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Some identities involving the Dirichlet L-function

by

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1. Introduction. Let $q \geq 3$ be an integer, χ be a Dirichlet character modulo q and

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

If χ is a primitive character, H. Walum [3] established a connection between $((x))$ and the Dirichlet L-function $L(s, \chi)$ as follows:

$$\sum_{a=1}^{q} \chi(a) \left(\left(\frac{a}{q} \right) \right) = \begin{cases} -\dfrac{\tau(\chi)}{\pi i} L(1, \overline{\chi}) & \text{if } \chi(-1) = -1, \\ 0 & \text{if } \chi(-1) = 1, \end{cases}$$

where

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right)$$

is the Gauss sum, and $e(y) = e^{2\pi i y}$. By using this connection he obtained the beautiful exact formula for the mean value of $L(1, \chi)$ in the case of $q = p$ a prime:

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 = \frac{(p-1)^2(p-2)}{12p^2} \pi^2.$$

For general q, the second author [5] got the following identity:

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1, \chi)|^2 = \frac{\pi^2}{12} \cdot \frac{\phi^2(q)}{q^2} \left(q \prod_{p \mid q} \left(1 + \frac{1}{p}\right) - 3 \right), \tag{1}$$

where $\phi(q)$ is the Euler function. In [6], another proof of (1) was given by the second author. This new method can also be used to calculate the general

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mean value

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^n}} L(n,\chi)L(m,\overline{\chi})$$

if $2\,|\,(n+m)$, where n, m are positive integers (see [2]). But for the case $2 \nmid (n+m)$, this method does not work even for $q = p$ and $m = 1$.

The present work deals mainly with some mean values involving the Dirichlet L-function by using the arithmetical properties of the character sums over short intervals and the periodic Bernoulli polynomials $\overline{B}_n(x)$, where $\overline{B}_n(x) = B_n(x - [x])$, $[x]$ denotes the greatest integer less than or equal to x, and $B_n(x)$ are the classical Bernoulli polynomials defined by the following exponential generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}.$$

Finally, some new connections between the Dirichlet L-function and the periodic Bernoulli polynomials are obtained. Namely, we shall prove the following results:

THEOREM 1. *Let $p \geq 5$ be a prime and n be a positive integer. Then we have the identities*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} L(1,\overline{\chi}\chi_4)L(n,\chi)$$

$$= -\frac{\pi\chi_4(p)(p-1)}{p}\left[\frac{\zeta(n)}{p^n}\left[\frac{p}{4}\right] + \frac{2^{n-1}(-1)^{n/2}\pi^n}{n!}\sum_{r\leq[p/4]}\overline{B}_n\left(\frac{r}{p}\right)\right]$$

if n is even, and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (2+\overline{\chi}(2)-\overline{\chi}(4))L(1,\overline{\chi})L(n,\chi) = \frac{(2i\pi)^{n+1}(p-1)}{2n!p}\sum_{r\leq[p/4]}\overline{B}_n\left(\frac{r}{p}\right)$$

if n is odd, where χ_4 denotes the primitive character modulo 4 and $\zeta(n)$ the Riemann zeta function.

THEOREM 2. *Let $p \geq 3$ be a prime and n be a positive integer. Then we have the identities*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} (\overline{\chi}(2)-2)L(1,\overline{\chi})L(n,\chi)$$

$$= -\frac{\pi(p-1)}{2ip}\left[\frac{\zeta(n)(p-1)}{p^n} + \frac{2^n(-1)^{n/2}\pi^n}{n!}\sum_{r\leq(p-1)/2}\overline{B}_n\left(\frac{r}{p}\right)\right]$$

if n is even, and

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (\overline{\chi}(2)-2)L(1,\overline{\chi})L(n,\chi) = \frac{(2i)^n\pi^{n+1}(p-1)}{2in!p}\sum_{r\le (p-1)/2}\overline{B}_n\left(\frac{r}{p}\right)$$

if n is odd.

In our theorems, the terms

$$\sum_{r\le [p/4]}\overline{B}_n\left(\frac{r}{p}\right)\quad \text{and}\quad \sum_{r\le (p-1)/2}\overline{B}_n\left(\frac{r}{p}\right)$$

could be easily calculated by using the explicit expression of the Bernoulli polynomials. In particular, for $n=1,2,3$, noting that

$$B_1(x)=x-\frac{1}{2},\quad B_2(x)=x^2-x+\frac{1}{6},\quad B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x,$$

we immediately get the following exact formulae:

COROLLARY 1. *Let $p\ge 5$ be a prime. Then we have the identities*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}} L(1,\overline{\chi}\chi_4)L(2,\chi) = \begin{cases} \dfrac{\pi^3(p-1)^2}{64p}\left(1-\dfrac{14}{3p}-\dfrac{5}{3p^2}\right) & \text{if } p\equiv 1 \pmod 4,\\ -\dfrac{\pi^3(p-3)}{64}\left(1-\dfrac{3}{p}-\dfrac{1}{p^2}+\dfrac{3}{p^3}\right) & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (2+\overline{\chi}(2)-\overline{\chi}(4))|L(1,\chi)|^2 = \begin{cases} \dfrac{3\pi^2(p-1)}{16}\left(1-\dfrac{1}{p}\right)^2 & \text{if } p\equiv 1 \pmod 4,\\ \dfrac{3\pi^2(p-3)}{16}\left(1-\dfrac{4}{3p}+\dfrac{1}{3p^2}\right) & \text{if } p\equiv 3 \pmod 4\end{cases}$$

and

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (2+\overline{\chi}(2)-\overline{\chi}(4))L(1,\overline{\chi})L(3,\chi) = \begin{cases} \dfrac{3\pi^4(p+3)(p-1)^2}{256p^2}\left(1-\dfrac{2}{3p}-\dfrac{1}{3p^2}\right) & \text{if } p\equiv 1 \pmod 4,\\ \dfrac{3\pi^4(p-3)}{256}\left(1+\dfrac{2}{3p}-\dfrac{4}{3p^2}-\dfrac{2}{3p^3}+\dfrac{1}{3p^4}\right) & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Corollary 2. *Let $p \geq 3$ be a prime. Then we have the identities*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} (\overline{\chi}(2)-2)L(1,\overline{\chi})L(2,\chi) = \frac{i\pi^3}{12}\left(1-\frac{1}{p}-\frac{1}{p^2}+\frac{1}{p^3}\right),$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (\overline{\chi}(2)-2)|L(1,\chi)|^2 = -\frac{\pi^2 p}{8}\left(1-\frac{1}{p}\right)^3,$$

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (\overline{\chi}(2)-2)L(1,\overline{\chi})L(3,\chi) = -\frac{\pi^4(p-1)^3(p+1)^2}{96p^4}.$$

2. Some lemmas. To prove the theorems, we need some lemmas.

Lemma 1. *Let χ be a primitive character modulo m with $\chi(-1)=-1$. Then*

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b) = \frac{i}{\pi}\,\tau(\chi)L(1,\overline{\chi}).$$

Proof. This can be easily deduced from Theorems 12.11 and 12.20 of [1].

Lemma 2 ([4, Lemma 2]). *Let $q \geq 5$ be an odd integer and χ be a primitive Dirichlet character modulo q such that $\chi(-1)=1$. Then*

$$\sum_{a=1}^{[q/4]} \chi(a) = -\frac{i\overline{\chi}(4)}{2\pi}\,\tau(\chi\chi_4)L(1,\overline{\chi}\chi_4),$$

where χ_4 is the primitive Dirichlet character modulo 4.

Lemma 3. *Let $q \geq 3$ be an odd integer. For any nonprincipal character χ mod q,*

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1-2\chi(2)} \sum_{a=1}^{(q-1)/2} \chi(a).$$

Proof. From the properties of Dirichlet characters, we have

$$\begin{aligned}
\sum_{a=1}^{q} 2a\chi(2a)
&= \sum_{a=1}^{(q-1)/2} 2a\chi(2a) + \sum_{a=(q+1)/2}^{q} 2a\chi(2a)\\
&= \sum_{a=1}^{(q-1)/2} 2a\chi(2a) + \sum_{a=1}^{(q+1)/2} (2a-1)\chi(q+2a-1) + q\sum_{a=1}^{(q+1)/2} \chi(2a-1)\\
&= \sum_{a=1}^{q} a\chi(a) + q\sum_{a=1}^{(q+1)/2} \chi(2a-1).
\end{aligned}$$

Noting that

$$\sum_{a=1}^{(q+1)/2} \chi(2a-1) + \sum_{a=1}^{(q-1)/2} \chi(2a) = \sum_{a=1}^{q} \chi(a) = 0,$$

we can write

$$\begin{aligned}(1-2\chi(2)) \sum_{a=1}^{q} a\chi(a) &= \sum_{a=1}^{q} a\chi(a) - \sum_{a=1}^{q} 2a\chi(2a) = q \sum_{a=1}^{(q-1)/2} \chi(2a) \\ &= \chi(2) q \sum_{a=1}^{(q-1)/2} \chi(a),\end{aligned}$$

as desired.

LEMMA 4. *Let $q \geq 3$ be an odd integer and χ be a primitive Dirichlet character modulo q such that $\chi(-1) = -1$. Then*

$$\sum_{a=1}^{[q/4]} \chi(a) = \frac{2 + \overline{\chi}(2) - \overline{\chi}(4)}{2i\pi} \tau(\chi) L(1, \overline{\chi}).$$

Proof. We consider two cases. First, we suppose $q \equiv 1 \pmod 4$. From the properties of the Dirichlet character modulo q, we can write

$$\begin{aligned}(2)\qquad & 4\chi(4) \sum_{a=1}^{q-1} a\chi(a) \\ &= \sum_{a=1}^{(q-1)/4} 4a\chi(4a) + \sum_{a=(q+3)/4}^{(2q-2)/4} 4a\chi(4a) \\ &\quad + \sum_{a=(2q+2)/4}^{(3q-3)/4} 4a\chi(4a) + \sum_{a=(3q+1)/4}^{q-1} 4a\chi(4a) \\ &= \sum_{a=1}^{(q-1)/4} 4a\chi(4a) + \sum_{a=1}^{(q-1)/4} (4a+q-1)\chi(4a-1) \\ &\quad + \sum_{a=1}^{(q-1)/4} (4a+2q-2)\chi(4a-2) + \sum_{a=1}^{(q-1)/4} (4a+3q-3)\chi(4a-3) \\ &= \sum_{a=1}^{q-1} a\chi(a) + \chi(4) q \sum_{a=1}^{(q-1)/4} \chi(a-\overline{4}) \\ &\quad + 2\chi(4) q \sum_{a=1}^{(q-1)/4} \chi(a - 2\cdot\overline{4}) + 3\chi(4) q \sum_{a=1}^{(q-1)/4} \chi(a - 3\cdot\overline{4}).\end{aligned}$$

Note that $\overline{4} \equiv (3q+1)/4 \pmod q$ if $q \equiv 1 \pmod 4$. So from (2), we have

$$
\begin{aligned}
(3)\qquad 4\chi(4)\sum_{a=1}^{q-1} a\chi(a)
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=(2q+2)/4}^{(3q-3)/4} \chi(a) \\
&\quad - 2\chi(4)q \sum_{a=(q+3)/4}^{(2q-2)/4} \chi(a) - 3\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a) \\
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=(q+3)/4}^{(2q-2)/4} \chi(a) - 3\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a) \\
&= \sum_{a=1}^{q-1} a\chi(a) - \chi(4)q \sum_{a=1}^{(q-1)/2} \chi(a) - 2\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a),
\end{aligned}
$$

where we have used the fact that $\chi(-1) = -1$ and

$$\sum_{a=(q+3)/4}^{(2q-2)/4} \chi(a) = - \sum_{a=(2q+2)/4}^{(3q-3)/4} \chi(a).$$

Now, from (3) and Lemma 3, we get

$$4\chi(4)\sum_{a=1}^{q-1} a\chi(a) = \sum_{a=1}^{q-1} a\chi(a) - (\chi(2) - 2\chi(4))\sum_{a=1}^{q-1} a\chi(a) - 2\chi(4)q \sum_{a=1}^{(q-1)/4} \chi(a).$$

That is,

$$\sum_{a=1}^{(q-1)/4} \chi(a) = \frac{\overline{\chi}(4) - \overline{\chi}(2) - 2}{2q} \sum_{a=1}^{q-1} a\chi(a) = \frac{\overline{\chi}(4) - \overline{\chi}(2) - 2}{2q} \sum_{a=1}^{q} a\chi(a).$$

Then from Lemma 1, we have

$$(4)\qquad \sum_{a=1}^{(q-1)/4} \chi(a) = \frac{2 + \overline{\chi}(2) - \overline{\chi}(4)}{2i\pi}\, \tau(\chi) L(1, \overline{\chi}).$$

This is the assertion of Lemma 4 in the case of $q \equiv 1 \pmod 4$. By the same method, we can also prove

$$(5)\qquad \sum_{a=1}^{(q-3)/4} \chi(a) = \frac{2 + \overline{\chi}(2) - \overline{\chi}(4)}{2i\pi}\, \tau(\chi) L(1, \overline{\chi}),$$

if $q \equiv 3 \pmod 4$. This completes the proof of Lemma 4.

Lemma 5. *Let q be any positive integer. Then for any positive integer r with $(r,q)=1$, we have the identity*

$$\overline{B}_n\left(\frac{r}{q}\right) = -\frac{2n!}{(2i\pi)^n}\, q^{-n} \sum_{d|q} \frac{d^n}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=(-1)^n}} G(r,\overline{\chi}) L(n,\chi),$$

where $G(r,\overline{\chi}) = \sum_{a=1}^{d} \overline{\chi}(a) e(ra/d)$ is the Gauss sum.

Proof. From Theorem 12.19 of [1], we know that

$$B_n(x) = -\frac{n!}{(2i\pi)^n} \sum_{\substack{t=-\infty\\ t\neq 0}}^{+\infty} \frac{e(tx)}{t^n}$$

if $0 < x \le 1$. So we can write

$$\overline{B}_n\left(\frac{r}{q}\right) = -\frac{n!}{(2i\pi)^n} \sum_{d|q} \sum_{\substack{t=-\infty\\ t\neq 0\\ (t,d)=1}}^{+\infty} \frac{e\left(\frac{r\frac{q}{d}t}{q}\right)}{\left(t\frac{q}{d}\right)^n}$$

$$= -\frac{n!}{(2i\pi)^n} \sum_{d|q} \left(\frac{d}{q}\right)^n \sum_{\substack{t=-\infty\\ t\neq 0\\ (t,d)=1}}^{+\infty} \frac{e(rt/d)}{t^n}.$$

Now from the orthogonality relation for Dirichlet characters χ mod d we immediately get

$$\sum_{d|q} \left(\frac{d}{q}\right)^n \sum_{\substack{t=-\infty\\ t\neq 0\\ (t,d)=1}}^{+\infty} \frac{e(rt/d)}{t^n} = q^{-n} \sum_{d|q} \frac{d^n}{\phi(d)} \sum_{\substack{t=-\infty\\ t\neq 0}}^{+\infty} \sum_{b=1}^{d} \sum_{\chi \bmod d} \frac{\chi(t\overline{b}) e(rb/d)}{t^n}$$

$$= 2q^{-n} \sum_{d|q} \frac{d^n}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=(-1)^n}} G(r,\overline{\chi}) L(n,\chi),$$

and the assertion follows.

3. Proof of the theorems. In this section, we complete the proofs of the theorems. Let $q=p>4$ be a prime. Noting that $G(r,\overline{\chi}) = \chi(r)\tau(\overline{\chi})$ if χ is a primitive character, from Lemma 5 we have

$$\overline{B}_n\left(\frac{r}{p}\right) = -\frac{2n!}{(2i\pi)^n}\, p^{-n} \sum_{d|p} \frac{d^n}{\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=(-1)^n}} \chi(r)\tau(\overline{\chi}) L(n,\chi) \tag{6}$$

$$
= \begin{cases} -\dfrac{2n!}{(2i\pi)^n}\, p^{-n}\left(\zeta(n) + \dfrac{p^n}{\phi(p)} \displaystyle\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \chi(r)\tau(\overline{\chi})L(n,\chi)\right) & \text{if } n \text{ is even,} \\ -\dfrac{2n!}{(2i\pi)^n\phi(p)} \displaystyle\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(r)\tau(\overline{\chi})L(n,\chi) & \text{if } n \text{ is odd.} \end{cases}
$$

First, we prove Theorem 1. If n is even, from (6) and Lemma 2 we can write

$$
\begin{aligned}
(7)\qquad & \sum_{r \le [p/4]} \overline{B}_n\left(\frac{r}{p}\right) \\
&= -\frac{2n!}{(2i\pi)^n}\, p^{-n} \\
&\quad \times \left(\zeta(n)\left[\frac{p}{4}\right] - \frac{ip^n}{2\pi\phi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \overline{\chi}(4)\tau(\chi\chi_4)\tau(\overline{\chi})L(1,\overline{\chi}\chi_4)L(n,\chi)\right).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\tau(\chi\chi_4) &= \sum_{a=1}^{4p} \chi\chi_4(a) e\left(\frac{a}{4p}\right) \\
&= \sum_{a=1}^{4}\sum_{b=1}^{p} \chi(4b+pa)\chi_4(4b+pa) e\left(\frac{4b+pa}{4p}\right) \\
&= \sum_{a=1}^{4}\sum_{b=1}^{p} \chi(4b)\chi_4(pa) e\left(\frac{b}{p} + \frac{a}{4}\right) \\
&= \chi(4)\chi_4(p)\left(\sum_{a=1}^{3} \chi_4(a) e\left(\frac{a}{4}\right)\right)\left(\sum_{b=1}^{p-1} \chi(b) e\left(\frac{b}{p}\right)\right) \\
&= \chi(4)\chi_4(p)\left(e\left(\frac{1}{4}\right) - e\left(\frac{3}{4}\right)\right)\left(\sum_{b=1}^{p-1} \chi(b) e\left(\frac{b}{p}\right)\right) \\
&= 2i\chi(4)\chi_4(p)\tau(\chi)
\end{aligned}
$$

and $\tau(\chi)\overline{\tau}(\chi) = p$ if $\chi(-1) = 1$, we have

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \overline{\chi}(4)\tau(\chi\chi_4)\tau(\overline{\chi})L(1,\overline{\chi}\chi_4)L(n,\chi) \\
&\qquad\qquad = 2i\chi_4(p)p \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} L(1,\overline{\chi}\chi_4)L(n,\chi).
\end{aligned}
$$

Now combining this with (7) we get

$$-\frac{2n!\chi_4(p)p}{(2i)^n\pi^{n+1}\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}} L(1,\overline{\chi}\chi_4)L(n,\chi) = \frac{2n!\zeta(n)}{(2i\pi)^n}\,p^{-n}\left[\frac{p}{4}\right] + \sum_{r\le[p/4]}\overline{B}_n\left(\frac{r}{p}\right).$$

If n is odd, similarly, from (6) and Lemma 4 we have

$$\begin{aligned}\sum_{r\le[p/4]}\overline{B}_n\left(\frac{r}{p}\right) &= -\frac{2n!}{(2i\pi)^n\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\frac{2+\overline{\chi}(2)-\overline{\chi}(4)}{2i\pi}\,\tau(\chi)L(1,\overline{\chi})\tau(\overline{\chi})L(n,\chi)\\ &= \frac{2n!p}{(2i\pi)^{n+1}\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}(2+\overline{\chi}(2)-\overline{\chi}(4))L(1,\overline{\chi})L(n,\chi).\end{aligned}$$

That is,

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}(2+\overline{\chi}(2)-\overline{\chi}(4))L(1,\overline{\chi})L(n,\chi) = \frac{(2i\pi)^{n+1}\phi(p)}{2n!p}\sum_{r\le[p/4]}\overline{B}_n\left(\frac{r}{p}\right),$$

where we have used the fact that $\tau(\chi)\tau(\overline{\chi}) = -p$ if $\chi(-1) = -1$.

Now we prove Theorem 2. From Lemmas 3 and 1, we easily get

$$\sum_{a=1}^{[p/2]}\chi(a) = \frac{(\overline{\chi}(2)-2)i}{\pi}\,\tau(\chi)L(1,\overline{\chi}).$$

Now by using the same method as in proving Theorem 1, we also obtain

$$-\frac{2in!p}{(2i)^n\pi^{n+1}\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=1}}(\overline{\chi}(2)-2)L(1,\overline{\chi})L(n,\chi) = \frac{n!\zeta(n)(p-1)}{(2i\pi p)^n} + \sum_{r\le(p-1)/2}\overline{B}_n\left(\frac{r}{p}\right)$$

if n is even, and

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}(\overline{\chi}(2)-2)L(1,\overline{\chi})L(n,\chi) = \frac{(2i)^n\pi^{n+1}\phi(p)}{2in!p}\sum_{r\le(p-1)/2}\overline{B}_n\left(\frac{r}{p}\right)$$

if n is odd. This completes the proof of the theorems.

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