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A criterion for periodicity of multi-continued fraction expansion of multi-formal Laurent series

by

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1. Introduction. We start with the classical continued fraction algorithm over the formal Laurent series field $F((z^{-1}))$. Let $\mathbb{Z}$ be the ring of integers and F be a field. Denote by

$$F((z^{-1})) = \Big\{\sum_{i=t}^{\infty} a_i z^{-i} \,\Big|\, a_i \in F,\, t \in \mathbb{Z}\Big\}$$

the formal Laurent series field over F in z^{-1}. For any non-zero element $r = \sum_{i=t}^{\infty} a_i z^{-i}$ in $F((z^{-1}))$ with $t \le 0$, set

$$\lfloor r \rfloor = \sum_{i=t}^{0} a_i z^{-i} \quad \text{and} \quad \{r\} = \sum_{i=1}^{\infty} a_i z^{-i}.$$

They are called the *polynomial part* and *remainder part* of r respectively. The *classical continued fraction algorithm* [14] over $F((z^{-1}))$ is recalled below:

Let $r \in F((z^{-1}))$. Initially, set $a_0 = \lfloor r \rfloor$ and $\alpha_0 = \{r\}$. Suppose that for $k \ge 1$, we have obtained $[a_0, a_1, \ldots, a_{k-1}]$ and α_{k-1}. If $\alpha_{k-1} = 0$, let $\mu = k-1$ and the algorithm terminates; otherwise, do the following steps iteratively:

(1) set $\varrho_k = 1/\alpha_{k-1}$,
(2) set $a_k = \lfloor \varrho_k \rfloor$, $\alpha_k = \varrho_k - a_k$.

If the above procedure never stops, let $\mu = \infty$.

The output $[a_0, a_1, \ldots]$ of the algorithm with input r is called the *classical continued fraction expansion* $C(r)$ of r.

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The continued fraction algorithm is a useful tool in dealing with many number-theoretic problems and numerical computation problems [9, 10, 12]. It is well-known that the continued fraction expansion $C(r)$ gives the optimal rational approximation of a single element r [14]. Many people have contrived to construct multi-dimensional continued fractions in dealing with the rational approximation problem for multi-reals. One construction is the Jacobi–Perron algorithm (JPA) [1]. This algorithm and its modification are extensively studied [7, 8, 11, 13]. These algorithms have been adapted to study the same problem for multi-dimensional formal Laurent series [4, 6]. But none of these algorithms guarantees optimal rational approximations to the general multi-dimensional formal Laurent series.

Let $C(r) = [a_0, a_1, \ldots]$ be the classical continued fraction expansion of r. It is called (λ, T)-*periodic* if there exist integers $\lambda \geq 1$ and $T \geq 1$ such that $a_{\lambda+T+k} = a_{\lambda+k}$ for all $k \geq 0$. We then also call $C(r)$ *periodic* for short. It is known that $C(r)$ is (λ, T)-periodic if and only if the $(\lambda-1)$th partial remainder $\alpha_{\lambda-1}$ and the $(\lambda+T-1)$th partial remainder $\alpha_{\lambda+T-1}$ are equal. In [2, 3], the classical continued fraction algorithm is generalized to an algorithm acting on a multi-formal Laurent series $\underline{r}$ in $F((z^{-1}))^m$, $m > 1$, and called the *multi-continued fraction algorithm* (or *multi-dimensional continued fraction algorithm*), m-CFA for short. Likewise, the m-CFA provides a multi-continued fraction expansion of a multi-series $\underline{r}$ in $F((z^{-1}))^m$ and a method of finding the optimal rational approximation of $\underline{r}$ as well. It is natural to ask whether the same criterion of periodicity is valid for multi-continued fraction expansions. Unfortunately, equality of the $(\lambda-1)$th and $(\lambda+T-1)$th partial remainders is not enough for $C(\underline{r})$ to be (λ, T)-periodic for $m > 1$.

In this paper, we provide a criterion to determine whether a multi-continued fraction expansion is (λ, T)-periodic.

This paper is organized as follows. In Section 2, some preliminaries are provided, which include the indexed valuation over $F((z^{-1}))^m$, the m-CFA, some parameters and some main properties of multi-continued fraction expansions. The main theorem, a criterion of periodicity of multi-continued fraction expansions, is stated in Section 3. Some preparatory lemmas and the proof of the main theorem are given in Sections 4 and 5 respectively.

2. Preliminaries. In this section we briefly recall some concepts such as the indexed valuation and m-CFA, all of which may be found in [2, 3].

2.1. *Indexed valuation over $F((z^{-1}))^m$.* Let F be a field and m be a positive integer. We denote by $F[z]^m$ and $F((z^{-1}))^m$ the spaces of column m-vectors over the polynomial ring $F[z]$ and over the formal Laurent series field

$F((z^{-1}))$ respectively. Before introducing the concept of indexed valuation over $F((z^{-1}))^m$, we first define an order over $Z_m \times \mathbb{Z}$, where Z_m denotes the set $\{1, \ldots, m\}$.

DEFINITION 2.1. For any (h, v) and (h', v') in $Z_m \times \mathbb{Z}$, we define $(h, v) < (h', v')$ if $v < v'$ or $v = v', h < h'$.

It is clear that the order defined above is linear [5].

Let $r = \sum_{i=t}^{\infty} a_i z^{-i}$ be a non-zero element in $F((z^{-1}))$. Then the integer t is called the *discrete valuation* [5] of r if $a_t \neq 0$, and is denoted by $v(r)$. By convention, $v(0) = \infty$.

DEFINITION 2.2. Let $\underline{r} = (r_1, \ldots, r_m)^\tau \in F((z^{-1}))^m \setminus \{0\}$, where τ means transpose. We define

$$Iv(\underline{r}) = (h, v),$$

where

$$v = \min\{v(r_j) \mid j \in Z_m\}, \qquad h = \min\{j \in Z_m \mid v(r_j) = v\}$$

and call $Iv(\underline{r})$ the *indexed valuation* of $\underline{r}$, v the *valuation* of $\underline{r}$, denoted by $v(\underline{r})$, and h the *index* of $\underline{r}$, denoted by $I(\underline{r})$. By convention, $Iv(\underline{0}) = (1, \infty)$.

For each $j \in Z_m$, let $\underline{e}_j = (e_{j,1}, \ldots, e_{j,m})^\tau$, where $e_{j,i} = 0$ for $i \neq j$ and $e_{j,j} = 1$, which is exactly the jth standard basis element in the column vector space of dimension m over $F((z^{-1}))$. For any non-zero element $\underline{r} = (r_1, \ldots, r_m)^\tau$ in $F((z^{-1}))^m$, where $r_j = \sum_i r_{j,i} z^{-i}$ for each $j \in Z_m$, if $Iv(\underline{r}) = (h, v)$, we call $r_{h,v} z^{-v} \underline{e}_h$ the *leading term* of $\underline{r}$, denoted by $Ld_0(\underline{r})$. The indexed valuation over $F((z^{-1}))^m$ has the following basic properties:

PROPOSITION 2.3. *Let* $\underline{\alpha}, \underline{\beta} \in F((z^{-1}))^m$. *Then*

(1) $Iv(\underline{\alpha}) \neq (1, \infty)$ *if and only if* $\underline{\alpha} \neq \underline{0}$.
(2) *If* $Iv(\underline{\alpha}) = (h, v)$, *then* $Iv(r\underline{\alpha}) = (h, v + v(r))$ *for any non-zero* r *in* $F((z^{-1}))$.
(3) $Iv(\underline{\alpha} - \underline{\beta}) \geq \min\{Iv(\underline{\alpha}), Iv(\underline{\beta})\}$, *and equality holds if and only if* $Ld_0(\underline{\alpha}) \neq Ld_0(\underline{\beta})$.

2.2. *m-CFA.* We first introduce some related notations and concepts. For any $\underline{r} = (r_1, \ldots, r_m)^\tau \in F((z^{-1}))^m$, we define

$$\lfloor \underline{r} \rfloor = (\lfloor r_1 \rfloor, \ldots, \lfloor r_m \rfloor)^\tau, \qquad \{\underline{r}\} = (\{r_1\}, \ldots, \{r_m\})^\tau,$$

which are called the *polynomial part* and *remainder part* of $\underline{r}$.

In this paper, we denote by $\mathrm{Diag}(\beta_1, \ldots, \beta_m)$ the diagonal matrix with the ith diagonal element being β_i.

The m-CFA can be described as below:

m-CFA: Let $\underline{r} \in F((z^{-1}))^m$. Initially, set $\underline{a}_0 = \lfloor \underline{r} \rfloor$, $\underline{\alpha}_0 = \{\underline{r}\}$, $\Delta_0 = I_m$. Suppose that for $k \geq 1$, we have obtained

$$\begin{bmatrix} & h_1 & \cdots & h_{k-1} \\ \underline{a}_0 & \underline{a}_1 & \cdots & \underline{a}_{k-1} \end{bmatrix},$$

$$\underline{\alpha}_{k-1} = (\alpha_{k-1,1}, \ldots, \alpha_{k-1,m}) \in F((z^{-1}))^m,$$

and

$$\Delta_{k-1} = \mathrm{Diag}(z^{-v_{k-1,1}}, \ldots, z^{-v_{k-1,m}}).$$

If $\underline{\alpha}_{k-1} = \underline{0}$, let $\mu = k-1$ and the algorithm terminates; otherwise, do the following steps iteratively:

(1) set $(h_k, v_k) = Iv(\Delta_{k-1}\underline{\alpha}_{k-1})$,
(2) set $\Delta_k = \mathrm{Diag}(z^{-v_{k,1}}, \ldots, z^{-v_{k,m}})$, where

$$v_{k,j} = \begin{cases} v_{k-1,j} & \text{if } j \neq h_k, \\ v_k & \text{if } j = h_k, \end{cases}$$

(3) set $\underline{\varrho}_k = (\varrho_{k,1}, \ldots, \varrho_{k,m})^\tau$, where

$$\varrho_{k,j} = \begin{cases} \alpha_{k-1,j}/\alpha_{k-1,h_k} & \text{if } j \neq h_k, \\ 1/\alpha_{k-1,h_k} & \text{if } j = h_k, \end{cases}$$

(4) set $\underline{a}_k = \lfloor \underline{\varrho}_k \rfloor$, $\underline{\alpha}_k = \{\underline{\varrho}_k\}$.

If the above procedure never stops, let $\mu = \infty$.

It is proved that m-CFA is well defined, that is, $\alpha_{k-1,h_k} \neq 0$ for $1 \leq k \leq \mu$. As a result of the m-CFA acting on $\underline{r}$, we obtain a sequence pair

$$C(\underline{r}) = (\underline{h}, \underline{a}) = \begin{bmatrix} & h_1 & \cdots & h_\mu \\ \underline{a}_0 & \underline{a}_1 & \cdots & \underline{a}_\mu \end{bmatrix},$$

where $\underline{h} = \{h_k\}_{1 \leq k \leq \mu}$, $\underline{a} = \{\underline{a}_k\}_{0 \leq k \leq \mu}$, $1 \leq h_k \leq m$ and $\underline{a}_k \in F[z]^m$. We call $C(\underline{r})$ the *multi-continued fraction expansion* of $\underline{r}$, and call μ the *length* of $C(\underline{r})$.

$C(\underline{r})$ provides an optimal rational approximation to $\underline{r}$ by the following procedure:

Let

$$A(\underline{a}_k) = \begin{pmatrix} I_m & \underline{a}_k \\ \mathbf{0} & 1 \end{pmatrix}, \qquad 0 \leq k \leq \mu,$$

and

$$B_0 = A(\underline{a}_0), \qquad B_k = B_{k-1} E_{h_k} A(\underline{a}_k), \qquad k \geq 1,$$

where I_m is the identity matrix of order m, and E_{h_k} is a permutation matrix of order $m+1$ obtained by exchanging the h_kth and $(m+1)$th columns of I_{m+1}.

Let

$$\begin{pmatrix} \underline{p}_k \\ q_k \end{pmatrix} = B_k \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix},$$

the rightmost column of B_k, where $\underline{p}_k \in F[z]^m$ and $q_k \in F[z]$. We call $\underline{p}_k/q_k$ the kth rational fraction of $C(\underline{r})$. In [2, 3], we proved that $\underline{p}_k/q_k$ is an optimal rational approximant of $\underline{r}$ for all $0 \le k \le \mu$.

To see how close to $\underline{r}$ the kth rational fraction $\underline{p}_k/q_k$ is and further study the properties of $C(\underline{r})$, we have to recall some parameters.

For all $1 \le k \le \mu$, let $\underline{a}_k = (a_{k,1}, \dots, a_{k,m})$ and denote by $\deg(a_{k,j})$ the degree of $a_{k,j}$. It is known [2, 3] that $a_{k,h_k} \neq 0$ and $\deg(a_{k,h_k}) \ge 1$.

For all $1 \le k \le \mu$, we define

$$t_k = \deg(a_{k,h_k}), \qquad d_k = \sum_{1\le i\le k} t_i.$$

Definition 2.4. Let $S \subseteq Z_m$. Denote by $D(S, z^{-c})$ the diagonal matrix

$$\mathrm{Diag}(z^{-c_1}, \dots, z^{-c_m})$$

where

$$c_j = \begin{cases} c & \text{if } j \in S, \\ 0 & \text{if } j \notin S. \end{cases}$$

When $S = \{h\}$, we simply denote $D(S, z^{-c})$ by $D(h, z^{-c})$.

Then we have

Theorem 2.5 ([2, 3]).

(1) *$\alpha_{k-1,h_k} \neq 0$ and $t_k = v(\alpha_{k-1,h_k}) \ge 1$ for all $1 \le k \le \mu$.*
(2) *For any $1 \le k \le \mu$, $1 \le j \le m$, we have*

$$v_{k,j} = \sum_{\substack{1\le i\le k \\ h_i = j}} t_i, \qquad v_k = v_{k,h_k}, \qquad v_{0,j} = 0.$$

As a consequence, $\Delta_k = \Delta_{k-1} D(h_k, z^{-t_k})$.

(3) *$Iv(\Delta_k \underline{a}_k) = Iv(\Delta_k \underline{\varrho}_k) = (h_k, v_{k-1,h_k}) < Iv(\Delta_k \underline{\alpha}_k) = (h_{k+1}, v_{k+1})$ for all $1 \le k \le \mu$, where we let $(h_{\mu+1}, v_{\mu+1}) = (1, \infty)$ if $\mu < \infty$.*
(4) *Let*

$$(-I_m, \underline{r}) B_k = (-R_{k-1}, \underline{r}_k)$$

for $k \ge 1$, where R_{k-1} is a square matrix of order m and $\underline{r}_k$ is an element in $F((z^{-1}))^m$. Then R_{k-1} is invertible and

$$\begin{cases} \underline{r}_k = \underline{r} q_k - \underline{p}_k = \{\underline{r} q_k\} = R_{k-1} \underline{\alpha}_k, \\ Iv(\underline{r}_k) = (h_{k+1}, v_{k+1}) \end{cases}$$

for any $0 \le k \le \mu$. As a consequence, $\underline{r}_k \neq \underline{0}$ for any $0 \le k < \mu$, and $\underline{r}_\mu = \underline{0}$ if $\mu < \infty$.

Theorem 2.6 ([2, 3]).

(1) $\deg(q_k) = d_k$ *and*

$$Iv(\underline{r} - \underline{p}_k/q_k) = (h_{k+1}, d_k + v_{k+1})$$

for all $0 \le k \le \mu$.

(2) *We have*

$$\underline{r} = \begin{cases} \lim_{k\to\infty} \dfrac{\underline{p}_k}{q_k} & \text{if } \mu = \infty, \\ \dfrac{\underline{p}_\mu}{q_\mu} & \text{if } \mu < \infty. \end{cases}$$

As a consequence, $\mu < \infty$ *if and only if* $\underline{r} \in F(z)^m$.

(3) $\underline{p}_k/q_k$ *is an optimal rational approximant of* $\underline{r}$. *Moreover*, *if* $\underline{p}/q$ *is an optimal rational approximant of* $\underline{r}$, *then* $\deg(q) = d_k$ *for some* k.

3. Periodicity of multi-continued fraction expansion. In the following, we only study infinite multi-continued fraction expansions, that is, we assume $\mu = \infty$.

Definition 3.1. We say that $C(\underline{r})$ is (λ, T)-*periodic*, where $\lambda \ge 1$ and $T \ge 1$, if

$$(h_{\lambda+k+T}, \underline{a}_{\lambda+k+T}) = (h_{\lambda+k}, \underline{a}_{\lambda+k}) \quad \text{for all } k \ge 0.$$

m-CFA is an iterative algorithm. A practical problem is how to determine whether $C(\underline{r})$ is (λ, T)-periodic. In this paper, we provide a criterion that permits one to determine whether $C(\underline{r})$ is (λ, T)-periodic only by means of the data obtained in the process of m-CFA.

When $m = 1$, the multi-continued fraction expansions are exactly the classical continued fraction expansions. In this case, we have $\underline{\varrho}_k = 1/\underline{\alpha}_{k-1}$. The continued fraction expansion $C(\underline{r})$ of $\underline{r}$ is (λ, T)-periodic if and only if $\underline{\alpha}_{\lambda-1} = \underline{\alpha}_{\lambda+T-1}$. However, when $m \ge 2$, the condition $\underline{\alpha}_{\lambda-1} = \underline{\alpha}_{\lambda+T-1}$ alone does not guarantee that $C(\underline{r})$ is (λ, T)-periodic.

Definition 3.2. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)^\tau \in F((z^{-1}))^m$. We define

$$J^*(\underline{\alpha}) = \{j \mid \alpha_j \ne 0,\ 1 \le j \le m\}.$$

For the given multi-continued fraction expansion $C(\underline{r})$ of $\underline{r}$ obtained by m-CFA, we give a simple criterion to decide whether $C(\underline{r})$ is periodic.

Theorem 3.3 (Main Theorem). *For* $\lambda \ge 1$ *and* $T \ge 1$, *the following three conditions are equivalent*:

(1) $C(\underline{r})$ *is* (λ, T)*-periodic.*

(2) $\begin{cases} \underline{\alpha}_{\lambda+T-1} = \underline{\alpha}_{\lambda-1}, \\ \Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c}), \end{cases}$

where c *is a positive integer.*

(3) $\begin{cases} \underline{\varrho}_{\lambda+T} = \underline{\varrho}_{\lambda}, \\ \Delta_{\lambda}^{-1}\Delta_{\lambda+T} = D(J^*(\underline{\varrho}_{\lambda}), z^{-c}), \end{cases}$

where c *is a positive integer.*

Remark. In Theorem 3.3, neither $\underline{\alpha}_{\lambda+T-1} = \underline{\alpha}_{\lambda-1}$ (resp. $\underline{\varrho}_{\lambda+T} = \underline{\varrho}_{\lambda}$) nor $\Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c})$ (resp. $\Delta_{\lambda}^{-1}\Delta_{\lambda+T} = D(J^*(\underline{\varrho}_{\lambda}), z^{-c})$) in condition (2) (resp. (3)) can be rejected, since the following examples show that $\underline{\alpha}_{\lambda+T-1} = \underline{\alpha}_{\lambda-1}$ (resp. $\underline{\varrho}_{\lambda+T} = \underline{\varrho}_{\lambda}$) does not imply $\Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c})$ (resp. $\Delta_{\lambda}^{-1}\Delta_{\lambda+T} = D(J^*(\underline{\varrho}_{\lambda}), z^{-c})$), and vice versa.

Example 3.4. Let $m = 2$ and $\underline{r} = \binom{r_1}{r_2} \in F_2((z^{-1}))^2$, where $r_1 = 1/(z + r_2)$ and $r_2 = \sum_i c_i z^{-i} \in F_2((z^{-1}))$, where

$$c_i = \begin{cases} 0 & \text{if } i < 4, \\ 1 & \text{if } i = 4, \\ \displaystyle\sum_{j=2}^{k-1} c_j c_{i-2j-4} & \text{if } i = 2k > 4, \\ \displaystyle c_{k-1} + c_k + \sum_{j=2}^{k-1} c_j c_{i-2j-4} & \text{if } i = 2k+1 > 4. \end{cases}$$

It is straightforward to check that r_2 is a root of the algebraic equation $X^3 + (z + z^3)X^2 + z^4X + 1 = 0$ over $F_2((z^{-1}))$, and $v(r_2) = 4$. By [4], this algebraic equation is irreducible over $F_2(z)$. So, by Theorem 2.6, $\mu = \infty$.

Then, by m-CFA, we have

Parameters obtained in the process of m-CFA with input $\underline{r}$

k	h_k	Δ_k	$\underline{a}_k$	$\underline{\alpha}_k$	$\underline{\varrho}_k$
0		I_2	$\underline{0}$	$\binom{r_1}{r_2}$	
1	1	$\mathrm{Diag}(z^{-1}, 1)$	$\binom{z}{0}$	$\binom{r_2}{r_1^{-1}r_2}$	$\binom{z+r_2}{r_1^{-1}r_2}$
2	2	$\mathrm{Diag}(z^{-1}, z^{-3})$	$\binom{0}{z^3}$	$\binom{r_1}{r_2}$	$\binom{r_1}{z^3+r_2}$
3	1	$\mathrm{Diag}(z^{-2}, z^{-3})$	$\binom{z}{0}$	$\binom{r_2}{r_1^{-1}r_2}$	$\binom{z+r_2}{r_1^{-1}r_2}$

From the above computations, we see that

$$\begin{cases} \underline{\alpha}_0 = \underline{\alpha}_2, \\ \Delta_0^{-1}\Delta_2 = \mathrm{Diag}(z^{-1}, z^{-3}) \neq D(J^*(\underline{\alpha}_0), z^{-c}) = z^{-c}I_2, \end{cases}$$

and

$$\begin{cases} \underline{\varrho}_1 = \underline{\varrho}_3, \\ \Delta_1^{-1}\Delta_3 = \mathrm{Diag}(z^{-1}, z^{-3}) \neq D(J^*(\underline{\varrho}_1), z^{-c}) = z^{-c}I_2. \end{cases}$$

Example 3.5. Let $m = 2$ and

$$\underline{\alpha}_0 = \underline{r} = \begin{pmatrix} (z^2+r_1)^{-1} \\ r_2(z^2+r_1)^{-1} \end{pmatrix} \in F_2((z^{-1}))^2,$$

where r_1 and r_2 are as in Example 3.4. Just as in Example 3.4, $C(\underline{r})$ is infinite, that is, $\mu = \infty$.

Then, by m-CFA, we have

Parameters obtained in the process of m-CFA with input $\underline{r}$

k	h_k	Δ_k	$\underline{a}_k$	$\underline{\alpha}_k$	$\underline{\varrho}_k$
0		I_2	$\underline{0}$	$\begin{pmatrix} (z^2+r_1)^{-1} \\ (z^2+r_1)^{-1}r_2 \end{pmatrix}$	
1	1	$\mathrm{Diag}(z^{-2}, 1)$	$\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$	$\begin{pmatrix} z^2+r_1 \\ r_2 \end{pmatrix}$
2	1	$\mathrm{Diag}(z^{-3}, 1)$	$\begin{pmatrix} z \\ 0 \end{pmatrix}$	$\begin{pmatrix} r_2 \\ r_1^{-1}r_2 \end{pmatrix}$	$\begin{pmatrix} z+r_2 \\ r_1^{-1}r_2 \end{pmatrix}$
3	2	$\mathrm{Diag}(z^{-3}, z^{-3})$	$\begin{pmatrix} 0 \\ z^3 \end{pmatrix}$	$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$	$\begin{pmatrix} r_1 \\ z^3+r_2 \end{pmatrix}$
4	1	$\mathrm{Diag}(z^{-4}, z^{-3})$	$\begin{pmatrix} z \\ 0 \end{pmatrix}$	$\begin{pmatrix} r_2 \\ r_1^{-1}r_2 \end{pmatrix}$	$\begin{pmatrix} z+r_2 \\ r_1^{-1}r_2 \end{pmatrix}$
5	2	$\mathrm{Diag}(z^{-4}, z^{-6})$	$\begin{pmatrix} 0 \\ z^3 \end{pmatrix}$	$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$	$\begin{pmatrix} r_1 \\ z^3+r_2 \end{pmatrix}$
6	1	$\mathrm{Diag}(z^{-5}, z^{-6})$	$\begin{pmatrix} z \\ 0 \end{pmatrix}$	$\begin{pmatrix} r_2 \\ r_1^{-1}r_2 \end{pmatrix}$	$\begin{pmatrix} z+r_2 \\ r_1^{-1}r_2 \end{pmatrix}$
7	1	$\mathrm{Diag}(z^{-9}, z^{-6})$	$\begin{pmatrix} z^4 \\ z \end{pmatrix}$	$\begin{pmatrix} z^3r_2 + r_1^{-1}r_2 \\ r_2 \end{pmatrix}$	$\begin{pmatrix} z^4 + z^3r_2 + r_1^{-1}r_2 \\ r_1^{-1} \end{pmatrix}$

From the above computations, we see that

$$\begin{cases} \underline{\alpha}_0 \neq \underline{\alpha}_3, \\ \Delta_0^{-1}\Delta_3 = \mathrm{Diag}(z^{-3}, z^{-3}) = D(J^*(\underline{\alpha}_0), z^{-3}) = z^{-3}I_2, \end{cases}$$

and

$$\begin{cases} \underline{\varrho}_2 \neq \underline{\varrho}_7, \\ \Delta_2^{-1}\Delta_7 = \mathrm{Diag}(z^{-6}, z^{-6}) = D(J^*(\underline{\varrho}_2), z^{-6}) = z^{-6}I_2. \end{cases}$$

4. Some lemmas. Before proving the main theorem, we need to prepare some lemmas.

LEMMA 4.1.

(1) *For all* $k \geq 1$, $\alpha_{k-1,h_{k+1}} \neq 0$.
(2) *If* $\alpha_{k,j} = 0$, *then* $\alpha_{l,j} = 0$ *for all* $l \geq k$. *As a consequence*, $J^*(\underline{\alpha}_{k+1}) \subseteq J^*(\underline{\alpha}_k)$ *for all* $k \geq 0$.
(3) $\lim_{k\to\infty} v_k = \infty$.
(4) *For all* $l > k \geq 1$, *define*

$$B_{k,l} = E_{h_{k+1}} A(\underline{a}_{k+1}) \cdots E_{h_l} A(\underline{a}_l).$$

Let $\binom{\underline{p}_{k,l}}{q_{k,l}}$ *be the last column of* $B_{k,l}$, *where* $\underline{p}_{k,l} \in F[z]^m$ *and* $q_{k,l} \in F[z]$. *Then* $q_{k,l} \neq 0$, *and*

$$\underline{\alpha}_k = \lim_{l\to\infty} \frac{\underline{p}_{k,l}}{q_{k,l}}.$$

Proof. (1) Noting that $\Delta_k \underline{\alpha}_k \neq \underline{0}$ and $h_{k+1} = I(\Delta_k \underline{\alpha}_k)$, we have $v_{k,h_{k+1}} + v(\alpha_{k,h_{k+1}}) = v(\Delta_k \underline{\alpha}_k) = v_{k+1} \in \mathbb{Z}$. Thus, $v(\alpha_{k,h_{k+1}}) = v_{k+1} - v_{k,h_{k+1}} \in \mathbb{Z}$. Hence, $\{\varrho_{k,h_{k+1}}\} = \alpha_{k,h_{k+1}} \neq 0$. So, $\varrho_{k,h_{k+1}} \neq 0$. By the definition of $\underline{\varrho}_k$, $\alpha_{k-1,h_{k+1}} \neq 0$.

(2) We have $\varrho_{k+1,j} = \alpha_{k,j}/\alpha_{k,h_k} = 0$. Thus, $\alpha_{k+1,j} = \{\varrho_{k+1,j}\} = 0$. Repeating this process, we have $\alpha_{l,j} = 0$ for any $l \geq k$.

(3) Let

$$H_\infty = \{j \mid \text{there exist infinitely many } k\text{'s such that } h_k = j, k \geq 1\},$$
$$K_j = \{k \mid h_k = j,\ k \geq 1\}.$$

By definition, $|K_j| = \infty$ for any $j \in H_\infty$. Moreover, there exists k_0 such that

$$\bigcup_{j\in H_\infty} K_j = \{k \mid k \geq k_0\} \cup S,$$

where S is a finite integer set.

By Theorem 2.5,

$$v_k = \sum_{\substack{1\leq i\leq k\\ h_i = h_k}} t_i, \quad \text{so} \quad \lim_{\substack{k\to\infty\\ k\in K_j}} v_k = \infty$$

for any $j \in H_\infty$. Further,

$$\lim_{k\to\infty} v_k = \lim_{\substack{k\to\infty\\ k\geq k_0}} v_k = \lim_{\substack{k\to\infty\\ k\in\bigcup_{j\in H_\infty} K_j}} v_k = \infty.$$

(4) Noting that

$$B_l = B_k B_{k,l} \quad \text{and} \quad (-R_{k-1}, \underline{r}_k) = (-I_m, \underline{r}) B_k,$$

we have

$$(-R_{l-1}, \underline{r}_l) = (-I_m, \underline{r})B_l = (-R_{k-1}, \underline{r}_k)B_{k,l}.$$

Then

$$\underline{r}_l = (-R_{k-1}, \underline{r}_k)\begin{pmatrix}\underline{p}_{k,l}\\ q_{k,l}\end{pmatrix}.$$

Suppose $q_{k,l} = 0$. Then $\underline{p}_{k,l} \neq 0$ and $\underline{p}_{k,l} = -R_{k-1}^{-1}\underline{r}_l$. Thus

$$v(-R_{k-1}^{-1}\underline{r}_l) = v(\underline{p}_{k,l}) \leq 0.$$

Since $\lim_{l\to\infty} v_l = \infty$, we have

$$\lim_{l\to\infty} v(\underline{r}_l) = \lim_{l\to\infty} v_{l+1} = \infty.$$

So there exists some integer l such that $v(-R_{k-1}^{-1}\underline{r}_l) > 0$, a contradiction. Hence, $q_{k,l} \neq 0$.

By Theorem 2.5, we have

$$\underline{r}_l = (-R_{k-1}, R_{k-1}\underline{\alpha}_k)\begin{pmatrix}\underline{p}_{k,l}\\ q_{k,l}\end{pmatrix} = R_{k-1}(-\underline{p}_{k,l} + \underline{\alpha}_k q_{k,l}).$$

Then since $\lim_{l\to\infty} v(\underline{r}_l) = \lim_{l\to\infty} v_{l+1} = \infty$, we have

$$\lim_{l\to\infty} v\left(\underline{\alpha}_k - \frac{\underline{p}_{k,l}}{q_{k,l}}\right) = \lim_{l\to\infty} v\left(\frac{R_{k-1}^{-1}\underline{r}_l}{q_{k,l}}\right) = \infty.$$

So,

$$\underline{\alpha}_k = \lim_{l\to\infty} \frac{\underline{p}_{k,l}}{q_{k,l}}. \blacksquare$$

LEMMA 4.2.

(1) *If* $h_{\lambda+T} = h_\lambda$ *and* $t_{\lambda+T} = t_\lambda$, *then*

$$\Delta_\lambda^{-1}\Delta_{\lambda+T} = \Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1}.$$

(2) *If* $h_{\lambda+T} = h_\lambda$, *then*

$$\underline{\varrho}_{\lambda+T} = \underline{\varrho}_\lambda \;\Leftrightarrow\; \underline{\alpha}_{\lambda+T-1} = \underline{\alpha}_{\lambda-1}.$$

(3) *Let* $\underline{\alpha} \in F((z^{-1}))^m$. *Then*

$$D(J^*(\underline{\alpha}), z^{-c})\underline{\alpha} = z^{-c}\underline{\alpha}.$$

(4) *If* $\underline{\alpha}_{k+l} = \underline{\alpha}_k$ *for some* $l \geq 1$, *then*

$$J^*(\underline{\alpha}_{k+i}) = J^*(\underline{\alpha}_k) \quad \textit{for any } 0 \leq i \leq l.$$

(5) $J^*(\underline{\varrho}_k) = J^*(\underline{\alpha}_{k-1})$ *for any* $k \geq 1$.

Proof. (1) By Theorem 2.5, we have

$$\Delta_{\lambda+T} = \Delta_{\lambda+T-1}D(h_{\lambda+T}, z^{-t_{\lambda+T}}), \quad \Delta_\lambda = \Delta_{\lambda-1}D(h_\lambda, z^{-t_\lambda}).$$

So,

$$\Delta_{\lambda}^{-1}\Delta_{\lambda+T} = \Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1}.$$

(2) & (3) Obvious.

(4) By Lemma 4.1, we have

$$J^*(\underline{\alpha}_k) = J^*(\underline{\alpha}_{k+l}) \subseteq \cdots \subseteq J^*(\underline{\alpha}_{k+1}) \subseteq J^*(\underline{\alpha}_k).$$

So, $J^*(\underline{\alpha}_{k+i}) = J^*(\underline{\alpha}_k)$ for any $0 \le i \le l$.

(5) Obvious. ∎

5. The proof of the main theorem. In this section, we will prove the main theorem.

(1)⇒(2). Since $C(\underline{r})$ is (λ, T)-periodic, we have $h_{\lambda+k} = h_{\lambda+k+T}$ and $B_{\lambda+k,\lambda+l} = B_{\lambda+k+T,\lambda+l+T}$ for any $l > k \ge 0$. Then

$$\begin{pmatrix} \underline{p}_{\lambda+k,\lambda+l} \\ q_{\lambda+k,\lambda+l} \end{pmatrix} = \begin{pmatrix} \underline{p}_{\lambda+k+T,\lambda+l+T} \\ q_{\lambda+k+T,\lambda+l+T} \end{pmatrix}.$$

Thus

$$\underline{\alpha}_{\lambda+k-1} = \lim_{l\to\infty} \frac{\underline{p}_{\lambda+k,\lambda+l}}{q_{\lambda+k,\lambda+l}} = \lim_{l\to\infty} \frac{\underline{p}_{\lambda+k+T,\lambda+l+T}}{q_{\lambda+k+T,\lambda+l+T}} = \underline{\alpha}_{\lambda+k+T-1}$$

for all $k \ge 0$ from Lemma 4.1.

Define $v_{\lambda+T-1,j} - v_{\lambda-1,j} = c_j$. Then

$$\Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = \mathrm{Diag}(z^{-c_1}, \ldots, z^{-c_m}), \quad \text{where} \quad c_j = \sum_{\substack{\lambda \le i < \lambda+T \\ h_i = j}} t_i.$$

Let $J = \{h_k \mid \lambda \le k < \lambda + T\}$. Then

$$(*) \qquad c_j \begin{cases} = 0 & \text{if } j \notin J, \\ > 0 & \text{if } j \in J. \end{cases}$$

We claim that c_j is a positive constant c for all $j \in J$, hence $\Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = D(J, z^{-c})$. In fact, for $k \ge \lambda, n \ge 1$ and $1 \le j \le m$, we have

$$\begin{aligned} (**) \qquad v_k + nc_{h_k} &= v_{k,h_k} + nc_{h_k} = v_{k+nT,h_k} = v_{k+nT,h_{k+nT}} = v_{k+nT} \\ &\le v_{k+nT-1,j} + v(\alpha_{k+nT-1,j}) = v_{k-1,j} + nc_j + v(\alpha_{k-1,j}), \end{aligned}$$

where the inequality comes from the fact that

$$\begin{aligned} (h_k, v_{k+nT}) = (h_{k+nT}, v_{k+nT}) &= Iv(\Delta_{k+nT-1}\underline{\alpha}_{k+nT-1}) \\ &\le (j, v_{k+nT-1,j} + v(\alpha_{k+nT-1,j})). \end{aligned}$$

Then taking $j = h_{k+1}$, we have

$$v_k + nc_{h_k} \le v_{k-1,h_{k+1}} + nc_{h_{k+1}} + t,$$

where $t = v(\alpha_{k-1,h_{k+1}}) \in \mathbb{Z}$, since $\alpha_{k-1,h_{k+1}} \neq 0$. This implies $c_{h_k} \leq c_{h_{k+1}}$ for any $k \geq \lambda$, since otherwise,

$$-n \geq n(c_{h_{k+1}} - c_{h_k}) \geq v_k - v_{k-1,h_{k+1}} - t$$

for all $n \geq 1$, a contradiction. So

$$c_{h_\lambda} \leq c_{h_{\lambda+1}} \leq \cdots \leq c_{h_{\lambda+T-1}} \leq c_{h_{\lambda+T}} = c_{h_\lambda}.$$

Thus $c_j = c_{h_\lambda}$ for any $j \in J$, i.e., c_j is equal to a positive constant c.

We claim that $J = J^*(\underline{\alpha}_{\lambda-1})$. In fact, if $j \in J$, then $j = h_k$ and $\alpha_{k-1,j} = \alpha_{k-1,h_k} \neq 0$ for some k satisfying $\lambda \leq k < \lambda + T$. Thus, $j \in J^*(\underline{\alpha}_{k-1}) \subseteq J^*(\underline{\alpha}_{\lambda-1})$ from Lemma 4.1.

If $j \notin J$, then

$$v_\lambda + nc = v_\lambda + nc_{h_\lambda} \leq v_{\lambda-1,j} + nc_j + v(\alpha_{\lambda-1,j}) = v_{\lambda-1,j} + v(\alpha_{\lambda-1,j})$$

for any $n \geq 1$ from $(*)$ and $(**)$. Noting that $c > 0$, we have $v(\alpha_{\lambda-1,j}) = \infty$ and $\alpha_{\lambda-1,j} = 0$. Thus, $j \notin J^*(\underline{\alpha}_{\lambda-1})$.

From the above we get immediately

$$\Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = D(J, z^{-c}) = D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c}).$$

(2)⇒(3). By Lemma 4.2,

$$\Delta_{\lambda+T-1}\underline{\alpha}_{\lambda+T-1} = \Delta_{\lambda-1}D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c})\underline{\alpha}_{\lambda-1} = z^{-c}\Delta_{\lambda-1}\underline{\alpha}_{\lambda-1}.$$

Then $h_{\lambda+T} = I(\Delta_{\lambda+T-1}\underline{\alpha}_{\lambda+T-1}) = I(z^{-c}\Delta_{\lambda-1}\underline{\alpha}_{\lambda-1}) = h_\lambda$. By Lemma 4.2, we have $\underline{\varrho}_{\lambda+T} = \underline{\varrho}_\lambda$. So, $\underline{a}_{\lambda+T} = \underline{a}_\lambda$, $\underline{\alpha}_{\lambda+T} = \underline{\alpha}_\lambda$ and $t_{\lambda+T} = t_\lambda$. By Lemma 4.2,

$$\Delta_\lambda^{-1}\Delta_{\lambda+T} = \Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c}) = D(J^*(\underline{\varrho}_\lambda), z^{-c}).$$

(3)⇒(2). By Lemma 4.2,

$$\Delta_{\lambda+T}\underline{\varrho}_{\lambda+T} = \Delta_\lambda D(J^*(\underline{\varrho}_\lambda), z^{-c})\underline{\varrho}_\lambda = z^{-c}\Delta_\lambda\underline{\varrho}_\lambda.$$

Then by Theorem 2.5,

$$h_{\lambda+T} = I(\Delta_{\lambda+T}\underline{\varrho}_{\lambda+T}) = I(z^{-c}\Delta_\lambda\underline{\varrho}_\lambda) = h_\lambda,$$

and

$$\begin{aligned} t_{\lambda+T} &= \deg(a_{\lambda+T,h_{\lambda+T}}) = \deg(\lfloor \varrho_{\lambda+T,h_{\lambda+T}} \rfloor) = \deg(\lfloor \varrho_{\lambda,h_{\lambda+T}} \rfloor) \\ &= \deg(\lfloor \varrho_{\lambda,h_\lambda} \rfloor) = \deg(a_{\lambda,h_\lambda}) = t_\lambda. \end{aligned}$$

Thus by Lemma 4.2,

$$\Delta_{\lambda-1}^{-1}\Delta_{\lambda+T-1} = \Delta_\lambda^{-1}\Delta_{\lambda+T} = D(J^*(\underline{\varrho}_\lambda), z^{-c}) = D(J^*(\underline{\alpha}_{\lambda-1}), z^{-c}).$$

(3)⇒(1). We already know that conditions (2) and (3) are equivalent, so we can suppose that they both hold. We claim that

(a) $(h_{\lambda+T}, \underline{a}_{\lambda+T}) = (h_\lambda, \underline{a}_\lambda)$;

(b) $\underline{\varrho}_{\lambda+1+T} = \underline{\varrho}_{\lambda+1}$, $\Delta_{\lambda+1}^{-1}\Delta_{\lambda+1+T} = D(J^*(\underline{\varrho}_{\lambda+1}), z^{-c})$.

Repeating this process, we find that $C(\underline{r})$ is (λ, T)-periodic.

In fact, in (3)⇒(2) above, we have proven $h_{\lambda+T} = h_\lambda$. By definitions, $\underline{\varrho}_{\lambda+T} = \underline{\varrho}_\lambda$ implies that

$$\underline{a}_{\lambda+T} = \underline{a}_\lambda, \qquad \underline{\alpha}_{\lambda+T} = \underline{\alpha}_\lambda.$$

Since (3) implies (2), we have $\underline{\alpha}_{\lambda+T-1} = \underline{\alpha}_{\lambda-1}$. Then $J^*(\underline{\alpha}_\lambda) = J^*(\underline{\alpha}_{\lambda-1}) = J^*(\underline{\varrho}_\lambda)$ from Lemma 4.2. So

$$\Delta_\lambda^{-1}\Delta_{\lambda+T} = D(J^*(\underline{\varrho}_\lambda), z^{-c}) = D(J^*(\underline{\alpha}_\lambda), z^{-c}).$$

Since (2) implies (3), from the above we conclude that

$$\underline{\varrho}_{\lambda+1+T} = \underline{\varrho}_{\lambda+1}, \qquad \Delta_{\lambda+1}^{-1}\Delta_{\lambda+1+T} = D(J^*(\underline{\varrho}_{\lambda+1}), z^{-c}). \ \blacksquare$$

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