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Inhomogeneous approximation with coprime integers and lattice orbits

by

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1. Introduction and results. Minkowski proved that for every real irrational ξ and every real $y \notin \mathbb{Z}\xi + \mathbb{Z}$, there exist infinitely many pairs of integers p, q such that

$$|q\xi + p - y| \le \frac{1}{4|q|}$$

(see for instance Theorem II in Chapter 3 of Cassels' monograph [4]). The statement is optimal in the sense that the approximating function $\ell \mapsto (4\ell)^{-1}$ cannot be decreased. Note that the restriction $y \notin \mathbb{Z}\xi + \mathbb{Z}$ can be dropped at the cost of replacing the upper bound $(4|q|)^{-1}$ by $c|q|^{-1}$ for any constant c greater than $1/\sqrt{5}$. When $y = 0$, the primitive point $(p/\gcd(p,q), q/\gcd(p,q))$ remains a solution to the above inequality, therefore we may moreover require that the integers p, q be coprime. However, for a non-zero y, this extra requirement is far from being obvious to satisfy. In this direction, Chalk and Erdős [6] obtained the following result:

Theorem (Chalk–Erdős). *Let ξ be an irrational real number and let y be a real number. There exists an absolute constant c such that the inequality*

$$|q\xi + p - y| \le \frac{c(\log q)^2}{q(\log\log q)^2} \tag{1}$$

holds for infinitely many pairs of coprime integers (p, q) with q positive.

We study more generally the diophantine inequality

$$|q\xi + p - y| \le \psi(|q|)$$

for coprime integers p and q, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a given function. Two types of questions naturally arise. First, finding unconditional results which are valid for every real pair (ξ, y) with ξ irrational as in (1), and secondly

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getting metrical results valid for almost all points (ξ, y). Here is an example of the first kind.

THEOREM 1. *Let ξ be an irrational real number and let y be a non-zero real number. There exist infinitely many integer quadruples (p_1, q_1, p_2, q_2) satisfying*

$$q_1p_2 - p_1q_2 = 1$$

and

$$|q_i\xi + p_i - y| \le \frac{c}{\max(|q_1|, |q_2|)^{1/2}} \le \frac{c}{\sqrt{|q_i|}} \qquad (i = 1, 2), \tag{2}$$

with $c = 2\sqrt{3}\max(1, |\xi|)^{1/2}|y|^{1/2}$.

Theorem 1 will be deduced in Section 2 from our results [10] on effective density for $\mathrm{SL}(2, \mathbb{Z})$-orbits in $\mathbb{R}^2$. The estimate (2) is best possible, up to the value of the constant c. However, the optimality of (1) remains unclear. We address the following

PROBLEM. *Can we replace the function $\psi(\ell) = c(\log \ell)^2/\ell(\log\log \ell)^2$ occurring in (1) by a smaller one, possibly $\psi(\ell) = c\ell^{-1}$?*

We shall further discuss this problem in Section 4 for the function $\psi(\ell) = 2\ell^{-1}$, offering some hints and indicating the difficulties which then arise. It turns out that the approximating function $\psi(\ell) = \ell^{-1}$ is permitted for almost all pairs (ξ, y) of real numbers relative to Lebesgue measure. The last assertion follows from the following metrical statement:

THEOREM 2. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. Assume that ψ is non-increasing, tends to 0 at infinity and that for every positive integer c there exists a positive real number c_1 satisfying*

$$\psi(c\ell) \ge c_1\psi(\ell), \qquad \forall \ell \ge 1. \tag{3}$$

Furthermore assume that

$$\sum_{\ell \ge 1} \psi(\ell) = +\infty.$$

Then for almost all pairs (ξ, y) of real numbers there exist infinitely many primitive points (p, q) such that

$$q \ge 1 \quad \text{and} \quad |q\xi + p - y| \le \psi(q). \tag{4}$$

If $\sum_{\ell\ge1}\psi(\ell)$ converges, the pairs (ξ, y) satisfying (4) for infinitely many primitive points (p, q) form a set of zero Lebesgue measure.

Note that we could have equivalently required in (4) that q be negative. Such a refinement could as well be achieved in the setting of Theorem 1, with a weaker approximating function of the form $\psi(\ell) = \ell^{-\mu}$ for any given real $\mu < 1/3$, by employing alternatively Theorem 5 in Section 9 of [10]. We leave the details of the proof, obtained by arguing as in Section 2, to the interested reader. For questions of density involving signs, see also [7].

The proof of Theorem 2 is given in Section 3. It combines standard tools from metrical number theory with the ergodic properties of the linear action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$ (see [13]). We refer to Harman's book [8] for closely related results. See also the recent overview [1] and the monographs [14], [15].

Theorem 2 is a metrical statement about pairs (ξ, y) of real numbers. A natural question is to understand what happens on each fiber when we fix either ξ or y. In this direction, here is a partial result which will be deduced from the explicit construction in Section 4.

THEOREM 3. *Let ξ be an irrational number and let $(p_k/q_k)_{k\geq 0}$ be the sequence of its convergents. Assume that the series*

$$\sum_{k\geq 0} \frac{1}{\max(1, \log q_k)} \tag{5}$$

diverges. Then for almost every real number y there exist infinitely many primitive points (p,q) satisfying

$$|q\xi + p - y| \leq 2/|q|.$$

Moreover the series (5) *diverges for almost every real ξ.*

We now turn to the second part of the paper devoted to density exponents for lattice orbits in $\mathbb{R}^2$. As already mentioned, the approximating function $\psi(\ell) = c\,\ell^{-1/2}$ occurring in Theorem 1 is directly connected to the density exponent $1/2$ for $\mathrm{SL}(2,\mathbb{Z})$-orbits. We intend to show that this exponent $1/2$ is best possible in general.

We work in the more general setting of *lattices Γ* in $\mathrm{SL}(2,\mathbb{R})$. Recall that a lattice Γ in $\mathrm{SL}(2,\mathbb{R})$ is a discrete subgroup for which the quotient $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ has finite Haar measure. We view $\mathbb{R}^2$ as a space of column vectors on which the group of matrices Γ acts by left multiplication. We equip $\mathbb{R}^2$ with the supremum norm $|\,|$, and for any matrix $\gamma \in \Gamma$, we denote also by $|\gamma|$ the maximum of the absolute values of the entries of γ. Let us first give

DEFINITION. Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x},\mathbf{y})$ the supremum, possibly infinite, of the exponents μ such that the inequality

$$|\gamma\mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu} \tag{6}$$

has infinitely many solutions $\gamma \in \Gamma$.

Note that for a fixed $\mathbf{x} \in \mathbb{R}^2$, the function $\mathbf{y} \mapsto \mu_\Gamma(\mathbf{x},\mathbf{y})$ is Γ-invariant. By the ergodicity of the action of Γ on $\mathbb{R}^2$ (see [13]), this function is therefore constant almost everywhere on $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x})$ its generic value, called the *generic density exponent* of the orbit $\Gamma\mathbf{x}$.

Theorem 4. *The upper bound $\mu_\Gamma(\mathbf{x}) \le 1/2$ holds true for any point $\mathbf{x} \in \mathbb{R}^2$ such that the orbit $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$.*

Equivalently Theorem 4 asserts that $\mu(\mathbf{x},\mathbf{y}) \le 1/2$ for almost all $\mathbf{y}\in\mathbb{R}^2$. This bound was already known in the case of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ as a consequence of Theorem 3 in [10].

One may optimistically conjecture that $\mu_\Gamma(\mathbf{x}) = 1/2$ for every $\mathbf{x}$ such that $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$, or at least for almost every $\mathbf{x} \in \mathbb{R}^2$. In this direction, it follows from [10] that

$$\mu_{\mathrm{SL}(2,\mathbb{Z})}(\mathbf{x}) \ge 1/3$$

for all points $\mathbf{x}$ in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ with irrational slope. Weaker lower bounds valid for any lattice $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ can also be deduced from [12]. Note that the function $\mathbf{x} \mapsto \mu_\Gamma(\mathbf{x})$ is Γ-invariant since $\mu_\Gamma(\mathbf{x})$ obviously depends only on the orbit $\Gamma\mathbf{x}$. Thus, the generic density exponent $\mu_\Gamma(\mathbf{x})$ takes the same value for almost all $\mathbf{x} \in \mathbb{R}^2$.

2. Proof of Theorem 1. We first state a result obtained in [10]. In this section, we denote by Γ the lattice $\mathrm{SL}(2,\mathbb{Z})$. For any point $\mathbf{x} = \binom{x_1}{x_2}$ in $\mathbb{R}^2$ with irrational slope x_1/x_2, the orbit $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$. We have obtained in [10] effective results concerning the density of such an orbit. In particular, our estimates are essentially optimal when the target point $\mathbf{y}$ has rational slope.

Lemma 1. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ with irrational slope and $\mathbf{y} = \binom{y}{y}$ a point on the diagonal with $y \ne 0$. Then there exist infinitely many matrices $\gamma \in \Gamma$ such that*

$$|\gamma\mathbf{x} - \mathbf{y}| \le c/|\gamma|^{1/2} \quad \text{with} \quad c = 2\sqrt{3}\,|\mathbf{x}|^{1/2}|y|^{1/2}. \tag{7}$$

Proof. The point $\mathbf{y}$ has rational slope 1. Apply Theorem 1(ii) of [10] with $a = b = 1$. ∎

Put $\mathbf{x} = \binom{\xi}{1}$. The point $\mathbf{x}$ has irrational slope ξ so that Lemma 1 may be applied. Write $\gamma = \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix}$ for a matrix provided by Lemma 1. Then (7) gives

$$\begin{aligned}\max(|q_1\xi + p_1 - y|, |q_2\xi + p_2 - y|) &\le \frac{c}{\max(|p_1|,|p_2|,|q_1|,|p_2|)^{1/2}} \\ &\le \frac{c}{\max(|q_1|,|q_2|)^{1/2}}.\end{aligned}$$

Therefore, both (p_1, q_1) and (p_2, q_2) satisfy (2), and since the determinant $q_1p_2 - q_2p_1$ is 1, these two integer points are primitive. As there exist infinitely many matrices γ satisfying (7), we thus find infinitely many solutions to (2).

Assume now that the irrational number ξ has bounded partial quotients. Then Theorem 4 in [10] gives us in the opposite direction a lower bound of the form

$$|\gamma\mathbf{x} - \mathbf{y}| \geq c'/|\gamma|^{1/2}$$

for every $\gamma \in \Gamma$, where the positive constant c' depends only upon (ξ, y). Since we have $|\gamma| \leq c'' \max(|q_1|, |q_2|)$ when (2) holds, the estimate (2) is optimal up to the value of c.

Remark. The single inequality $|q_1\xi + p_1 - y| \leq \psi(|q_1|)$ geometrically means that the point $\gamma\mathbf{x}$ falls inside a neighborhood of the vertical line $x_1 = y$. A better understanding of the shrinking target problem for the dense orbit $\Gamma\mathbf{x}$, not to a point $\mathbf{y}$ as in [10] but to a line in $\mathbb{R}^2$, may possibly lead to a refinement of (1).

3. Proof of Theorem 2. It is convenient to view the pairs (ξ, y) occurring in Theorem 2 as column vectors $\binom{\xi}{y}$ in $\mathbb{R}^2$. We are concerned with the set $\mathcal{E}(\psi)$ of vectors $\binom{\xi}{y} \in \mathbb{R}^2$ for which there exist infinitely many primitive integer points (p, q) such that

$$q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q). \tag{8}$$

For fixed p, q, denote by $\mathcal{E}_{p,q}(\psi)$ the strip

$$\mathcal{E}_{p,q}(\psi) := \left\{ \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2;\ |q\xi + p - y| \leq \psi(q) \right\},$$

and for every positive integer q, let

$$\mathcal{E}_q(\psi) := \bigcup_{\substack{p\in\mathbb{Z} \\ \gcd(p,q)=1}} \mathcal{E}_{p,q}(\psi)$$

be the union of all relevant strips involved in (8) for fixed q. Without loss of generality, we shall assume that $\psi(q) \leq 1/2$, so that the above union is disjoint. Then $\mathcal{E}(\psi)$ is equal to the limsup set

$$\mathcal{E}(\psi) = \bigcap_{Q\geq 1} \bigcup_{q \geq Q} \mathcal{E}_q(\psi).$$

As usual when dealing with limsup sets in metrical theory, we first estimate the Lebesgue measure of pairwise intersections of the subsets $\mathcal{E}_q(\psi)$, $q \geq 1$. We next establish a new kind of zero-one law.

3.1. Measuring intersections. In this section, we restrict our attention to points in the unit square $[0,1]^2$. We denote by φ the Euler totient function and by λ the Lebesgue measure on $\mathbb{R}^2$.

Lemma 2. *Let $\psi : \mathbb{N} \to [0, 1/2]$ be a function.*

(i) *For every positive integer q, we have*

$$\lambda(\mathcal{E}_q(\psi) \cap [0,1]^2) = 2\varphi(q)\psi(q)/q.$$

(ii) *Let q and s be distinct positive integers. Then*

$$\lambda(\mathcal{E}_q(\psi)\cap\mathcal{E}_s(\psi)\cap[0,1]^2)\le 4\psi(q)\psi(s).$$

Proof. Denote by χ_q the characteristic function of $[-\psi(q),\psi(q)]$. Then the characteristic function $\chi_{\mathcal{E}_q(\psi)}$ of the subset $\mathcal{E}_q(\psi)\subset\mathbb{R}^2$ is equal to

$$\chi_{\mathcal{E}_q(\psi)}(\xi,y)=\sum_{\substack{p\in\mathbb{Z}\\ \gcd(p,q)=1}}\chi_q(q\xi+p-y)=\sum_{\substack{p\in\mathbb{Z}\\ \gcd(p,q)=1}}\chi_q(q\xi-p-y).$$

Observe that if $\binom{\xi}{y}\in[0,1]^2$, the indices p of non-vanishing terms occurring in the last sum satisfy $-1\le p\le q$. Integrating first with respect to x, we find

$$\begin{aligned}
\lambda(\mathcal{E}_q(\psi)\cap[0,1]^2)&=\int_0^1\!\!\int_0^1\chi_{\mathcal{E}_q(\psi)}(x,y)\,dx\,dy\\
&=\sum_{\substack{p\in\mathbb{Z}\\ -1\le p\le q,\,\gcd(p,q)=1}}\int_0^1\!\!\int_0^1\chi_q(qx-p-y)\,dx\,dy\\
&=\int_{1-\psi(q)}^{1}\frac{-1+y+\psi(q)}{q}\,dy+\sum_{\substack{1\le p\le q-2\\ \gcd(p,q)=1}}\int_0^1\frac{2\psi(q)}{q}\,dy\\
&\quad+\int_0^{1-\psi(q)}\frac{2\psi(q)}{q}\,dy+\int_{1-\psi(q)}^{1}\frac{1-y+\psi(q)}{q}\,dy\\
&=\frac{2\varphi(q)\psi(q)}{q}.
\end{aligned}$$

The first term appearing in the third equality of the above formula corresponds to the summation index $p=-1$ and the last two to $p=q-1$. We have thus proved (i).

For the second assertion, we majorize

$$\begin{aligned}
\lambda(\mathcal{E}_q(\psi)\cap\mathcal{E}_s(\psi)\cap[0,1]^2)&=\int_0^1\!\!\int_0^1\chi_{\mathcal{E}_q(\psi)}(x,y)\chi_{\mathcal{E}_s(\psi)}(x,y)\,dx\,dy\\
&\le\int_0^1\!\!\int_0^1\Big(\sum_{p\in\mathbb{Z}}\chi_q(qx+p-y)\Big)\Big(\sum_{r\in\mathbb{Z}}\chi_s(sx+r-y)\Big)\,dx\,dy\\
&=\int_0^1\!\!\int_0^1\chi_q(\|qx-y\|)\chi_s(\|sx-y\|)\,dx\,dy,
\end{aligned}$$

where $\|\cdot\|$ stands as usual for the distance to the nearest integer. Now, (ii) follows from the probabilistic independence formula

$$\int_0^1\int_0^1 \chi_q(\|qx-y\|)\chi_s(\|sx-y\|)\,dx\,dy = 4\psi(q)\psi(s),$$

obtained by Cassels [4, p. 124, Proof (ii)]. ∎

3.2. A zero-one law. We say that a subset of $\mathbb{R}^2$ is a *null set* if it has Lebesgue measure 0. A set whose complement is a null set is called a *full set*. The goal of this section is to prove

PROPOSITION. *Let ψ be an approximating function as in Theorem 2. Then $\mathcal{E}(\psi)$ is either a null set or a full set.*

To prove the proposition, it is convenient to introduce the larger set

$$\mathcal{E}'(\psi) = \bigcup_{k\ge 1} \mathcal{E}(k\psi).$$

In other words, $\mathcal{E}'(\psi)$ is the set of all points $\binom{\xi}{y}$ in $\mathbb{R}^2$ for which there exist a positive real κ, depending possibly on $\binom{\xi}{y}$, and infinitely many primitive points (p,q) satisfying

$$q \ge 1 \quad \text{and} \quad |q\xi + p - y| \le \kappa\psi(q). \tag{9}$$

Observe that $\mathcal{E}(k\psi) \subseteq \mathcal{E}(k'\psi)$ if $1 \le k \le k'$. In particular, $\mathcal{E}(\psi) \subset \mathcal{E}'(\psi)$.

LEMMA 3. *Assume that the approximating function $\psi : \mathbb{N} \to \mathbb{R}^+$ tends to zero at infinity. Then $\mathcal{E}'(\psi) \setminus \mathcal{E}(\psi)$ is a null set.*

Proof. We show that all sets $\mathcal{E}(k\psi)$, $k \ge 1$, have the same Lebesgue measure. For every real y, denote by $\mathcal{E}(\psi, y) \subseteq \mathbb{R}$ the section of $\mathcal{E}(\psi)$ on the horizontal line $\mathbb{R} \times \{y\}$, i.e.

$$\mathcal{E}(\psi, y) = \left\{\xi \in \mathbb{R};\ \begin{pmatrix}\xi\\ y\end{pmatrix} \in \mathcal{E}(\psi)\right\}.$$

Then, using (8), we can express

$$\mathcal{E}(\psi, y) = \bigcap_{Q\ge 1}\bigcup_{q\ge Q}\bigcup_{\substack{p\in\mathbb{Z}\\ \gcd(p,q)=1}} \left[\frac{-p+y-\psi(q)}{q}, \frac{-p+y+\psi(q)}{q}\right]$$

as a limsup set of intervals. If we restrict ourselves to a bounded part of $\mathcal{E}(\psi, y)$, the above union over p reduces to a finite one. Observe that the centers $(-p+y)/q$ of these intervals do not depend on ψ, and that their length is multiplied by the constant factor k when replacing ψ by $k\psi$. Appealing now to a result due to Cassels [5], we infer that all limsup sets $\mathcal{E}(k\psi, y)$, $k \ge 1$, have the same Lebesgue measure. See also [8, Corollary of Lemma 2.1, p. 30]. Notice that for fixed k, the length $2k\psi(q)/q$ of the relevant intervals tends to 0 as q tends to infinity, as required by Lemma 2.1. By Fubini, the fibered sets

$$\mathcal{E}(k\psi) = \coprod_{y\in\mathbb{R}} (\mathcal{E}(k\psi, y) \times \{y\}), \qquad k \geq 1,$$

all have the same Lebesgue measure in $\mathbb{R}^2$ as well. ∎

LEMMA 4. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a non-increasing function satisfying* (3). *Then $\mathcal{E}'(\psi)$ is either a null set or a full set.*

Proof. The proof is based on the following observation. Let $\binom{\xi}{y} \in \mathcal{E}'(\psi)$ and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ be such that $c\xi + d > 0$. Then the point $\binom{\xi'}{y'}$ with coordinates

$$\xi' = \frac{a\xi + b}{c\xi + d} \quad \text{and} \quad y' = \frac{y}{c\xi + d}$$

belongs to $\mathcal{E}'(\psi)$. Indeed, substituting

$$q = aq' + cp', \qquad p = bq' + dp' \tag{10}$$

in (9) and dividing by $c\xi + d$, we obtain

$$q' \geq 1 \quad \text{and} \quad |q'\xi' + p' - y'| \leq \frac{\kappa}{c\xi + d}\psi(q) \leq \kappa'\psi(q'), \tag{11}$$

for some $\kappa' > 0$ independent of q'. The positivity of q' is proved as follows. Note that (9) implies the estimate

$$p = -q\xi + \mathcal{O}_{\xi,y}(1).$$

Then, inverting the linear substitution (10), we find

$$q' = dq - cp = q(c\xi + d) + \mathcal{O}_{\gamma,\xi,y}(1).$$

Since we have assumed that $c\xi + d > 0$, the term $q(c\xi + d)$ is arbitrarily large when q is large enough. The condition (3) now shows that $\psi(q) \asymp \psi(q')$. Thus (11) is satisfied for infinitely many primitive points (p', q'), since the linear substitution (10) is unimodular. We have shown that $\binom{\xi'}{y'} \in \mathcal{E}'(\psi)$.

We now prove that $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is either a full subset or a null subset of the half-plane $\mathbb{R} \times \mathbb{R}^+$. To that end, we consider the map

$$\Phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+ \quad \text{defined by} \quad \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x/y \\ 1/y \end{pmatrix}.$$

Clearly Φ is a continuous involution of $\mathbb{R} \times \mathbb{R}^+$. The image

$$\Omega := \Phi(\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+))$$

is formed by all points of the type

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi/y \\ 1/y \end{pmatrix},$$

where $\binom{\xi}{y}$ ranges over $\mathcal{E}'(\psi)\cap(\mathbb{R}\times\mathbb{R}^+)$. Now, the above condition $c\xi+d>0$ is obviously equivalent to $cu+dv>0$ since y is positive. Then the point

$$\Phi\begin{pmatrix} au+bv \\ cu+dv \end{pmatrix} = \begin{pmatrix} \frac{au+bv}{cu+dv} \\ \frac{1}{cu+dv} \end{pmatrix} = \begin{pmatrix} \frac{a\xi+b}{c\xi+d} \\ \frac{y}{c\xi+d} \end{pmatrix}$$

belongs to $\mathcal{E}'(\psi)\cap(\mathbb{R}\times\mathbb{R}^+)$, by the preceding observation. Applying the involution Φ, we find that

$$\Phi\left(\begin{pmatrix} \frac{a\xi+b}{c\xi+d} \\ \frac{y}{c\xi+d} \end{pmatrix}\right) = \begin{pmatrix} au+bv \\ cu+dv \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to Ω. In other words, setting $\Gamma=\mathrm{SL}(2,\mathbb{Z})$, we have established the inclusion

$$(\Gamma\Omega)\cap(\mathbb{R}\times\mathbb{R}^+)\subseteq\Omega.$$

Since the reverse inclusion is obvious, we have $\Omega=(\Gamma\Omega)\cap(\mathbb{R}\times\mathbb{R}^+)$. Assuming that Ω is not a null set, the ergodicity of the linear action of Γ on $\mathbb{R}^2$ [13] shows that $\Gamma\Omega$ is a full set in $\mathbb{R}^2$. Hence Ω is a full set in the half-plane $\mathbb{R}\times\mathbb{R}^+$. Transforming now Ω by Φ, we find that

$$\Phi(\Omega)=\mathcal{E}'(\psi)\cap(\mathbb{R}\times\mathbb{R}^+)$$

is also a full set in $\mathbb{R}\times\mathbb{R}^+$, thus proving the claim.

We finally use another transformation to carry the zero-one law from the positive half-plane $\mathbb{R}\times\mathbb{R}^+$ to the negative one $\mathbb{R}\times\mathbb{R}^-$. Writing (9) in the equivalent form

$$q\geq 1 \quad\text{and}\quad |q(-\xi)+(-p)-(-y)|\leq\kappa\psi(q)$$

shows that $\mathcal{E}'(\psi)$ is invariant under the symmetry $\binom{\xi}{y}\mapsto\binom{-\xi}{-y}$ which maps $\mathbb{R}\times\mathbb{R}^+$ onto $\mathbb{R}\times\mathbb{R}^-$. Therefore $\mathcal{E}'(\psi)\cap(\mathbb{R}\times\mathbb{R}^-)$ is a null set or a full set in $\mathbb{R}\times\mathbb{R}^-$ whenever $\mathcal{E}'(\psi)\cap(\mathbb{R}\times\mathbb{R}^+)$ is, respectively, a null set or a full set in $\mathbb{R}\times\mathbb{R}^+$. ■

Now, the combination of Lemmas 3 and 4 obviously yields our proposition.

3.3. Concluding the proof of Theorem 2. Assume first that $\sum\psi(\ell)$ converges. We have to show that the set

$$\mathcal{E}(\psi)=\limsup_{q\to\infty}\mathcal{E}_q(\psi)$$

has zero Lebesgue measure. Lemma 2 shows that the partial sums

$$\sum_{q=1}^{Q}\lambda(\mathcal{E}_q(\psi)\cap[0,1]^2)=2\sum_{q=1}^{Q}\frac{\varphi(q)\psi(q)}{q}\leq 2\sum_{q=1}^{Q}\psi(q)$$

converge ([1]). Then the Borel–Cantelli Lemma ensures that the limsup set $\mathcal{E}(\psi) \cap [0,1]^2$ is a null set. Thus $\mathcal{E}(\psi)$ cannot be a full set. Now, the above proposition tells us that $\mathcal{E}(\psi)$ is a null set.

We now consider the case of a divergent series $\sum \psi(\ell)$. Observe that

$$\frac{1}{2}\sum_{q=1}^{Q} \psi(q) \le \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \le \sum_{q=1}^{Q} \psi(q) \tag{12}$$

for any large integer Q, since the sequence $(\psi(\ell))_{\ell\ge 1}$ is non-increasing. The right inequality is obvious, while the left one easily follows by Abel summation. See for instance Chapter 2 of [8], where full details are provided. By Lemma 2 and (12), the sums

$$\sum_{q=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap [0,1]^2) = 2\sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \ge \sum_{q=1}^{Q} \psi(q)$$

are then unbounded. Then, using a classical converse to the Borel–Cantelli Lemma, we have the lower bound

$$\begin{aligned} \lambda(\mathcal{E}(\psi) \cap [0,1]^2) &= \lambda\Big(\limsup_{q\to\infty}(\mathcal{E}_q(\psi) \cap [0,1]^2)\Big) \\ &\ge \limsup_{Q\to\infty} \frac{(\sum_{q=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap [0,1]^2))^2}{\sum_{q=1}^{Q}\sum_{s=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0,1]^2)}. \end{aligned} \tag{13}$$

See for instance Lemma 2.3 in [8]. Lemma 2 and (12) now show that the numerator on the right hand side of (13) equals

$$4\Big(\sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q}\Big)^2 \ge \Big(\sum_{q=1}^{Q} \psi(q)\Big)^2$$

when Q is large, while the denominator is bounded from above by

$$4\sum_{\substack{q=1,\, s=1\\ q\neq s}}^{Q} \psi(q)\psi(s) + 2\sum_{q=1}^{Q} \psi(q) \le 4\Big(\sum_{q=1}^{Q} \psi(q)\Big)^2 + 2\sum_{q=1}^{Q} \psi(q).$$

Thus (13) yields the lower bound

$$\lambda(\mathcal{E}(\psi) \cap [0,1]^2) \ge 1/4.$$

Hence $\mathcal{E}(\psi)$ is not a null set; it is thus a full set according to our proposition.

4. An approach to our problem. In this section, we apply a transference principle between homogeneous and inhomogeneous approximation,

([1]) Here again we assume without loss of generality that $\psi(q) \le 1/2$ for every $q \ge 1$, so that Lemma 2 may be applied.

as displayed in Chapter V of [4] and in [3], for constructing explicit integer solutions of the inequality

$$|q\xi + p - y| \le 2/|q|. \tag{14}$$

Let $(p_k/q_k)_{k\ge 0}$ be the sequence of convergents to the irrational number ξ. The theory of continued fractions (see for instance the monograph [9]) tells us that

$$|q_k\xi - p_k| \le 1/q_{k+1} \quad \text{and} \quad p_kq_{k+1} - p_{k+1}q_k = (-1)^{k+1} \tag{15}$$

for any $k \ge 0$. Setting $\nu_k = (-1)^{k+1}q_ky$, we thus have the relations

$$\nu_kq_{k+1} + \nu_{k+1}q_k = 0 \quad \text{and} \quad \nu_k(q_{k+1}\xi - p_{k+1}) + \nu_{k+1}(q_k\xi - p_k) = y. \tag{16}$$

Now, let n_k be either $\lfloor \nu_k \rfloor$ or $\lceil \nu_k \rceil$ [2]. Then

$$|\nu_k - n_k| < 1, \tag{17}$$

and n_k is either equal to $(-1)^{k+1}\lfloor yq_k \rfloor$ or to $(-1)^{k+1}\lceil yq_k \rceil$. Setting

$$p = -n_kp_{k+1} - n_{k+1}p_k \quad \text{and} \quad q = n_kq_{k+1} + n_{k+1}q_k, \tag{18}$$

we deduce from (16) the expressions

$$\begin{aligned} q\xi + p - y &= n_k(q_{k+1}\xi - p_{k+1}) + n_{k+1}(q_k\xi - p_k) - y \\ &= (n_k - \nu_k)(q_{k+1}\xi - p_{k+1}) + (n_{k+1} - \nu_{k+1})(q_k\xi - p_k) \end{aligned} \tag{19}$$

and

$$q = (n_k - \nu_k)q_{k+1} + (n_{k+1} - \nu_{k+1})q_k. \tag{20}$$

Recall that $q_k\xi - p_k$ and $q_{k+1}\xi - p_{k+1}$ have opposite signs. Assuming that $n_k - \nu_k$ and $n_{k+1} - \nu_{k+1}$ have the same sign, we infer from (19), (20) and (15), (17) that

$$|q\xi + p - y| < 1/q_{k+1} \quad \text{and} \quad |q| < 2q_{k+1}. \tag{21}$$

Otherwise, we have

$$|q\xi + p - y| < 2/q_{k+1} \quad \text{and} \quad |q| < q_{k+1}. \tag{22}$$

The inequalities (21) and (22) obviously imply (14).

Since the linear substitution (18) is unimodular, the integers p and q are coprime if and only if n_k and n_{k+1} are coprime. Recall that the two choices $n_k = \lfloor \nu_k \rfloor$ and $n_k = \lceil \nu_k \rceil$ are admissible, both for n_k and n_{k+1}. It thus remains to find indices k for which at least one of the coprimality conditions

$$\begin{aligned} &\gcd(\lfloor yq_k \rfloor, \lfloor yq_{k+1} \rfloor) = 1, \quad \gcd(\lceil yq_k \rceil, \lceil yq_{k+1} \rceil) = 1, \\ &\gcd(\lfloor yq_k \rfloor, \lceil yq_{k+1} \rceil) = 1, \quad \gcd(\lceil yq_k \rceil, \lfloor yq_{k+1} \rfloor) = 1 \end{aligned} \tag{23}$$

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[2] As usual $\lfloor x \rfloor$ and $\lceil x \rceil$ stand respectively for the floor and the ceiling of the real number x. Then $\lceil x \rceil = \lfloor x \rfloor + 1$, unless x is an integer in which case $\lfloor x \rfloor = \lceil x \rceil = x$.

is satisfied. Note that obviously there is no such $k \geq 0$ when y is an integer not equal to 1 or to -1. Otherwise, the existence of infinitely many indices k satisfying (23) is a non-trivial problem that we leave hanging.

Let us mention that the proof of (1) in [6] follows the same idea, finding a primitive integer point inside the square centered at the point $(\nu_k, \nu_{k+1}) \in \mathbb{R}^2$ with side $C \log|\nu_k|/\log\log|\nu_k|$ for some suitable large absolute constant C.

4.1. Proof of Theorem 3. We quote the following metrical result due to Harman (Theorem 8.3 in [8]). Assume that the series (5) diverges. Then for almost all positive real numbers y, there exist infinitely many indices k such that the integer part $\lfloor yq_k \rfloor$ is a prime number. These indices k satisfy (23) since, assuming for simplicity that y is irrational, either $\lfloor yq_{k+1} \rfloor$ or $\lceil yq_{k+1} \rceil = \lfloor yq_{k+1} \rfloor + 1$ is not divisible by $\lfloor yq_k \rfloor$ and is thus relatively prime to $\lfloor yq_k \rfloor$. Hence (14) has infinitely many coprime solutions (p, q) for almost every positive real number y. Writing now (14) in the equivalent form

$$|(-q)\xi + (-p) - (-y)| \leq 2/|q|$$

shows that, given ξ, the set of all real numbers y for which (14) has infinitely many coprime solutions is invariant under the symmetry $y \mapsto -y$. The first assertion is thus established. To complete the proof, note that

$$\lim_{k\to\infty} \frac{\log q_k}{k} = \frac{\pi^2}{12\log 2}$$

for almost every ξ by the Khintchine–Lévy Theorem (see equation (4.18) in [2]). Thus the series (5) diverges for almost every ξ.

5. Generic density exponents. In this section we prove Theorem 4, as a consequence of the Borel–Cantelli Lemma combined with the following counting result.

LEMMA 5. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ whose orbit $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$. For every symmetric compact set Ω in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ there exists $c > 0$ such that*

$$\mathrm{Card}\{\gamma \in \Gamma;\ \gamma\mathbf{x} \in \Omega,\ |\gamma| \leq T\} \leq cT$$

for any real $T \geq 1$.

Proof. Ledrappier [11] has shown that the limit formula

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\gamma\in\Gamma,\,|\gamma|\leq T} f(\gamma\mathbf{x}) = \frac{4}{|\mathbf{x}|\,\mathrm{vol}(\Gamma\backslash\mathrm{SL}(2,\mathbb{R}))} \int \frac{f(\mathbf{y})}{|\mathbf{y}|}\,d\mathbf{y}$$

holds for any even continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ having compact support on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, with a suitable normalisation of Haar measure on $\mathrm{SL}(2,\mathbb{R})$. Approximating uniformly from above and from below the characteristic function of Ω by even continuous functions, we deduce that

$$\lim_{T\to\infty} \frac{\mathrm{Card}\{\gamma\in\Gamma;\ \gamma\mathbf{x}\in\Omega,\ |\gamma|\le T\}}{T} = \frac{4}{|\mathbf{x}|\,\mathrm{vol}(\Gamma\backslash \mathrm{SL}(2,\mathbb{R}))}\int_{\Omega}\frac{d\mathbf{y}}{|\mathbf{y}|}.$$

Lemma 5 immediately follows. ■

For any $\mathbf{y}\in\mathbb{R}^2$ and any positive real number r, we denote by

$$B(\mathbf{y},r) = \{\mathbf{z}\in\mathbb{R}^2;\ |\mathbf{z}-\mathbf{y}|\le r\}$$

the closed disc centered at $\mathbf{y}$ with radius r.

Lemma 6. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ whose orbit $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$, Ω a symmetric compact set in $\mathbb{R}^2\setminus\{\mathbf{0}\}$ and μ a real number $>1/2$. For every integer $n\ge 1$, put*

$$\mathcal{B}_n = \bigcup_{\substack{\gamma\in\Gamma\\ |\gamma|=n,\ \gamma\mathbf{x}\in\Omega}} B(\gamma\mathbf{x}, n^{-\mu}).$$

Then

$$\mathcal{B} := \limsup_{n\to\infty}\mathcal{B}_n = \bigcap_{N\ge1}\bigcup_{n\ge N}\mathcal{B}_n = \bigcap_{N\ge1}\bigcup_{\substack{\gamma\in\Gamma\\ |\gamma|\ge N,\ \gamma\mathbf{x}\in\Omega}} B(\gamma\mathbf{x},|\gamma|^{-\mu})$$

is a null set.

Proof. We apply the Borel–Cantelli Lemma to prove that the series $\sum_{n\ge1}\lambda(\mathcal{B}_n)$ converges if $\mu>1/2$.

For every positive integer n, set

$$M_n = \mathrm{Card}\{\gamma\in\Gamma;\ \gamma\mathbf{x}\in\Omega,\ |\gamma|=n\}.$$

Lemma 5 gives us the upper bound

$$M_1+\cdots+M_n = \mathrm{Card}\{\gamma\in\Gamma;\ \gamma\mathbf{x}\in\Omega,\ |\gamma|\le n\}\le cn \tag{24}$$

for some $c>0$ independent of $n\ge1$. Since a ball of radius r has Lebesgue measure $4r^2$, we trivially bound from above

$$\lambda(\mathcal{B}_n)\le \sum_{\substack{\gamma\in\Gamma\\ |\gamma|=n,\ \gamma\mathbf{x}\in\Omega}} 4n^{-2\mu} = 4M_n n^{-2\mu}.$$

Summing by parts, we deduce from (24) that

$$\begin{aligned}\sum_{n=1}^{N}\frac{M_n}{n^{2\mu}} &= \sum_{n=1}^{N-1}(M_1+\cdots+M_n)\left(\frac{1}{n^{2\mu}}-\frac{1}{(n+1)^{2\mu}}\right)+\frac{M_1+\cdots+M_N}{N^{2\mu}}\\ &\le c\sum_{n=1}^{N-1} n\left(\frac{1}{n^{2\mu}}-\frac{1}{(n+1)^{2\mu}}\right)+\frac{cN}{N^{2\mu}} = c\sum_{n=1}^{N}\frac{1}{n^{2\mu}}.\end{aligned}$$

The partial sums

$$\sum_{n=1}^{N} \lambda(\mathcal{B}_n) \le 4 \sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} \le 4c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}$$

thus converge if $\mu > 1/2$. ∎

5.1. Proof of Theorem 4. Suppose on the contrary that $\mu_\Gamma(\mathbf{x}) > 1/2$. Fix a real μ with $1/2 < \mu < \mu_\Gamma(\mathbf{x})$. Then for almost all $\mathbf{y} \in \mathbb{R}^2$, we have $\mu(\mathbf{x}, \mathbf{y}) > \mu$. This means that there exist infinitely many $\gamma \in \Gamma$ satisfying (6), or equivalently that $\mathbf{y}$ belongs to infinitely many balls of the form $B(\gamma\mathbf{x}, |\gamma|^{-\mu})$. We now restrict our attention to points $\mathbf{y}$ with $\mu(\mathbf{x}, \mathbf{y}) > \mu$ lying in an annulus

$$\Omega' = \{\mathbf{z} \in \mathbb{R}^2;\ a' \le |\mathbf{z}| \le b'\},$$

where $b' > a' > 0$ are arbitrarily fixed. Since $\mathbf{y} \in \Omega' \cap B(\gamma\mathbf{x}, |\gamma|^{-\mu})$, the triangle inequality yields

$$a' - |\gamma|^{-\mu} \le |\gamma\mathbf{x}| \le b' + |\gamma|^{-\mu}.$$

If $a < a'$ and $b > b'$, then the center $\gamma\mathbf{x}$ lies in the larger annulus

$$\Omega = \{\mathbf{z} \in \mathbb{R}^2;\ a \le |\mathbf{z}| \le b\},$$

provided that $|\gamma|$ is large enough. It follows that $\mathbf{y}$ falls inside the union of balls

$$\bigcup_{\substack{\gamma\in\Gamma \\ |\gamma|\ge N,\, \gamma\mathbf{x}\in\Omega}} B(\gamma\mathbf{x}, |\gamma|^{-\mu})$$

considered in Lemma 6 for every integer N large enough, and thus $\mathbf{y} \in \mathcal{B}$. However, Lemma 6 asserts that $\mathcal{B}$ is a null set, which is a contradiction.

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