a_n}{q^n - \alpha} \tag{1}$$

with $\alpha \in \{1, -1\}$; note that $f(\underline{a}, 1) = F(\underline{a})$. The arithmetic part of the proof depends essentially on the value of α, and the new case $\alpha = -1$ turns out to be much more interesting than the earlier $\alpha = 1$. Our present extensions also give new applications.

2. Notations and results. Let q be an algebraic number satisfying $|q| > 1$, and let $K := \mathbb{Q}(q)$, $d := [K : \mathbb{Q}]$. We shall consider linear independence (over K) of 1, $f(\underline{a}, \alpha)$ and $f(\underline{b}, \alpha)$, where $\underline{a} = (a_n)$ and $\underline{b} = (b_n)$ are linearly independent periodic sequences in $K^{\mathbb{N}}$ of period length ≤ 2.

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We shall use the absolute height of $\beta \in K$ defined by

$$h(\beta) := \prod_w \max(1, |\beta|_w)^{d_w/d},$$

where the product is taken over all places w of K, with $|\ |_w$ denoting the valuation corresponding to w, normalized in the usual way, K_w the completion of K at w, and $d_w := [K_w : \mathbb{Q}_w]$. For a vector $\underline{\beta} := (\beta_1, \ldots, \beta_n) \in K^n$ we define

$$|\underline{\beta}|_w := \max_i |\beta_i|_w, \qquad h(\underline{\beta}) := \prod_w \max(1, |\underline{\beta}|_w)^{d_w/d}.$$

Further, let us denote by v the infinite place with $|q| = |q|_v$.

To formulate our results we need the quantity

$$\lambda := \frac{d \log h(q)}{d_v \log |q|_v}. \tag{2}$$

Clearly $\lambda \geq 1$ holds always, and $\lambda = 1$ if and only if $|q|_w \leq 1$ for all places $w \neq v$. For example, if K is $\mathbb{Q}$ or an imaginary quadratic number field and q is an integer in such a K, then $\lambda = 1$. Furthermore, $\lambda = 1$ holds for all algebraic integers such that $|q|_w \leq 1$ for all infinite places $w \neq v$. Examples of such algebraic integers are the elements of the classes S (nowadays called Pisot, or Pisot–Vijayaraghavan, or PV numbers) and T (now called Salem numbers) in terms of Salem's monograph [6] (Chap. I and III, respectively).

In our results we shall use an upper bound $\lambda(\alpha)$ for λ defined by

$$\lambda(\alpha) := \begin{cases} 3\pi^2/(2\pi^2+8) = 1.067399\ldots & \text{if } \alpha = 1, \\ 27\pi^2/(19\pi^2+66) = 1.051107\ldots & \text{if } \alpha = -1. \end{cases} \tag{3}$$

THEOREM 1. *Let q be an algebraic number such that $|q|_v > 1$ and $|q|_w \neq 1$ for all infinite places $w \neq v$ of $K = \mathbb{Q}(q)$, and assume that $\underline{a} = (a_n)$ and $\underline{b} = (b_n)$ are linearly independent periodic sequences in $K^{\mathbb{N}}$ of period length ≤ 2. If $\lambda < \lambda(\alpha)$, then the numbers*

$$1, \quad f(\underline{a}, \alpha), \quad f(\underline{b}, \alpha) \tag{4}$$

are linearly independent over K. Moreover, for any $\varepsilon \in \mathbb{R}_+$, there exists a positive constant $H_0 = H_0(|q|_v, \underline{a}, \underline{b}, \alpha, \varepsilon)$ such that

$$|\ell_0 + \ell_1 f(\underline{a}, \alpha) + \ell_2 f(\underline{b}, \alpha)|_v > |(\ell_1, \ell_2)|_v H^{-m_1(\alpha)-\varepsilon} \tag{5}$$

for all non-zero $\underline{\ell} \in K^3$, where $H := \max(h(\underline{\ell}), H_0)$ and $m_1(\alpha)$ is defined by

$$\frac{d_v}{d} m_1(\alpha) = \begin{cases} 3\pi^2/(3\pi^2 - 2\lambda(\pi^2+4)) & \textit{if } \alpha = 1, \\ 27\pi^2/(27\pi^2 - \lambda(19\pi^2+66)) & \textit{if } \alpha = -1. \end{cases}$$

REMARK. From the proof of this theorem we see that the ε in (5) can be replaced by a positive function of H of size $O((\log\log H)/(\log H)^{1/2})$ if $\alpha = 1$, or of size $O((\log\log H)^2/(\log H)^{1/2})$ if $\alpha = -1$.

As noted after (2), $\lambda = 1$ happens in some cases. In such a situation we have

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied, and suppose further that* $|q|_w \le 1$ *for all places* $w \neq v$ *of* K. *Then the numbers* (4) *are linearly independent over* K, *and inequality* (5) *holds with* $m_1(\alpha)$ *replaced by* $m_2(\alpha)$, *which is defined by*

$$\frac{d_v}{d}\, m_2(\alpha) = \begin{cases} 3\pi^2/(\pi^2 - 8) & \text{if } \alpha = 1, \\ 27\pi^2/(8\pi^2 - 66) & \text{if } \alpha = -1. \end{cases}$$

This theorem has the following corollary containing Tachiya's Theorem 2 (see [7]).

COROLLARY 1. *If* q *is an algebraic integer*, $|q| = |q|_v > 1$ *and* $|q|_w < 1$ *for all infinite places* $w \neq v$ *of* K, *then the claims of Theorem* 2 *are true. In particular*, *for any non-zero periodic sequence* $\underline{a} = (a_n)$ *of period length* ≤ 2 *the number* $f(\underline{a}, \alpha)$ *is not in* K, *and*, *for any* $\varepsilon > 0$, *the inequality*

$$|\ell_0 + \ell_1 f(\underline{a}, \alpha)| > |\ell_1| H^{-m_2(\alpha)-\varepsilon}$$

holds for all non-zero $\underline{\ell} \in K^2$ *with* $H = \max(h(\underline{\ell})) \ge H_0(|q|_v, \underline{a}, \alpha, \varepsilon)$.

We next give as a theorem the special case of Theorem 1 where $K = \mathbb{I}$, the field $\mathbb{Q}$ or an imaginary quadratic number field. Note that if the components of $\underline{\ell}$ are integers in $\mathbb{I}$, then $h(\underline{\ell}) = \max(|\ell_i|)$.

THEOREM 3. *Let the hypotheses of Theorem* 1 *be satisfied, and suppose that* $\mathbb{Q}(q) = \mathbb{I}$, *and* q *is an integer in* $\mathbb{I}$. *Then the numbers* (4) *are linearly independent over* $\mathbb{I}$ *and*, *for any* $\varepsilon > 0$, *the inequality*

$$|\ell_0 + \ell_1 f(\underline{a}, \alpha) + \ell_2 f(\underline{b}, \alpha)| > h^{-m_3(\alpha)-\varepsilon}$$

holds for all $\underline{\ell} \in \mathbb{I}^3$ *with integer components satisfying* $h = \max(|\ell_1|, |\ell_2|) \ge H_0(|q|, \underline{a}, \underline{b}, \alpha, \varepsilon)$, *where*

$$m_3(\alpha) := \begin{cases} 2(\pi^2 + 4)/(\pi^2 - 8) & \text{if } \alpha = 1, \\ (19\pi^2 + 66)/(8\pi^2 - 66) & \text{if } \alpha = -1. \end{cases}$$

The case $\alpha = 1$ of this theorem is Theorem 2 of [3]. We can also give a generalization of Corollary 1 of [3] and Theorem 1 and its corollary in [8]. For this, we introduce the q-logarithm, the first formula defining L_q in $|z| < |q|$ only:

$$L_q(z) = \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1} = z \sum_{n=1}^{\infty} \frac{1}{q^n - z}.$$

COROLLARY 2. *Let the hypotheses of Theorem* 1 *be satisfied. If* $\lambda < \lambda(\alpha)$, *then*, *for* $\alpha \in \{1, -1\}$, *any of the following sets of three numbers is linearly independent over* K:

$$\{1, L_q(\alpha), L_q(-\alpha)\}, \quad \{1, L_q(\alpha), L_{q^2}(\alpha)\}, \quad \{1, L_{q^2}(\alpha), L_{q^2}(\alpha/q)\}.$$

Moreover, the lower bound given in (5) *holds true for linear forms in any of these triples of numbers.*

Remark. For obvious reasons, we also call $L_q(1)$ a *q-harmonic series* (and denote it by $\zeta_q(1)$), and $L_q(-1)$ a *q-analogue* of $\log 2$, sometimes denoted by $\log_q 2$. Of course, we could as well give the above triples in these terms.

Our theorems also have some further interesting corollaries. For the next one, let $\tau_{\mathrm{o}}(n)$ and $\tau_{\mathrm{e}}(n)$ denote, respectively, the number of odd and even positive integral divisors d of the positive integer n. Then we have

Corollary 3. *Let the assumptions of Theorem* 1 *be satisfied. If* $\lambda < \lambda(1)$, *then the numbers*

$$1, \quad \sum_{n\geq 1} \tau_{\mathrm{o}}(n) q^{-n}, \quad \sum_{n\geq 1} \tau_{\mathrm{e}}(n) q^{-n}$$

are linearly independent over K. *Moreover, the lower bound given in* (5) *holds true for linear forms in these numbers.*

For our next result we define the arithmetical functions

$$s(n) := \sum_{d|n} (-1)^{d-1}, \qquad t(n) := \sum_{d|n} (-1)^{d+n/d}.$$

We shall see in Lemma 4 that these are multiplicative functions intimately connected with the classical divisor function τ.

Corollary 4. *Let the assumptions of Theorem* 1 *be satisfied. If* $\lambda < \lambda(-1)$, *then the numbers*

$$1, \quad \sum_{n=1}^{\infty} s(n) q^{-n}, \quad \sum_{n=1}^{\infty} t(n) q^{-n}$$

are linearly independent over K *and the lower bound given in* (5) *holds true for linear forms in these numbers.*

In the following we work in $\mathbb{Q}(\sqrt{5})$, where we choose $q := -(3+\sqrt{5})/2$. Then $|q|_w = (3-\sqrt{5})/2 < 1$ for the other infinite place of K, and therefore $\lambda = 1$. Since

$$\frac{1}{q^n - 1} = \frac{\beta^n}{\sqrt{5} F_n} \quad \text{and} \quad \frac{1}{q^n + 1} = \frac{\beta^n}{L_n},$$

where $\beta := (1-\sqrt{5})/2$ and (F_n) is the Fibonacci and (L_n) the Lucas sequence, Theorem 2 immediately implies the following

Corollary 5. *Let* (a_n) *and* (b_n) *be linearly independent periodic sequences in* $\mathbb{Q}(\sqrt{5})^{\mathbb{N}}$ *of period length* ≤ 2. *Then the sets*

$$\left\{1, \sum_{n=1}^{\infty} \frac{a_n\beta^n}{F_n}, \sum_{n=1}^{\infty} \frac{b_n\beta^n}{F_n}\right\}, \quad \left\{1, \sum_{n=1}^{\infty} \frac{a_n\beta^n}{L_n}, \sum_{n=1}^{\infty} \frac{b_n\beta^n}{L_n}\right\}, \quad \left\{1, \sum_{n=1}^{\infty} \frac{\beta^n}{F_n}, \sum_{n=1}^{\infty} \frac{\beta^n}{L_n}\right\}$$

are linearly independent over $\mathbb{Q}(\sqrt{5})$. *Moreover*, *the lower bound given in* (5) *holds true for linear forms in any of these triples*, *where* $m_1(\alpha)$ *has to be replaced by* $6\pi^2/(\pi^2-8)$ *or* $27\pi^2/(4\pi^2-33)$ *in the first and third case*, *or in the second case*, *respectively.*

We note that the irrationality measures 2.874 and 7.652 for $\sum_{n\ge1}\beta^n/F_n$ and $\sum_{n\ge1}\beta^n/L_n$, respectively, were proved by Matala-aho and Prévost [5].

3. Analytic construction. To prove Theorems 1 and 2 we consider a linear form in 1, $f(\underline{a},\alpha)$ and $f(\underline{b},\alpha)$, with $\alpha=\pm1$ and linearly independent $\underline{a}$ and $\underline{b}$, say

$$L := \ell_0 + \ell_1 f(\underline{a},\alpha) + \ell_2 f(\underline{b},\alpha), \tag{6}$$

where $\underline{\ell} := (\ell_0,\ell_1,\ell_2) \in K^3 \setminus \{0\}$. We assume that $(\ell_1,\ell_2)\neq(0,0)$ (the case $\ell_1=\ell_2=0$ being trivial). Clearly L is of the form

$$L = \ell_0 + f(\underline{d},\alpha),$$

where $\underline{d} := \ell_1\underline{a} + \ell_2\underline{b}$ is a periodic sequence of period length ≤ 2. Since $\underline{a}$ and $\underline{b}$ are linearly independent, we have $\underline{d}\neq\underline{0}$.

We now construct approximations to $f(\underline{d},\alpha)$ similarly to [3], see also [2]. For this we use the complex integral

$$J(N) := \frac{1}{2\pi i}\oint_{|z|=1} \frac{\prod_{k=1}^{2N-\delta}(\alpha z - q^k)}{z^{2N}\prod_{n=1}^{N}(1-q^{2n}z)}\, f(\underline{d},\alpha z)\,dz, \tag{7}$$

where $\delta := 0$ if $|d_1|_v \ge |d_2|_v$, $\delta := 1$ if $|d_1|_v < |d_2|_v$, and $N\in\mathbb{N}$ is a parameter to be fixed later. By using the equalities

$$f(\underline{d},\alpha q^{-2n}) = q^{2n}\biggl(f(\underline{d},\alpha) - \sum_{m=1}^{2n}\frac{d_m}{q^m-\alpha}\biggr),$$

$$\frac{f^{(\nu)}(\underline{d},\alpha z)}{\nu!}\bigg|_{z=0} = \alpha^\nu\,\frac{d_1q^{\nu+1}+d_2}{q^{2(\nu+1)}-1}$$

and the residue theorem we obtain

$$J(N) = P(N)f(\underline{d},\alpha) + Q(N,\underline{d}), \tag{8}$$

where

$$\begin{aligned} P(N) &:= P(N,q,\alpha) \\ &:= \sum_{n=1}^{N}(-1)^{N+n+1+\delta}\,\frac{q^{n(n-1)+2\delta n}\prod_{k=1}^{2N-\delta}(q^{k+2n}-\alpha)}{\prod_{\nu=1}^{n-1}(q^{2\nu}-1)\prod_{\nu=1}^{N-n}(q^{2\nu}-1)}, \end{aligned} \tag{9}$$

the single summands here being denoted later by $p_n(N,q,\alpha)$, and

$$Q(N,\underline{d}) := Q(N,\underline{d},q,\alpha) := -\sum_{n=1}^{N} p_n(N,q,\alpha)\sum_{m=1}^{2n}\frac{d_m}{q^m-\alpha} \tag{10}$$
$$+\sum_{\kappa+\mu+\nu=2N-1}\Big(\sum_{1\le k_1<\cdots<k_{2N-\delta-\kappa}\le 2N-\delta}\alpha^{\kappa}(-1)^{\delta+\kappa}q^{k_1+\cdots+k_{2N-\delta-\kappa}}\Big)$$
$$\times\Big(\sum_{\mu_1+\cdots+\mu_N=\mu}(-1)^{\mu}q^{2(1\mu_1+\cdots+N\mu_N)}\Big)\alpha^{\nu}\,\frac{d_1q^{\nu+1}+d_2}{q^{2(\nu+1)}-1}.$$

Furthermore, as in the proof of Theorem 2 in [3], we have

$$|J(N)|_v = c_1|\underline{d}|_v|q|_v^{-3N^2-(2+\delta)N}(1+O(|q|_v^{-N})), \tag{11}$$

where c_1 (as also $c_2, c_3, \ldots$ later) and the implied constant in the O-notation are positive constants depending on α and q, but not on $\underline{d}$ or N.

4. Arithmetic considerations. In the analytic part there was no essential difference between the cases $\alpha = 1$ and $\alpha = -1$. Now we consider the arithmetic properties, in particular the denominators, of $P(N,q,\alpha)$ and $Q(N,\underline{d},q,\alpha)$, and here the situation is different. The case $\alpha = 1$ is rather standard, but the case $\alpha = -1$ is a highly delicate one.

The case $\alpha = 1$ is treated in [3], and in this case (9) implies

$$P(N,q,1) = \sum_{n=1}^{N}(-1)^{N+n+1+\delta}q^{n(n-1)/2}\begin{bmatrix}N-1\\ n-1\end{bmatrix}_{q^2}\begin{bmatrix}2N+2n-1\\ 2n\end{bmatrix}_{q} \tag{12}$$
$$\times (q^{2N+2n}-1)^{1-\delta}\prod_{\nu=1}^{N}(q^{2\nu-1}-1),$$

where the q-binomial coefficients are in $\mathbb{Z}[q]$. Therefore $P(N,q,1)\in\mathbb{Z}[q]$, and if

$$D(N,q,1) := \mathrm{lcm}(q^2-1, q^4-1, \ldots, q^{4N}-1),$$

then $D(N,q,1)Q(N,\underline{d},q,1)\in K[q]$, by (10). Furthermore, by using the cyclotomic polynomials Φ_ν we have

$$D(N,q,1) = \prod_{\nu=1}^{2N}\Phi_\nu(q^2)$$

and

$$|D(N,q,1)|_v = |q|_v^{24\pi^{-2}N^2+O(N\log N)}, \tag{13}$$

where the O-constant depends at most on q.

In the case $\alpha = -1$, we certainly need $D(N,q,1)$ in the common denominator, but this is not enough. After multiplying by $D(N,q,1)$, we still need a common denominator for all $n = 1, \dots, N$ of

$$\frac{\prod_{k=1+2n}^{2N-\delta+2n}(q^k+1)\prod_{l=1}^{2N}\Phi_l(q^2)}{\prod_{k=1}^{N-1}(q^{2k}-1)}\cdot\frac{1}{q^j+1}\qquad(j=1,\dots,2n);$$

see (9), (10) and the above $D(N,q,1)$. Because of

$$\prod_{k=1}^{N}(q^k-1)=\prod_{k=1}^{N}\prod_{l|k}\Phi_l(q)=\prod_{l=1}^{N}\Phi_l(q)^{[N/l]}$$

the above expression is, for each $j \in \{1, \dots, 2n\}$, of the form

$$(14)\qquad \frac{\prod_{k=1+2n}^{2N-\delta+2n}(q^{2k}-1)\cdot\prod_{l=1}^{2N}\Phi_l(q^2)}{\prod_{k=1+2n}^{2N-\delta+2n}(q^k-1)\cdot\prod_{k=1}^{N-1}(q^{2k}-1)}\cdot\frac{q^j-1}{q^{2j}-1}$$
$$=\frac{\prod_{k=1+2n}^{2N-\delta+2n}\prod_{l|2k}\Phi_l(q)\cdot\prod_{l=N}^{2N}\Phi_l(q^2)\cdot\prod_{l|j}\Phi_l(q)}{\prod_{k=1+2n}^{2N-\delta+2n}\prod_{l|k}\Phi_l(q)\cdot\prod_{l=1}^{[(N-1)/2]}\Phi_l(q^2)^{[(N-1)/l]-1}\cdot\prod_{l|2j}\Phi_l(q)}.$$

LEMMA 1. *The product*

$$\Psi_N(q) := \prod_{\substack{l=1\\ 2\nmid l}}^{[(N-1)/2]}\Phi_l(q)^{[(N-1)/l]-1}$$

is a common denominator of all rational expressions (14).

Proof. Since

$$\Phi_l(q^2)=\begin{cases}\Phi_{2l}(q) & \text{for even } l,\\ \Phi_{2l}(q)\Phi_l(q) & \text{for odd } l,\end{cases}$$

(compare, e.g., [4, Chap. 2]) our lemma follows if all quotients

$$\frac{\prod_{k=1+2n}^{2N-\delta+2n}\prod_{l|2k}\Phi_l(q)}{\prod_{k=1+2n}^{2N-\delta+2n}\prod_{l|k}\Phi_l(q)}\cdot\frac{\prod_{l=N}^{2N}\Phi_l(q^2)}{\prod_{l=1}^{[(N-1)/2]}\Phi_{2l}(q)^{[(N-1)/l]-1}}\cdot\frac{\prod_{l|j}\Phi_l(q)}{\prod_{l|2j}\Phi_l(q)}$$

are in $\mathbb{Z}[q]$. Obviously all $\Phi_l(q)$ with odd l appearing in the denominator can be cancelled by the nominator.

The polynomial Φ_l with even l appears at least

$$A := \left[\frac{2N+2n-\delta}{l/2}\right]-\left[\frac{2n}{l/2}\right]$$

times in the nominator and at most

$$B := \left[\frac{2N+2n-\delta}{l}\right]-\left[\frac{2n}{l}\right]+\left[\frac{N-1}{l/2}\right]-1+1$$

times in the denominator. On putting $l' := l/2$ and writing $N = \nu l' + \nu'$,

$n = \mu l' + \mu'$ with $\nu', \mu' \in \{0, 1, \ldots, l'-1\}$ we obtain

$$A - B = \left[\frac{2\nu' + 2\mu' - \delta}{l'}\right] - \left[\frac{2\mu'}{l'}\right] - \left[\frac{\nu' + \mu' - \delta/2}{l'}\right] - \left[\frac{\nu' - 1}{l'}\right].$$

If $\nu' = 0$, then clearly $A - B \geq 0$. If $\nu' \in \{1, \ldots, l'-1\}$ and $\nu' + \mu' - \delta/2 < l'$, then

$$A - B = \left[\frac{2\nu' + 2\mu' - \delta}{l'}\right] - \left[\frac{2\mu'}{l'}\right] \geq 0.$$

Finally, if $\nu' + \mu' - \delta/2 \geq l'$, then

$$A - B \geq 2 - \left[\frac{2\mu'}{l'}\right] - 1 \geq 0.$$

This proves our lemma.

Lemma 2. *We have the following asymptotic formula*:

$$|\Psi_N(q)|_v = |q|_v^{(1/3 - 2/\pi^2)N^2 + O(N \log^2 N)}.$$

Proof. It is obviously enough to prove that

$$\sum_{l \leq n/2}{}' \varphi(l)\left(\left[\frac{n}{l}\right] - 1\right) = \left(\frac{1}{3} - \frac{2}{\pi^2}\right)n^2 + O(n \log^2 n), \tag{15}$$

where $\sum'$ denotes the summation over all odd positive integers specified under the sum sign. Namely we have

$$\sum_{l \leq n/2}{}' \varphi(l)\left(\left[\frac{n}{l}\right] - 1\right) = \sum_{k=1}^{n} k \sum_{n/(k+1) < l \leq n/k}{}' \varphi(l) - \sum_{l \leq n/2}{}' \varphi(l) =: \Sigma_1 - \Sigma_2.$$

From

$$\sum_{j=1}^{m} \varphi(2j-1) = \frac{8}{\pi^2} m^2 + O(m \log m), \tag{16}$$

we immediately see $\Sigma_2 = 2\pi^{-2} n^2 + O(n \log n)$. On the other hand, we find

$$\Sigma_1 = \sum_{k=1}^{n} \sum_{l \leq n/k}{}' \varphi(l) = \sum_{k \leq H} \sum_{l}{}' + \sum_{H < k \leq n} \sum_{l}{}' =: \Sigma_3 + \Sigma_4,$$

where $H := [n/\log^2 n] + 1$. In Σ_4, we estimate trivially

$$\sum_{l \leq n/k}{}' \varphi(l) \leq \sum_{l \leq n/k}{}' l \leq \frac{n^2}{k^2}, \quad \text{hence} \quad \Sigma_4 < n^2 \sum_{k > H} \frac{1}{k^2} < \frac{n^2}{H} < n \log^2 n.$$

In contrast, Σ_3 contributes to the main term in (15). Namely, by (16), we have

$$\Sigma_3 = \sum_{k\le H}\left(\frac{2}{\pi^2}\,\frac{n^2}{k^2} + O\left(\frac{n}{k}\log\frac{n}{k}\right)\right)$$
$$= \frac{2}{\pi^2}\,n^2\left(\frac{\pi^2}{6} - \sum_{k>H}\frac{1}{k^2}\right) + O\left(n(\log n)\sum_{k\le H}\frac{1}{k}\right)$$
$$= \frac{n^2}{3} + O\left(\frac{n^2}{H}\right) + O(n(\log n)(\log H)).$$

Combination of our above considerations yields (15) and thus Lemma 2.

By Lemmas 1 and 2 we now know that in the case $\alpha = -1$ we have a common denominator

$$D(N,q,-1) = \Psi_N(q)D(N,q,1)$$

satisfying

$$|D(N,q,-1)|_v = |q|_v^{(22/\pi^2+1/3)N^2+O(N\log^2 N)}. \tag{17}$$

We now multiply (8) by $D(N,q,\alpha)$ and get

$$r_N(q,\alpha) := D(N,q,\alpha)J(N) = s_N(q,\alpha)f(\underline{d},\alpha) + t_N(\underline{d},q,\alpha), \tag{18}$$

where s_N and t_N are polynomials in $K[q]$. By using (9), (10), (11), (13) and (17) we now obtain the following

LEMMA 3. *We have*

$$|r_N(q,\alpha)|_v = |\underline{d}|_v|q|_v^{-b(\alpha)N^2+O(N\log^{\nu(\alpha)}N)}, \tag{19}$$

and for all places w of K,

$$\max\left(|s_N(q,\alpha)|_w, \frac{|t_N(\underline{d},q,\alpha)|_w}{|\underline{d}|_w}\right) \le 2^{\delta(w)N\log N}(\max(1,|q|_w))^{a(\alpha)N^2+O(N\log^{\nu(\alpha)}N)}, \tag{20}$$

where

$$a(1) := 6 + \frac{24}{\pi^2}, \qquad b(1) := 3 - \frac{24}{\pi^2}, \qquad \nu(1) := 1,$$
$$a(-1) := \frac{19}{3} + \frac{22}{\pi^2}, \quad b(-1) := \frac{8}{3} - \frac{22}{\pi^2}, \quad \nu(-1) := 2,$$

and $\delta(w) = 0$ for finite w, but $\delta(w) = 1$ for infinite w.

Proof. We easily get (19) from (11), (13) and (17). For (20) we first note that

$$\left|\begin{bmatrix} n \\ k \end{bmatrix}_q\right|_w \le 2^{\delta(w)n}(\max(1,|q|_w))^{nk-k^2}$$

and

$$|\Phi_n(q)|_w \le 2^{\delta(w)O(1)}(\max(1,|q|_w))^{\varphi(n)}.$$

Then the use of (12) immediately gives an estimate

$$(21)\quad |p_n(N,q,1)|_w \le 2^{\delta(w)O(N)}(\max(1,|q|_w))^{6N^2+O(N)} \qquad (n=1,\dots,N).$$

The use of the above bound for $|\Phi_n(q)|_w$ then implies, as in (13), that

$$|D(N,q,1)|_w \le 2^{\delta(w)O(N)}(\max(1,|q|_w))^{24\pi^{-2}N^2+O(N\log N)},$$

and this gives (20) in the case $\alpha=1$. The case $\alpha=-1$ follows similarly on noting that

$$p_n(N,q,-1) = (-1)^{N+n+1+\delta}q^{n(n-1)/2}\begin{bmatrix} N-1 \\ n-1 \end{bmatrix}_{q^2} \frac{\prod_{k=1+2n}^{2N-\delta+2n}(q^k+1)}{\prod_{k=1}^{N-1}(q^{2k}-1)}.$$

The bound for $|t_N(\underline{d},q,\alpha)|_w$ then follows immediately from (10) and the inequalities (see the notations in (10))

$$\begin{aligned} k_1+\dots+k_{2N-\delta-\kappa}+2(1\mu_1+\dots+N\mu_N) &\le (2N-\delta)+(2N-\delta-1)+\dots+\kappa+2N\mu \\ &\le 2N^2-\frac{1}{2}\kappa(\kappa-1)+2N(2N-1-\kappa) \le 6N^2. \end{aligned}$$

5. Proofs of the theorems. Now we shall prove a lower bound for the linear form (6), i.e. for

$$L = \ell_0+\ell_1 f(\underline{a},\alpha)+\ell_2 f(\underline{b},\alpha) = \ell_0 + f(\underline{d},\alpha),$$

where $\underline{d}=\ell_1\underline{a}+\ell_2\underline{b}$. By our assumption on linear independence of $\underline{a}$ and $\underline{b}$ we have, for all places w of K,

$$(22)\qquad |\underline{d}|_w \le |\underline{\gamma_1}|_w|(\ell_1,\ell_2)|_w, \qquad |(\ell_1,\ell_2)|_w \le |\underline{\gamma_2}|_w|\underline{d}|_w$$

for some constant non-zero vectors $\underline{\gamma_1}$ and $\underline{\gamma_2}$ depending only on $\underline{a}$ and $\underline{b}$.

From (18) we find

$$(23)\quad s_N(q,\alpha)L = \ell_0 s_N(q,\alpha)-t_N(\underline{d},q,\alpha)+r_N(q,\alpha) =: \Delta(q,\alpha)+r_N(q,\alpha).$$

Assume now that $\Delta_N(q,\alpha)\ne 0$ and

$$(24)\qquad |r_N(q,\alpha)|_v \ge \frac{1}{2}\,|\Delta_N(q,\alpha)|_v.$$

By Lemma 3, (22) and the product formula we then obtain

$$\begin{aligned} &\frac{d_v}{d}(\log_+|\underline{\gamma_1}|_v+\log_+|\underline{\ell}|_v-b(\alpha)N^2\log|q|_v+c_2N(\log N)^{\nu(\alpha)}\log|q|_v) \\ &\qquad \ge \frac{d_v}{d}\log|\Delta_N(q,\alpha)|_v = -\sum_{w\ne v}\frac{d_w}{d}\log|\Delta_N(q,\alpha)|_w \\ &\qquad \ge -\sum_{w\ne v}\frac{d_w}{d}(\log_+|\underline{\gamma_1}|_w+\log_+|\underline{\ell}|_w+a(\alpha)N^2\log_+|q|_w \\ &\qquad\qquad + c_3N(\log N)^{\nu(\alpha)}\log_+|q|_w+\delta(w)c_4N\log N) \end{aligned}$$

for all $N \geq c_5$, where $\log_+ x := \log\max(1, x)$. By (2), this yields

$$\log h(\underline{\ell}) \geq (a(\alpha) + b(\alpha) - \lambda a(\alpha))\,\frac{d_v}{d}\,N^2 \log|q|_v - c_6 N(\log N)^{\nu(\alpha)}. \tag{25}$$

From the assumption $\lambda < \lambda(\alpha)$ it follows that $a(\alpha) + b(\alpha) - \lambda a(\alpha)$ is positive. We now fix N to be the smallest positive integer such that

$$\log H < (a(\alpha) + b(\alpha) - \lambda a(\alpha))\,\frac{d_v}{d}\,n^2 \log|q|_v - c_6 n(\log n)^{\nu(\alpha)} \tag{26}$$

for all $n \geq N$, where $H := \max(h(\underline{\ell}), H_0)$ and H_0 is a sufficiently large constant to guarantee $N \geq c_5$. For this N, (25) and thus also (24) cannot hold, which implies $|r_N(q,\alpha)|_v < |\Delta_N(q,\alpha)|_v/2$. By (23) we then obtain

$$|s_N(q,\alpha)L|_v \geq |r_N(q,\alpha)|_v. \tag{27}$$

If $\Delta_N(q,\alpha) = 0$, then (23) gives $s_N(q,\alpha)L = r_N(q,\alpha)$ and (27) is also true. Thus we have (27) in both cases if N is fixed as before. The above choice of N also gives

$$(a(\alpha)+b(\alpha)-\lambda a(\alpha))\,\frac{d_v}{d}\,(N-1)^2 \log|q|_v - c_6(N-1)(\log(N-1))^{\nu(\alpha)} \leq \log H,$$

and therefore

$$N^2 \log|q|_v \leq \frac{d\log H}{d_v(a(\alpha) + b(\alpha) - \lambda a(\alpha))} + c_7 N(\log N)^{\nu(\alpha)}.$$

This result together with Lemma 3, (22) and (27) implies

$$\begin{aligned}
\log|L|_v &\geq \log|r_N(q,\alpha)|_v - \log|s_N(q,\alpha)|_v \\
&\geq \log|\underline{d}|_v - (a(\alpha) + b(\alpha))N^2 \log|q|_v - c_8 N(\log N)^{\nu(\alpha)} \\
&> \log|(\ell_1, \ell_2)|_v - \frac{d(a(\alpha) + b(\alpha))}{d_v(a(\alpha) + b(\alpha) - \lambda a(\alpha))} \log H \\
&\quad - c_9(\log H)^{1/2}(\log\log H)^{\nu(\alpha)}.
\end{aligned}$$

Theorems 1–3 are now immediately obtained by using the values of $a(\alpha)$ and $b(\alpha)$ given in Lemma 3.

6. Proofs of the corollaries. Clearly we need to consider only Corollaries 3 and 4. Since

$$\tau(n) = \sum_{d|n} 1 = \tau_{\mathrm{o}}(n) + \tau_{\mathrm{e}}(n) \quad \text{and} \quad \sum_{d|n} (-1)^{d-1} = \tau_{\mathrm{o}}(n) - \tau_{\mathrm{e}}(n),$$

we have

$$\sum_{m\geq 1} \frac{1}{q^m - 1} = \sum_{n\geq 1} \tau(n) q^{-n} = \sum_{n\geq 1} \tau_{\mathrm{o}}(n) q^{-n} + \sum_{n\geq 1} \tau_{\mathrm{e}}(n) q^{-n}$$

and

$$\sum_{m\ge1}\frac{(-1)^{m-1}}{q^m-1}=\sum_{n\ge1}q^{-n}\sum_{d|n}(-1)^{d-1}=\sum_{n\ge1}\tau_{\mathrm{o}}(n)q^{-n}-\sum_{n\ge1}\tau_{\mathrm{e}}(n)q^{-n}$$

if $|q|>1$. This proves Corollary 3.

Further, again if $|q|>1$, we see

$$\sum_{m\ge1}\frac{1}{q^m+1}=-L_q(-1)=\sum_{m\ge1}\frac{(-1)^{m-1}}{q^m-1}=\sum_{n\ge1}s(n)q^{-n},$$

$$\sum_{m\ge1}\frac{(-1)^{m-1}}{q^m+1}=\sum_{m\ge1}\frac{(-1)^{m-1}}{q^m}\sum_{\nu\ge0}\frac{(-1)^{\nu}}{q^{m\nu}}=\sum_{d,m\ge1}\frac{(-1)^{d+m}}{q^{dm}}=\sum_{n\ge1}t(n)q^{-n}.$$

Thus the case $\alpha=-1$ of Theorem 1 implies Corollary 4.

We finally investigate more closely the connection of the functions s and t with the divisor function τ. For this purpose we give the following

LEMMA 4. *The arithmetical functions s and t are multiplicative. For odd n, we have $s(n)=t(n)=\tau(n)$, whereas for even n, we have*

$$s(n)=-\frac{\nu_2(n)-1}{\nu_2(n)+1}\,\tau(n),\qquad t(n)=\frac{\nu_2(n)-3}{\nu_2(n)+1}\,\tau(n),$$

$\nu_2(n)\in\mathbb{N}$ *denoting the exact exponent of* 2 *in* n.

Proof. We first remark that the arithmetical function $r(n):=(-1)^{n-1}$ is multiplicative. Namely, if $n_1,n_2\in\mathbb{N}$ are coprime, then at least one of these numbers is odd, and hence the congruence $n_1n_2-1=(n_1-1)(n_2-1)+(n_1-1)+(n_2-1)\equiv(n_1-1)+(n_2-1)$ modulo 2 holds, yielding $r(n_1n_2)=r(n_1)r(n_2)$. Denoting by $*$ the Dirichlet convolution on the set of arithmetical functions, we see $s=r*\mathbf{1}$ (with $\mathbf{1}(n):=1$ for all $n\in\mathbb{N}$) and $t=r*r$, and thus s and t are also multiplicative (compare, e.g., [1, Chap. 2]).

Since $r(n)=1$ for odd n, we have $s(n)=\sum_{d|n}r(d)=\tau(n)$, and further $t(n)=\sum_{d|n}r(d)r(n/d)=\tau(n)$ for such n's. From $r(2^\nu)=-1$ for each $\nu\in\mathbb{N}$ we see $s(2^\nu)=1-\nu$, $t(2^\nu)=\nu-3$, and this proves our formulae for $s(n),t(n)$ if n is even.

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Mathematical Institute
University of Cologne
Weyertal 86-90
50931 Köln, Germany
E-mail: pb@math.uni-koeln.de

Department of Mathematical Sciences
University of Oulu
P.O. Box 3000
90014 Oulu, Finland
E-mail: kvaanane@sun3.oulu.fi

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