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The diophantine equation $x^2 + C = y^n$, II

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The problem is for given C to determine possible solutions in positive integers x, y and $n \geq 3$. As in the first paper in the series [2], it suffices to consider only odd prime values of n, say p. Much of the previous paper was concerned with the question of determining possible solutions of the equation

$$\pm 1 = \sum_{r=0}^{(p-1)/2} \binom{p}{2r+1} a^{p-2r-1}(-C)^r. \tag{A}$$

As a result of the ground-breaking [1], this is now easily treated, for it is equivalent to

$$\pm 1 = \frac{\alpha^p - \beta^p}{\alpha - \beta} \quad \text{where } \alpha, \beta = a \pm \sqrt{-C}$$

and it then follows that there are no solutions for any $p > 5$. This removes the need for most of the third section of [2] except for Lemma 7, and enables Theorem 1 to be improved to

Theorem. *Let* $C > 0$, $C = cd^2$, c *square-free*, $c \not\equiv 7 \pmod 8$ *and* h *be the class number of the field* $\mathbb{Q}[\sqrt{-c}]$. *Then a solution of the equation of the title in coprime positive integers* x *and* y *can exist only in the following cases*:

(a) *there exist integers* a *and* b *with* $b \mid d$ *and* $b \neq \pm d$ *such that* $y = a^2 + b^2 c$ *and* $\pm x + d\sqrt{-c} = (a + b\sqrt{-c})^p$; *or*

(b) $c \equiv 3 \pmod 8$, $p = 3$ *and there exist odd integers* A *and* B *with* $B \mid d$ *such that* $y = \frac{1}{4}(A^2 + B^2 c)$ *and* $\pm x + d\sqrt{-c} = \frac{1}{8}(A + B\sqrt{-c})^p$; *or*

(c) $p \mid h$; *or*

(d) $p = 3$ *if* $C = 3a^2 \pm 8$ *with* $x = a^3 \pm 3a$ *or if* $C = 3a^2 \pm 1$ *with* $x = 8a^2 \pm 3a$; *or*

(e) $p = 5$ *if* $C = 19$ *with* $x = 22434$ *or if* $C = 341$ *with* $x = 2759646$.

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Secondly, two problems left open in [2], viz. the cases $C = 74$ and 86 with $p = 5$, have been completely solved in [3].

Finally, we note that we are unaware of any progress in dealing with the very difficult problems in which the principal ideals generated by $x \pm d\sqrt{-c}$ may have common factors, in particular for the values $C = 7$ or 25.

References

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[3] M. Mignotte and B. M. M. de Weger, *On the diophantine equations* $x^2+74=y^5$ *and* $x^2+86=y^5$, Glasgow Math. J. 38 (1996), 77–85.

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