ACTA ARITHMETICA
109.2 (2003)

Metric properties of alternating Oppenheim expansions

by

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1. Introduction. Let $a_n(j)$ and $b_n(j)$, $n \geq 1$, be two sequences of positive integer-valued functions of the positive integers $j \geq 1$, and set

$$h_n(j) = \frac{a_n(j)}{b_n(j)}\, j(j-1), \qquad j \geq 2. \tag{1}$$

Assume that $h_n(j)$ is integer-valued $(n \geq 1,\ j \geq 2)$.

The algorithm $0 < x \leq 1,\ x = x_1$, and, for $n \geq 1$, with positive integers d_n,

$$\frac{1}{d_n} < x_n \leq \frac{1}{d_n - 1}, \qquad x_{n+1} = \left(x_n - \frac{1}{d_n}\right)\frac{b_n(d_n)}{a_n(d_n)} \tag{2}$$

leads to the series representation

$$x = \frac{1}{d_1} + \frac{a_1(d_1)}{b_1(d_1)}\,\frac{1}{d_2} + \ldots + \frac{a_1(d_1)\ldots a_n(d_n)}{b_1(d_1)\ldots b_n(d_n)}\,\frac{1}{d_{n+1}} + \ldots \tag{3}$$

The algorithm (2) implies

$$d_{n+1} > h_n(d_n), \tag{4}$$

which in turn yields $d_n \geq 2$ for each $n \geq 1$. The algorithm (2) never terminates, and (3) with (4) is equivalent to (2). The representation (3) under (4) is unique.

The representation (3) under (2) or (4) was first studied by Oppenheim [8] who established the arithmetical properties, including the question of rationality, of the expansion (2)–(4). Oppenheim's work was first distributed in the form of lecture notes. These notes were used by Galambos [1] where the foundations of the metric theory of (2)–(4) are laid down. Further development, with several new results, can be found in a monograph of Galambos [2]. The expansion (2)–(4) became known as Oppenheim expansion.

2000 *Mathematics Subject Classification*: Primary 11K55; Secondary 11A67.

The terms in (3) are all positive. A modification of (2) to the algorithm $0 < x \le 1$, $x = x_1$, and

$$\frac{1}{D_n + 1} < x_n \le \frac{1}{D_n}, \quad x_{n+1} = \left(\frac{1}{D_n} - x_n\right)\frac{b_n(D_n)}{a_n(D_n)}, \quad D_n \text{ integer}, \ n \ge 1, \tag{5}$$

generates an alternating series representation. Indeed, if we rearrange the recursive formula in (5) to

$$x_n = \frac{1}{D_n} - \frac{a_n(D_n)}{b_n(D_n)}\, x_{n+1} \tag{6}$$

we get

$$x = \frac{1}{D_1} - \frac{a_1(D_1)}{b_1(D_1)}\,\frac{1}{D_2} + \ldots + (-1)^n\,\frac{a_1(D_1)\ldots a_n(D_n)}{b_1(D_1)\ldots b_n(D_n)}\, x_{n+1}. \tag{7}$$

The algorithm at (5) terminates for rational x but never for irrationals, that is, (7) always extends to an infinite sum for irrational x. In (5) we get

$$x_{n+1} < \left(\frac{1}{D_n} - \frac{1}{D_n + 1}\right)\frac{b_n(D_n)}{a_n(D_n)} = \frac{1}{H_n(D_n)}, \tag{8}$$

say, yielding

$$D_{n+1} \ge H_n(D_n), \qquad n \ge 1, \tag{9}$$

where

$$H_n(j) = \frac{a_n(j)}{b_n(j)}\, j(j+1), \qquad j \ge 1. \tag{10}$$

We once again assume that $H_n(j)$ is integer-valued for $n \ge 1,\ j \ge 1$.

Note the difference between (1) and (10). Even though (9) implies a faster growth for the sequence $D_n = D_n(x)$ that what is already known for $d_n = d_n(x)$ at (2) and (4), we shall establish a remarkable similarity between the metric properties of the two sequences d_n and D_n, valid for almost all x (Lebesgue measure). As usual for metric theory, we shall use Lebesgue measure on the Borel subsets A of the interval $(0, 1]$, and write $P(A)$ for this measure.

Very little is known about the metric theory of the sequence D_n for general sequences $a_n(j)$ and $b_n(j)$. In fact, only the special cases $a_n(j) = 1$ and $b_n(j) = j(j+1)$ for all $n \ge 1$ and $j \ge 1$ (the Lüroth case; see Kalpazidou *et al.* [6] and Indlekofer *et al.* [5]), and $a_n(j) = 1$ and $b_n(j) = j$ (alternating Engel series or Pierce expansions; see Shallit [9]). We now fill in this gap. For the arithmetical properties of the sequence D_n, see Indlekofer *et al.* [5] and their references.

2. The basic distributional properties. We assume that x is irrational, so the algorithm at (5) does not terminate. It is immediate from (5) that the set of x for which $D_1 = u$, $u \geq 1$, is an interval of length $1/(u(u+1))$, and the set of those x's for which $D_t = u_t$, $1 \leq t \leq s$, is once again an interval, whose length, by (7), equals

$$\frac{a_1(u_1)\dots a_{s-1}(u_{s-1})}{b_1(u_1)\dots b_{s-1}(u_{s-1})}\,\frac{1}{u_s(u_s+1)}$$

where we assumed that the $u_t, 1 \leq t \leq s$, are in conformity with (9). We record these simple observations as

THEOREM 1. *The sequence* $D_n, n \geq 1$, *satisfies* $P(D_1 = u) = 1/(u(u+1))$ *for* $u \geq 1$, *and*, *for* u_t, $1 \leq t \leq s$, $s \geq 2$,

$$P(D_1 = u_1, \dots, D_s = u_s) = \frac{a_1(u_1)\dots a_{s-1}(u_{s-1})}{b_1(u_1)\dots b_{s-1}(u_{s-1})}\,\frac{1}{u_s(u_s+1)} \tag{11}$$

whenever any two consecutive u_t *satisfy* (9). *Consequently*, *the sequence* D_n, $n \geq 1$, *forms a Markov chain with transition probabilities*

$$P(D_{n+1} = u \mid D_n = v) = \frac{H_n(v)}{u(u+1)}, \qquad u \geq H_n(v). \tag{12}$$

Only (12) needs proof. By definition

$$P(D_{n+1} = u \mid D_j = u_j,\ 1 \leq j \leq n-1,\ D_n = v) = \frac{P(D_1 = u_1, \dots, D_{n-1} = u_{n-1}, D_n = v, D_{n+1} = u)}{P(D_1 = u_1, \dots, D_{n-1} = u_{n-1}, D_n = v)}.$$

By (11), the right hand side above does reduce to the right hand side of (12). Since this latter form depends only on u and v, both the Markovian property and (12) follow.

Put

$$y_{n+1} = H_n(D_n)x_{n+1}. \tag{13}$$

Just as in the case of Oppenheim expansions with positive terms, we have

THEOREM 2. *For* $n \geq 2$, y_n *is uniformly distributed on the interval* $(0,1]$, *and* y_n *is stochastically independent of the vector* $(D_1, \dots, D_{n-1})$.

Proof. Let $0 < c \leq 1$ be an arbitrary real number. We have to prove

$$P(D_1 = u_1, \dots, D_{n-1} = u_{n-1}, y_n \leq c) = cP(D_1 = u_1, \dots, D_{n-1} = u_{n-1}), \tag{14}$$

where the u_t are integers satisfying (9). We once again refer to (7) and (5) and note that the set of those x's which appear on either side of (14) inside $P(\,)$ is an interval and the length of the interval on the left hand side is exactly c times the length of the one on the right hand side (see also (11)).

This completes the independence part of the theorem. Upon summing over all possible values of the u_t in (14), we now obtain $P(y_n \le c) = c$. The proof is complete.

THEOREM 2a. *Let each c_j be the reciprocal of an integer ≥ 2. Then the events $\{y_j \le c_j\}$, $1 \le j \le n$, are independent with $P(y_j \le c_j) = c_j$.*

Proof. Decompose $P(y_1 \le c_1, y_2 \le c_2, \ldots, y_n \le c_n)$ as the sum of terms appearing in (11). Upon observing that $y_j \le 1/k_j$, with $k_j \ge 2$ integer, is equivalent to $D_j \ge k_j H_{j-1}(D_{j-1})$, the summation introduced in the previous sentence yields $1/(k_1 k_2 \ldots k_n)$. Hence, by the first part of Theorem 2, the proof is complete.

We record a special case of Theorem 2a as

THEOREM 3. *The integer parts V_j of the ratios $D_j/H_{j-1}(D_{j-1})$ are stochastically independent with distribution*

$$P(V_j = k) = \frac{1}{k(k+1)}, \qquad k \ge 1 \text{ and } j \ge 1, \tag{15}$$

where V_1 is just D_1.

Theorem 3 is a very powerful tool for analyzing the growth rate of the sequence $D_n = D_n(x)$ when we seek the growth rate valid for almost all x. Note also that for the alternating Lüroth expansions, $H_n(j) = 1$ for all n and j, so $V_n = D_n$ in this case. Therefore, the subsequent statements on the sequence V_n reestablish the results of Kalpazidou *et al.* [6] on the Lüroth case.

Set $R_1 = D_1$, and for $n \ge 2$,

$$R_n = D_n/H_{n-1}(D_{n-1}). \tag{16}$$

By definition, V_n and R_n differ by at most one, hence several statements on the sequence V_n immediately transform to similar statements on the sequence R_n. We make use of this possibility in what follows, without repeatedly referring to this relation between V_n and R_n.

3. Asymptotic results on V_n and R_n. Because the expected value of V_n is infinity, we have from probability theory

$$\lim \frac{1}{n}(V_1 + \ldots + V_n) = \infty \quad \text{for almost all } x \ (n \to \infty) \tag{A1}$$

and, for any $a > 0$,

$$\lim P\left(\left|\frac{1}{n \log n}(V_1 + \ldots + V_n) - 1\right| > a\right) = 0 \quad (n \to \infty), \tag{A2}$$

which limit result cannot be extended to a limit for almost all x (see Galambos [3, p. 63]).

However, if one turns to logarithm, the strong law of large numbers applies and one gets

$$\text{(A3)}\qquad \lim(\log V_1 + \ldots + \log V_n)/n = E > 0 \quad \text{finite} \quad \text{as } n \to \infty.$$

All three limits above remain valid for the sequence R_j in the place of V_j, $1 \le j \le n$.

When one analyzes the speed of convergence at (7), as $n \to \infty$, one needs an estimate on the product of $a_j(D_j)/b_j(D_j)$, $1 \le j \le n$. We turn to logarithm. By the definitions at (10) and (16),

$$\begin{aligned}\log \frac{a_j(D_j)}{b_j(D_j)} &= \log H_j(D_j) - \log D_j - \log(D_j + 1)\\ &= -\log R_{j+1} + \log D_{j+1} - \log D_j - \log(D_j + 1).\end{aligned}$$

Upon summing the extreme sides above over j, from 1 to n, we get

$$\text{(17)}\qquad \sum_{j=1}^{n} \log \frac{a_j(D_j)}{b_j(D_j)} = \log D_{n+1} - \sum_{j=2}^{n+1} \log R_j - \log D_1 - \sum_{j=1}^{n} \log(D_{j+1} + 1).$$

The sum of $\log R_j$, when divided by n, converges to a finite number $A > 0$ as $n \to \infty$ (see (A3)). So far we ignored x_{n+1} in the error term at (7) (the last term of (7)). However, by (5), $1/2 \le x_{n+1} D_{n+1} \le 1$, whose logarithm is bounded. Hence, if $N(n)$ is a sequence of numbers tending to infinity faster than n, we have

THEOREM 4. *If $N(n)$ goes to infinity faster than n, for the error term at* (7) *we have*

$$\frac{1}{N(n)}\bigg(\log x_{n+1} + \sum_{j=1}^{n} \log \frac{a_j(D_j)}{b_j(D_j)}\bigg) = -\frac{1}{N(n)} \sum_{j=1}^{n} \log(D_j + 1).$$

To every classical expansion with positive terms such as that of Engel, Sylvester, Lüroth, Cantor and others (see Galambos [2, pp. 14–19] for a list) there is a corresponding alternating series expansion. For each of these, except for the case of Lüroth, $N(n)$ is indeed of a larger magnitude than n when one seeks a finite nonzero limit on the right hand side of Theorem 4; consequently, Theorem 4 applies to all of the classical cases. For the exceptional case of Lüroth, the limits (A1)–(A3) correspond to Theorem 4. Since the basic distributional properties of the alternating series are similar to those in the case of expansions with positive terms, we refer to [2, pp. 106–109] for the choice of $N(n)$ for the classical cases.

4. The real role of the fundamental inequality (9). The fundamental inequality (9) is a necessary and sufficient condition for obtaining the digits D_j at (7) by the expansion (5) and (6). In particular, there are numbers x for which equality holds at (9) infinitely many times.

However, if we allow a rule to apply for all x except on a set of measure zero, then a significant change occurs at (9). We easily deduce from Theorem 3 that the set of x for which

$$D_{n+1} \geq 2H_n(D_n) \tag{18}$$

for all large n is of measure zero because it would imply that, for such x's, $V_j = 1$ would occur only a finite number of times, while Theorem 3 implies that V_j should be one in about 50% of the cases (by one more application of the strong law of large numbers). What is then the real magnitude of growth for D_n in the light of (9) and (18)? In one direction we get the following rule from the Borel–Cantelli lemmas.

THEOREM 5. *The inequalities* $D_{n+1} \geq k_n H_n(D_n)$ *for arbitrary real numbers* $k_n > 1$ *occur infinitely often either with probability one or with probability zero. It is of measure zero if, and only if, the sum of* $1/k_n$ *over* n *is finite.*

In the opposite direction we extend a recent result of Lee [7] who proved that, for the newly introduced Daróczy–Kátai–Birthday (DKB) expansions (see Galambos [4]), one can get very close to the opposite inequality at (18). The DKB expansion of real numbers in an Oppenheim expansion with $h_n(j) = j^2(j-1)$. Its modification to alternating DKB expansion is the one obtained by the algorithm (5) and (6) with

$$H_n(j) = j^2(j+1). \tag{19}$$

This is a cubic equation. We shall deal with expansions whose $H_n(j)$ is a polynomial of degree $t \geq 3$ with leading coefficient one. The cases $t = 1$ or 2 have similar growth rates to those established below but their proof deviates somewhat from the one that follows. We therefore do not deal with those cases here.

THEOREM 6. *Let* $H_n(j) = H(j)$ *in* (10) *be a polynomial of degree* $t \geq 3$ *with leading coefficient one and assume that it is the same function for all* n. *Then, for almost all* $0 < x \leq 1$,

$$D_{n+1} \geq D_n^t + D_n^t/(\log D_n)^c \quad \textit{for all } n \geq n_0(x), \tag{20}$$

where $c > 0$ *is an arbitrary constant.*

Proof. Let $u_j,\ 1 \leq j \leq n$, be integers which are possible values of the D_j, i.e. for which the inequalities (9) hold. We shall refer to such values as realizable sequences (by the algorithm (5) and (6)). We fix n and u_n, and estimate

$$P(D_{n+1} < D_n^t + D_n^t/(\log D_n)^c). \tag{21}$$

The value in (21) is the sum over all realizable values $u_j,\ 1 \leq j \leq n+1$, where u_{n+1} is further limited by (21). For a fixed set of u_j, Theorem 1 applies. Next, we observe that

$$\frac{a_j(u_j)}{b_j(u_j)} = \frac{H(u_j)}{j(j+1)},$$

which, by assumption, is $j^{t-2}+O(j^{t-3})$, with constant in $O(\ )$ not dependent on j. We do summation over a single u_j, $1 \le j \le n-1$, starting with u_1. This way, the summands will be u_j^{t-2} (with the appropriate error term that we shall take care of soon), and in view of (9), u_j is limited by $u_{j+1}^{1/t}$. We increase these sums if we sum over all integers for u_j not exceeding $u_{j+1}^{1/t}$, getting the bounds in the jth summation

$$(u_{j+1})^{e(j)+1}/(e(j)+1)$$

where the exponents $e(j)$ are defined by the recursive relation $e(1) = t-2$, and $e(j+1) = (e(j)+1)/t + t - 2$. Upon solving this difference equation we get $e(j) = t-1-t^{-j+1}$. Finally, upon observing that u_{n+1} satisfies both (9) and (21), summation over these values yields the estimate (recall (11) with $s = n+1$)

$$1/(u_{n+1}(u_{n+1}+1)) \le u_n^{-2t}(1+1/u_n^t)$$

multiplied by the number of terms between (9) and (21) which equals $u_n^t/(\log u_n)^c$.

When we collect the above terms for fixed u_n we obtain on (21) the upper estimate

$$\prod_{j=1}^{n-1}(e(j)+1)^{-1}\sum\frac{u_n^{e(n)}}{u_n^t(\log u_n)^c}$$

where the summation is over all possible values of u_n. Now, since u_n is larger than the values generated by (9) with equality for every n, the smallest value of u_n satisfies $\log(u_n) > gt^n$ with some constant $g > 0$ (see Lee [7], whose argument easily extends from $t = 3$ to any $t \ge 3$). From this fact one easily sees that the sum above is finite. By the Borel–Cantelli lemma, the inequality in (21) fails for all large n with probability one, i.e. (20) applies. The theorem is established.

Since the argument above applies in the case of expansions with positive terms as well, our result extends that of Lee [7], whose result is for $t = 3$ (the DKB expansions), and his result is not for arbitrary $c > 0$.

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Received on 3.4.2002
and in revised form on 2.10.2002 (4251)