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# Arithmetics of beta-expansions

by

L. S. GUIMOND (Montréal), Z. MASÁKOVÁ (Praha)
and E. PELANTOVÁ (Praha)

**1. Beta-expansions.** Let $\beta$ be a real number strictly greater than 1. A real number $x \geq 0$ can be represented using a sequence $(x_i)_{k \geq i > -\infty}$, $x_i \in \mathbb{Z}$, $0 \leq x_i < \beta$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \ldots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \ldots$$

for some $k \in \mathbb{Z}$. We then write

$$(x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots$$

A particular representation is the $\beta$-expansion of $x$ (see [7]). The digits $x_i$ of the $\beta$-expansion are computed by the "greedy" algorithm: Let $[y]$ denote the largest integer smaller than or equal to $y$. Find $k \in \mathbb{Z}$ for which $\beta^k \leq x < \beta^{k+1}$. Put $x_k = [x/\beta^k]$ and $r_k = x/\beta^k \bmod 1$. For $i \in \mathbb{Z}$, $i < k$ put $x_i = [\beta r_{i+1}]$ and $r_i = \beta r_{i+1} \bmod 1$. If $k < 0$, i.e. $0 < x < 1$, we put $x_0, x_1, \ldots, x_{k+1} = 0$ and write $(x)_\beta = 0 \bullet 00\ldots 0 x_k x_{k-1} \ldots$ If an expansion ends in infinitely many zeros, it is said to be *finite* and the final zeros are omitted.

We denote by $\mathrm{Fin}(\beta)$ the set of all $x$ for which $|x|$ has a finite $\beta$-expansion. The $\beta$-*expansion* of every $x \in \mathrm{Fin}(\beta)$ has therefore the form

$$(x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots x_{-l},$$

where $x_k x_{k-1} \ldots x_1 x_0 \bullet$ is the $\beta$-*integer part* and $\bullet x_{-1} x_{-2} \ldots x_{-l}$ is the $\beta$-*fractional part* of $x$. We usually call it simply the *integer* and the *fractional part* of $x$. The length of the fractional part of $x$ is denoted by $\mathrm{fp}_\beta(x)$. Elements of $\mathrm{Fin}(\beta)$ with vanishing fractional part (i.e. $\mathrm{fp}_\beta(x) = 0$) are called $\beta$-*integers*. The set of $\beta$-integers is denoted by $\mathbb{Z}_\beta$.

The sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$ are generally not closed under addition and multiplication. In spite of that it is sometimes useful in computer science to consider these operations in $\beta$-arithmetics. That is why it is important to

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study what fractional parts may appear as a result of addition and multiplication of $\beta$-integers.

DEFINITION 1.1. Let $\beta > 1$. We define

$$L_{\oplus}(\beta) := \min\{L \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_\beta,\ x + y \in \mathrm{Fin}(\beta) \Rightarrow \mathrm{fp}_\beta(x+y) \le L\},$$
$$L_{\odot}(\beta) := \min\{L \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_\beta,\ xy \in \mathrm{Fin}(\beta) \Rightarrow \mathrm{fp}_\beta(xy) \le L\}.$$

The minimum of an empty set is defined to be $+\infty$.

The aim of this paper is to give some quantitative results for $L_{\oplus}(\beta)$ and $L_{\odot}(\beta)$. Let us mention some of the known results. Frougny and Solomyak in [4] showed that $L_{\oplus}(\beta)$ is finite if $\beta$ is a Pisot number. A *Pisot number* $\beta$ is an algebraic integer such that $\beta > 1$ and all its algebraic conjugates are of modulus smaller than 1. Let us mention that to our knowledge no example is known of a $\beta$ such that $L_{\oplus}(\beta)$ or $L_{\odot}(\beta)$ is infinite.

Results for the special case of quadratic Pisot units are found in [3]. The authors gave exact values for $L_{\oplus}(\beta)$ and $L_{\odot}(\beta)$ when $\beta > 1$ is a solution either of the equation $x^2 = mx - 1$, $m \in \mathbb{N}$, $m \ge 3$ or of the equation $x^2 = mx + 1$, $m \in \mathbb{N}$. In the first case $L_{\oplus}(\beta) = L_{\odot}(\beta) = 1$; in the second case $L_{\oplus}(\beta) = L_{\odot}(\beta) = 2$.

In this article we provide estimates on $L_{\oplus}(\beta)$ and $L_{\odot}(\beta)$ for those algebraic numbers $\beta > 1$ that have at least one conjugate of modulus smaller than 1. Other results are valid for Pisot numbers $\beta$. The last part of the paper is devoted to quadratic Pisot numbers. We recover the results of [3] as a special case.

**2. Beta-integers and cut-and-project sequences.** The Rényi development of unity plays an important role in the description of the properties of the sets $\mathbb{Z}_\beta$ and $\mathrm{Fin}(\beta)$. For its definition we introduce the transformation $T_\beta(x) := \{\beta x\}$ for $x \in [0,1]$. The *Rényi development of unity* is defined as

$$d(1,\beta) := t_1 t_2 \ldots t_i \ldots, \quad \text{where} \quad t_i := [\beta T_\beta^{i-1}(1)].$$

Parry [6] showed that $x = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} \ldots x_{-p}$ is a $\beta$-expansion if and only if $x_i x_{i-1} \ldots x_{-p}$ is lexicographically smaller than $t_1 t_2 \ldots t_i \ldots$ for every $-p \le i \le k$.

$\mathrm{Fin}(\beta)$ and $\mathbb{Z}_\beta$ are centrally symmetric sets. While $\mathrm{Fin}(\beta)$ is dense in $\mathbb{R}$, $\mathbb{Z}_\beta$ has no accumulation points. Distances between consecutive points in $\mathbb{Z}_\beta$ take values in $\{0 \bullet t_i t_{i+1} \ldots \mid i \in \mathbb{N}\}$. It is obvious that if $d(1,\beta)$ is eventually periodic, then $\mathbb{Z}_\beta$ has a finite number of distances between consecutive points. Numbers $\beta$ with this property are called *beta-numbers*. Some results and conjectures on beta-numbers are given in [2, 8]; a description of beta-numbers is provided in [9]. Note that every Pisot number $\beta$ is a beta-number.

The set $\mathbb{Z}_\beta$ of $\beta$-integers forms a ring only if $\beta$ is a rational integer, $\beta > 1$. If $\beta$ is an algebraic integer of order $q \geq 2$, then $\mathbb{Z}_\beta$ can be naturally embedded into the ring $\mathbb{Z}[\beta]$ defined as

$$\mathbb{Z}[\beta] := \{n_0 + n_1\beta + \ldots + n_{q-1}\beta^{q-1} \mid n_i \in \mathbb{Z}\}\,.$$

Note that the ring $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$. In certain cases $\mathbb{Z}[\beta]$ coincides with $\mathrm{Fin}(\beta)$, i.e. $\mathrm{Fin}(\beta)$ is a ring (see [4]). Let us show that for $\beta$ an algebraic integer, the ring $\mathbb{Z}[\beta]$ is a projection of an integer lattice $\mathbb{Z}^q \subset \mathbb{R}^q$ on a one-dimensional subspace $V_1$ for a suitable decomposition $V_1 \oplus V_2$ of the space $\mathbb{R}^q$. A similar construction can be found in [1].

Denote by $\beta^{(1)} = \beta$, $\beta^{(2)}$, …, $\beta^{(s)}$ the real roots of the minimal polynomial of $\beta$ and by $\beta^{(s+1)}$, $\beta^{(s+2)}$, …, $\beta^{(q-1)}$, $\beta^{(q)}$ the non-real conjugates of $\beta$. We have ordered the complex roots in such a way that $\overline{\beta^{(s+1)}} = \beta^{(s+2)}$, …, $\overline{\beta^{(q-1)}} = \beta^{(q)}$.

First we have to find (possibly) complex vectors

$$(\vec{x}^{(1)})^T = (x_0^{(1)}, x_1^{(1)}, \ldots, x_{q-1}^{(1)}), \quad \ldots, \quad (\vec{x}^{(q)})^T = (x_0^{(q)}, x_1^{(q)}, \ldots, x_{q-1}^{(q)}),$$

such that for any $\vec{x} = (n_0, n_1, \ldots, n_{q-1}) \in \mathbb{R}^q$ we have

$$\begin{aligned}\vec{x} = \Big(\sum_{i=0}^{q-1} n_i(\beta^{(1)})^i\Big)\vec{x}^{(1)} + \Big(\sum_{i=0}^{q-1} n_i(\beta^{(2)})^i\Big)\vec{x}^{(2)} + \ldots \\ + \Big(\sum_{i=0}^{q-1} n_i(\beta^{(q)})^i\Big)\vec{x}^{(q)}.\end{aligned} \tag{1}$$

Denote by $\mathbb{X}$ the $q \times q$ matrix with $(\mathbb{X})_{ij} = x_j^{(i)}$. Then (1) holds for each $\vec{x}$ if and only if

$$I_q = \mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)}) \cdot \mathbb{X},$$

where $\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)})$ is the Vandermonde matrix in variables $\beta^{(1)}, \ldots, \beta^{(q)}$,

$$\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)}) := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \beta^{(1)} & \beta^{(2)} & \ldots & \beta^{(q)} \\ (\beta^{(1)})^2 & (\beta^{(2)})^2 & \ldots & (\beta^{(q)})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (\beta^{(1)})^{q-1} & (\beta^{(2)})^{q-1} & \ldots & (\beta^{(q)})^{q-1} \end{pmatrix}.$$

The determinant of $\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)})$ is equal to $\prod_{q\geq i>j\geq 1}(\beta^{(i)} - \beta^{(j)})$. Since all conjugates are distinct, the determinant is non-zero.

Using the Cramer rule to compute $x_j^{(i)}$, we find that $\vec{x}^{(i)}$ is real if $\beta^{(i)}$ is real, and if $\beta^{(j)}$ and $\beta^{(j+1)}$ are complex conjugate roots then $\vec{x}^{(j)} = \overline{\vec{x}^{(j+1)}}$.

Thus we can define a real basis $\vec{y}^{(1)},\dots,\vec{y}^{(q)}$ of $\mathbb{R}^q$ in such a way that $\vec{y}^{(i)}=\vec{x}^{(i)}$ if $\vec{x}^{(i)}$ is a real vector, and $\vec{y}^{(j)}=\vec{x}^{(j)}+\overline{\vec{x}^{(j)}}$, $\vec{y}^{(j+1)}=i(\vec{x}^{(j)}-\overline{\vec{x}^{(j)}})$, if $\vec{x}^{(j)}$ and $\vec{x}^{(j+1)}=\overline{\vec{x}^{(j)}}$ are complex conjugate vectors.

Note that the coordinates of a vector $\vec{x}=(n_0,n_1,\dots,n_{q-1})\in\mathbb{R}^q$ with respect to the basis $\vec{y}^{(1)},\dots,\vec{y}^{(q)}$ are

$$\begin{array}{ll} \displaystyle\sum_{p=0}^{q-1} n_p(\beta^{(i)})^p & \text{if } \vec{y}^{(i)}=\vec{x}^{(i)},\\ \displaystyle\Re\Big[\sum_{p=0}^{q-1} n_p(\beta^{(j)})^p\Big] & \text{if } \vec{y}^{(j)}=\vec{x}^{(j)}+\overline{\vec{x}^{(j)}},\\ \displaystyle\Im\Big[\sum_{p=0}^{q-1} n_p(\beta^{(j)})^p\Big] & \text{if } \vec{y}^{(j)}=i(\vec{x}^{(j)}-\overline{\vec{x}^{(j)}}). \end{array}$$

If we put $V_1=\mathbb{R}\vec{y}^{(1)}$ and $V_2=\mathbb{R}\vec{y}^{(2)}+\mathbb{R}\vec{y}^{(3)}+\dots+\mathbb{R}\vec{y}^{(q)}$, then the set $\mathbb{Z}[\beta]$ is the projection of $\mathbb{Z}^q$ on $V_1$ along $V_2$.

Projections of crystallographic lattices and non-crystallographic lattices are studied by the theory of cut-and-project sets. Let us recall a special case of their definition, which will be used here.

Definition 2.1. Let $U_1$ and $U_2$ be linear subspaces of $\mathbb{R}^d$ such that $\dim U_1=1$, $\dim U_2=d-1$ and $U_1\oplus U_2=\mathbb{R}^d$. Denote by $\pi_1$ the projection on $U_1$ along $U_2$ and by $\pi_2$ the projection on $U_2$ along $U_1$. Let $\Omega\subset U_2$ be a bounded set with non-empty interior $\Omega^\circ$, such that the closures of $\Omega$ and $\Omega^\circ$ coincide. If the mapping $\pi_1:\mathbb{Z}^q\to\pi_1(\mathbb{Z}^d)$ is one-to-one and $\pi_2(\mathbb{Z}^d)$ is dense in $V_2$, then the set $\Sigma(\Omega)=\{\pi_1(x)\mid x\in\mathbb{Z}^d,\ \pi_2(x)\in\Omega\}$ is called a *cut-and-project set* with *acceptance window* $\Omega$.

Basic properties of cut-and-project sets can be found in [5]. For us the most important property is that $\Sigma(\Omega)$ is relatively dense and uniformly discrete, i.e. there exists a real increasing sequence $(\alpha_n)_{n\in\mathbb{Z}}$ and constants $r,R>0$ such that $\Sigma(\Omega)=\{\alpha_n\vec{y}\mid n\in\mathbb{Z}\}$ and $r\le\alpha_{n+1}-\alpha_n\le R$ for all $n\in\mathbb{Z}$. In particular, the distances between consecutive points of $\Sigma(\Omega)$ take only finitely many values, i.e. the set $\{\alpha_{n+1}-\alpha_n\mid n\in\mathbb{Z}\}$ is finite.

Let us consider again an algebraic integer $\beta$ of order $q$ and the decomposition $\mathbb{R}^q=V_1\oplus V_2$ as described above. As shown by Akiyama [1], the projection $\pi_1(\mathbb{Z}^2)=\mathbb{Z}[\beta]$ of $\mathbb{Z}^2$ on $V_1$ is one-to-one and the projection $\pi_1(\mathbb{Z}^2)$ on $V_2$ is dense in $V_2$. For $\alpha\in\mathbb{Q}[\beta]$ we denote by $\alpha^{(k)}$ the image of $\alpha$ under the $k$th Galois isomorphism $\mathbb{Q}[\beta]\to\mathbb{Q}[\beta^{(k)}]$ induced by the assignment $\beta\mapsto\beta^{(k)}$, i.e. if $\alpha=\sum_{i=0}^{q-1}n_i\beta^i$ for $n_i\in\mathbb{Q}$, then $\alpha^{(k)}=\sum_{i=0}^{q-1}n_i(\beta^{(k)})^i$.

We shall focus on specific acceptance windows $\Omega(h)\subset V_2$ for $h>0$. As the acceptance window $\Omega(h)\subset V_2$ we choose the cartesian product of one-dimensional line-segments $\{t\vec{y}^{(i)}\mid |t|<h\}$ if $\beta^{(i)}$ is real and two-dimensional

ellipses $\{t\vec{y}^{(j)} + s\vec{y}^{(j+1)} \mid t^2 + s^2 < h^2\}$ if $\beta^{(j)}$ and $\beta^{(j+1)}$ are complex conjugate. Such an acceptance window $\Omega(h)$ satisfies the assumptions of Definition 2.1.

The point $\alpha\vec{y}^{(1)}$ belongs to $\Sigma(\Omega(h))$ if and only if $\alpha \in \mathbb{Z}[\beta]$ and $|\alpha^{(k)}| < h$ for $k = 2, \ldots, q$. In other words, we have the following proposition.

Proposition 2.2. *Let $\beta$ be an algebraic integer of order $q$. If $h > 0$, then the set*

$$\Sigma(h) = \{\alpha \in \mathbb{Z}[\beta] \mid |\alpha^{(k)}| < h,\ k = 2, \ldots, q\}$$

*is relatively dense and uniformly discrete and the distances in $\Sigma(h)$ take only finitely many values.*

In the following, the sets $\Sigma(h)$ are called *cut-and-project sequences*. In the case that $\beta$ is a Pisot number, we show the relation between cut-and-project sequences and $\beta$-integers $\mathbb{Z}_\beta$.

Proposition 2.3. *Let $\beta$ be a Pisot number of order $q$. Set*

$$l = [\beta] \max\{(1 - |\beta^{(i)}|)^{-1} \mid i = 2, \ldots, q\}.$$

*Then*

$$\mathbb{Z}_\beta \subset \Sigma(l), \qquad \mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \Sigma(2l), \qquad \mathbb{Z}_\beta\mathbb{Z}_\beta \subset \Sigma(l^2).$$

*Proof.* Let $x \in \mathbb{Z}_\beta$, i.e. $x = \pm\sum_{i=0}^{n} x_i\beta^i$ for some $n$. Then

$$|x^{(j)}| \le \sum_{i=0}^{n} [\beta]|\beta^{(j)}|^i < [\beta]\,\frac{1}{1 - |\beta^{(j)}|} \le l \qquad \text{for } j = 2, \ldots, q.$$

The statement follows easily. ∎

**3. Sufficient conditions for finiteness of $L_\oplus$ and $L_\odot$.** In this section we provide sufficient conditions on $\beta$ so that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite. First we demonstrate Theorem 3.1 stating that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite for a Pisot $\beta$. The statement for $L_\oplus$ has been proven in [4], but we provide a different and simpler proof. We further show that this condition is not necessary. Theorem 3.3 provides a different sufficient condition together with bounds on $L_\oplus(\beta)$ and $L_\odot(\beta)$. In the next section we apply Theorem 3.3 to the case of quadratic Pisot numbers.

Theorem 3.1. *Let $\beta$ be a Pisot number. Then $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite.*

*Proof.* Let $x, y \in \mathbb{Z}_\beta$. To determine $L_\odot(\beta)$ it suffices to consider $x, y > 0$. Set $z_0 = \max\{z \in \mathbb{Z}_\beta \mid z \le xy\}$ and $r := xy - z_0$. Since distances in $\mathbb{Z}_\beta$ are bounded by 1, we have $0 \le r < 1$. Therefore obviously the remainder $r$ is the fractional part of the $\beta$-expansion of $xy$, i.e. $xy \in \mathrm{Fin}(\beta)$ if and only if $r \in \mathrm{Fin}(\beta)$. Since $l > 1$, we have $\Sigma(l) \subset \Sigma(l^2)$ and according to Proposition 2.3 both $xy$ and $z_0$ belong to $\Sigma(l^2)$.

According to Proposition 2.2 distances in $\Sigma(l^2)$ take only finitely many values, say $f_1, \dots, f_T$. The gap $r$ between $z_0$ and $xy$ must be composed of these distances. Therefore $1 > r = xy - z_0 = \sum h_i f_i$, where $h_i \in \mathbb{N}_0$. Fractional parts of all results of multiplication $xy$ belong to the set

$$F := \Big\{ \sum_i h_i f_i < 1 \;\Big|\; h_i \in \mathbb{N}_0 \Big\},$$

which is finite and therefore

$$L_{\odot}(\beta) \le \max\{\mathrm{fp}_\beta(r) \mid r \in F \cap \mathrm{Fin}(\beta)\}.$$

To derive the finiteness of $L_{\oplus}(\beta)$ one uses an analogous argument. ∎

A simple consequence of the above proof is that $\mathbb{Z}_\beta$ is a Meyer set.

Corollary 3.2. *Let $\beta$ be a Pisot number. Then there exists a finite set $F$ such that*

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F, \qquad \mathbb{Z}_\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F.$$

Theorem 3.1 gives a sufficient condition for finiteness of $L_{\oplus}(\beta)$ and $L_{\odot}(\beta)$. An upper bound on $L_{\oplus}(\beta)$ is determined in [10] using some complicated techniques. However, their result applies only to a class of Pisot numbers. The condition that $\beta$ is Pisot is however not necessary. In the following theorem we provide a similar estimate on $L_{\oplus}(\beta)$ with less restrictive criteria for $\beta$. Moreover, we determine an upper bound for $L_{\odot}(\beta)$.

Theorem 3.3. *Let $\beta > 1$ be an irrational algebraic number such that at least one of its conjugates, say $\beta'$, is of modulus smaller than* 1. *Define*

$$H = \sup\{|z'| \mid z \in \mathbb{Z}_\beta\}, \qquad K = \inf\{|z'| \mid z \in \mathbb{Z}_\beta,\ z \notin \beta\mathbb{Z}_\beta\}.$$

*If $K > 0$, then $L_{\oplus}(\beta)$ and $L_{\odot}(\beta)$ are finite and*

$$\left(\frac{1}{|\beta'|}\right)^{L_{\oplus}(\beta)} < \frac{2H}{K}, \tag{2}$$

$$\left(\frac{1}{|\beta'|}\right)^{L_{\odot}(\beta)} < \frac{H^2}{K}. \tag{3}$$

*Proof.* Let $x, y \in \mathbb{Z}_\beta$ and $x + y \in \mathrm{Fin}(\beta)$, $x + y = \sum_{i=-L}^{k} a_i \beta^i$, $a_{-L} \ge 1$. Then $\beta^L(x+y) \in \mathbb{Z}_\beta$ and $\beta^L(x+y) \notin \beta\mathbb{Z}_\beta$. Thus

$$K \le |\beta'|^L |x' + y'| \le |\beta'|^L (|x'| + |y'|) < 2H|\beta'|^L,$$

which implies (2). Note that the supremum $H$ is never attained, i.e. $|z'| < H$ for all $z \in \mathbb{Z}_\beta$. The proof for multiplication is similar. ∎

Remark 3.4. 1. Using the same inequalities as in the proof of Proposition 2.3 we obtain

$$H \le [\beta]\,\frac{1}{1 - |\beta'|}.$$

2. If $\beta' \in (0,1)$, then $K = 1$. Indeed, for $z = \sum_{i=0}^{n} z_i\beta^i$, $z_0 \neq 0$, one has

$$z' = \sum_{i=0}^{n} z_i(\beta')^i \geq z_0 \geq 1.$$

COROLLARY 3.5. *Let $\beta > 1$ be an algebraic integer such that at least one of its conjugates, say $\beta'$, belongs to $(0,1)$. Then*

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus(\beta)} < \frac{2[\beta]}{1-\beta'}, \qquad \left(\frac{1}{|\beta'|}\right)^{L_\odot(\beta)} < \frac{[\beta]^2}{(1-\beta')^2}.$$

**4. Theorem 3.3 for quadratic Pisot numbers.** So far we have been interested in $L_\oplus(\beta)$ and $L_\odot(\beta)$ for general algebraic integers $\beta$. From now on we shall focus on quadratic Pisot numbers. In the quadratic case the Pisot condition implies that $\beta$ is a solution of an equation

$$\begin{aligned} x^2 &= mx - n, \qquad m, n \in \mathbb{N},\ m \geq n+2, \\ x^2 &= mx + n, \qquad m, n \in \mathbb{N},\ m \geq n. \end{aligned}$$

We shall try to apply Theorem 3.3 for such $\beta$ and derive the corresponding bounds on $L_\oplus(\beta)$ and $L_\odot(\beta)$. It will be seen that the situation drastically differs for the two types of quadratic equations.

Note that for $n = 1$, the root $\beta$ is a quadratic Pisot unit. For such $\beta$ the values of $L_\oplus(\beta)$ and $L_\odot(\beta)$ have been determined in [3].

Let us now study the case of $\beta > 1$ solving the equation $x^2 = mx - n$, $m, n \in \mathbb{N}$, $m \geq n+2$. Note that $[\beta] = m-1$, thus the digits in $\beta$-expansions are $0, 1, \ldots, m-1$. The conjugate $\beta'$ of $\beta$ satisfies $\beta' \in (0,1)$, and the $\beta$-development of unity is $d(1,\beta) = (m-1)(m-n-1)^\omega$. For $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^{n} z_i\beta^i$ we have

$$\begin{aligned} z' = \sum_{i=0}^{n} z_i(\beta')^i &< (m-1) + (m-2)\beta' + (m-2)\beta'^2 + \ldots \\ &= 1 + (m-2)\frac{1}{1-\beta'} = \frac{\beta(\beta-1)}{\beta-n} = H. \end{aligned} \tag{4}$$

Clearly, $\beta(\beta-1)/(\beta-n)$ above is the desired supremum $H$ of Theorem 3.3, since we can construct a sequence of numbers

$$z_n = (m-1)\beta^0 + \sum_{i=1}^{n}(m-2)\beta^i \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$$

such that $\lim_{n\to\infty} |z_n'| = H$. For the relation (4) we have considered the admissibility of sequences of digits in $\beta$-expansions. According to Remark 3.4 we have $K = 1$, and hence we can use Theorem 3.3 to derive results for $L_\oplus(\beta)$ and $L_\odot(\beta)$.

PROPOSITION 4.1. *Let* $\beta^2 = m\beta - n$, $m \geq n+2$. *Then*

$$L_\oplus(\beta) \leq 3m\ln m, \qquad L_\odot(\beta) \leq 4m\ln m.$$

*In particular*, *if* $n = 1$, *then* $L_\oplus(\beta) = L_\odot(\beta) = 1$.

*Proof.* Since $K = 1$ and $H = \beta(\beta-1)/(\beta-n) = (\beta-1)^2/(m-n-1)$ we can estimate

$$\left(\frac{m-1}{n}\right)^{L_\oplus} < \left(\frac{\beta}{n}\right)^{L_\oplus} = \left(\frac{1}{\beta'}\right)^{L_\oplus} < 2\,\frac{(\beta-1)^2}{m-n-1} < 2\,\frac{(m-1)^2}{m-n-1}.$$

For $n = 1$ we obtain directly $L_\oplus \leq 1$. For general $n \leq m-2$ we estimate the left hand side of the inequality by

$$\left(\frac{m-1}{n}\right)^{L_\oplus} \geq \left(\frac{m-1}{m-2}\right)^{L_\oplus} > e^{L_\oplus/m},$$

where we have used $(1+1/k)^{k+1} > e$ for $k \in \mathbb{N}$. The right hand side of the inequality is estimated by $m^3$. Altogether we get $L_\oplus(\beta) \leq 3m\ln m$. The estimate for $L_\odot(\beta)$ is derived analogously, the first step for $n = 1$ being

$$\beta^{L_\odot} = \left(\frac{1}{\beta'}\right)^{L_\odot} < \left(\frac{\beta(\beta-1)}{\beta-1}\right)^2 = \beta^2 \;\Rightarrow\; L_\odot \leq 1.$$

In order to show that for $n = 1$ we have $L_\oplus(\beta) = L_\odot(\beta) = 1$ it suffices to observe that

$$((m-1)+(m-1))_\beta = (2\cdot(m-1))_\beta = \left(\beta + (m-2) + \frac{1}{\beta}\right)_\beta = 1(m-2)\bullet 1. \;\blacksquare$$

Let us now study the case of $\beta > 1$ solving the equation $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. Note that $[\beta] = m$. Therefore the digits in $\beta$-expansions are $0, 1, \ldots, m$. The $\beta$-development of unity is $d(1,\beta) = mn$. Now the conjugate $\beta'$ of $\beta$ satisfies $\beta' \in (-1, 0)$. If $w \in \mathbb{Z}_\beta$, $w = \sum_{i=0}^{n} w_i\beta^i$, we have

$$\ldots + m\beta'^3 + m\beta' < w' < m + m\beta'^2 + m\beta'^4 + \ldots,$$
$$-1 < w' < \frac{m}{1-\beta'^2} = \frac{\beta^2 m}{m\beta + n - n^2} = H.$$

Unfortunately, in this case $K = 0$ for all $n \in \mathbb{N}$ except $n = 1$. Therefore only for $n = 1$ can we use Theorem 3.3 to find $L_\oplus(\beta)$ and $L_\odot(\beta)$. In this case for $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^{n} z_i\beta^i$ with $z_0 \neq 0$, we have

$$\begin{aligned} z' &\geq z_0 + z_1\beta' + z_3\beta'^3 + z_5\beta'^5 + \ldots \\ &\geq 1 + (m-1)\beta' + m\beta'^3 + m\beta'^5 + \ldots \\ &= 1 - \beta' + \frac{m\beta'}{1-\beta'} = -\beta' = \frac{1}{\beta} = K. \end{aligned}$$

Note that $H$ is equal to $\beta$ for $n = 1$. Using (2) and (3), we obtain for $m \geq 2$

$$\left.\begin{array}{l} \beta^{L_\oplus} < 2\beta^2 < \beta^3 \\ \beta_{L_\odot} < \beta^3 \end{array}\right\} \Rightarrow \begin{array}{l} L_\oplus(\beta) \leq 2, \\ L_\odot(\beta) \leq 2. \end{array}$$

To prove that $L_\oplus(\beta) = L_\odot(\beta) = 2$ we calculate

$$(m+m)_\beta = (2\cdot m)_\beta = \left(\beta + (m-1) + \frac{m-1}{\beta} + \frac{1}{\beta^2}\right)_\beta = 1(m-1)\bullet(m-1)1.$$

For $m = 1$, i.e. $\beta$ the golden ratio, it is not true that $2\beta^2 < \beta^3$. A slightly finer discussion is necessary to obtain the exact bound on the number of fractional digits of the sum $x + y$.

In the above considerations we are not able to derive any estimates on $L_\oplus(\beta)$ and $L_\odot(\beta)$ if $\beta$ is a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n \geq 2$. Therefore in the rest of the paper we focus on such quadratic Pisot numbers. First we give an estimate on $L_\odot(\beta)$ using $L_\oplus(\beta)$ and then we determine the value of $L_\oplus(\beta)$.

**5. Relation of $L_\oplus$ and $L_\odot$ for quadratic Pisot numbers.** In Section 2 we have shown that $\mathbb{Z}_\beta$ can be embedded into a cut-and-project sequence with a suitably chosen window. In our case $\beta$ is a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n \geq 2$. Therefore we choose $\Sigma(H)$, where $H = m/(1 - \beta'^2)$. We show that a cut-and-project set with arbitrary window can be embedded into a finite union of shifted copies of $\mathbb{Z}_\beta$, where the shifts belong to $\mathbb{Z}[\beta]$. In fact, a product $xy$ of $x, y \in \mathbb{Z}_\beta$ can be expressed as a sum of a $\beta$-integer and a small rational integer and therefore we can find an upper estimate of $L_\odot(\beta)$ using $L_\oplus(\beta)$. A similar result can also be proven for non-quadratic Pisot $\beta$. The demonstration is however rather technical.

THEOREM 5.1. *Let $\beta > 1$ be a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$, and let $h > 0$. Then there exists $p \in \mathbb{N}$ such that*

$$\Sigma(h) \subset \mathbb{Z}_\beta + \{-p, -p+1, \ldots, -1, 0, 1, \ldots, p-1, p\},$$

*where*

$$p \leq h - \beta' H = h - \beta' \frac{m}{1 - \beta'^2}.$$

*Proof.* Since $\beta$ is a quadratic integer, we can write every power $\beta^k$ as an integer combination of 1 and $\beta$. Define $F_k$, $G_k$ by

$$\beta^k = F_k\beta + G_k.$$

Since $\beta^{k+1} = \beta(F_k\beta + G_k) = F_k m\beta + F_k n + G_k\beta$, the sequences $(F_k)_{k\in\mathbb{N}_0}$, $(G_k)_{k\in\mathbb{N}_0}$ satisfy $F_{k+1} = mF_k + G_k$, $G_{k+1} = nF_k$, which gives a recurrence relation

$$F_{k+2} = mF_{k+1} + nF_k, \quad \text{where} \quad F_0 = 0,\ F_1 = 1.$$

It is easy to see that every $x \in \mathbb{N}$ can be written in the form $x = \sum_{i=1}^{j} c_i F_i$, where $c_i \in \{0, 1, \ldots, m\}$ and $c_i c_{i-1}$ is lexicographically smaller than $mn$. The coefficients $c_j, c_{j-1}, \ldots, c_1$ can be found by the so-called greedy algorithm. Thus $j$ is a number for which $F_j \le x < F_{j+1}$ and $c_j := [xF_j^{-1}]$. We obtain the coefficients $c_i$, $i < j$, by applying the same steps to the integer $\widetilde{x} = x - c_j F_j$.

Let $z \in \Sigma(h)$, i.e. $z = a + b\beta$ and $|z'| < h$. Since both $\Sigma(h)$ and $\mathbb{Z}_\beta$ are symmetric with respect to the origin, it suffices to show the statement for $b \ge 0$. Let $b = \sum_{i=1}^{j} c_i F_i$. Then

$$z = \sum_{i=1}^{j} c_i(F_i\beta + G_i) - \sum_{i=1}^{j} c_i G_i + a = z_1 + z_2, \tag{5}$$

where $z_2 := a - \sum_{i=1}^{j} c_i G_i \in \mathbb{Z}$ and $z_1 := \sum_{i=1}^{j} c_i \beta^i \in \beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. Applying the Galois automorphism to the equality $z = z_1 + z_2$ gives $z_2 = z' - z_1'$. Since $|z'| < h$ and $|z_1'| < -\beta' H$, the integer $z_2$ belongs to the interval $(-h + \beta' H, h - \beta' H)$. ∎

Corollary 5.2.

$$\mathbb{Z}_\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{-p, \ldots, p\}, \quad \textit{where} \quad p \le (m+2)^4/4.$$

*Proof.* Since $\mathbb{Z}_\beta \subset \Sigma(H)$, we have $\mathbb{Z}_\beta\mathbb{Z}_\beta \subset \Sigma(H^2)$. The proof will be completed if we verify that $H^2 - \beta' H \le \frac{1}{4}(m+2)^4$. Let us first show that

$$\frac{1}{1-\beta'^2} < \frac{m+3}{2}. \tag{6}$$

We have $-\beta' = n/\beta$, thus for $n \le m-1$,

$$1 - \beta'^2 = 1 - \frac{n^2}{\beta^2} > 1 - \frac{n^2}{m^2} \ge 1 - \frac{(m-1)^2}{m^2} = \frac{2m-1}{m^2} \ge \frac{2}{m+3}.$$

For $n = m$ the inequality (6) is verified directly using $\beta' = \frac{1}{2}(m - \sqrt{m^2 + 4m})$. Therefore

$$\begin{aligned} H^2 - \beta' H \le H^2 + H &= \frac{m^2}{(1-\beta'^2)^2} + \frac{m}{1-\beta'^2} < \frac{m^2(m+3)^2}{4} + \frac{m(m+3)}{2} \\ &= \frac{1}{4} m(m+1)(m+2)(m+3) \le \frac{1}{4}(m+2)^4. \ \blacksquare \end{aligned}$$

The above corollary states that a product of two $\beta$-integers can be written as a sum of a $\beta$-integer and a rational integer. Let us derive the number of fractional digits of the $\beta$-expansion of a rational integer $p$.

Lemma 5.3. *Let $p \in \mathbb{N}$. Then*

$$\mathrm{fp}_\beta(p) \le (1 + \log_2 p) L_\oplus(\beta).$$

*Proof.* The proof is based on the simple observation that

$$\mathrm{fp}_\beta(x+y) \le \max\{\mathrm{fp}_\beta(x), \mathrm{fp}_\beta(y)\} + L_\oplus(\beta), \tag{7}$$

which in particular gives $\mathrm{fp}_\beta(2x) \le \mathrm{fp}_\beta(x) + L_\oplus(\beta)$. Applying the latter $k$ times we obtain $\mathrm{fp}_\beta(2^k) \le kL_\oplus(\beta)$. We use induction on $j$ to prove that if $p$ has a binary expansion $p = \sum_{i=0}^{j} a_i 2^i$ then $\mathrm{fp}_\beta(p) \le (j+1)L_\oplus(\beta)$. Using the hypothesis for $p = \sum_{i=0}^{j} a_i 2^i = 2^j + \sum_{i=0}^{j-1} a_i 2^i$ we obtain

$$\begin{aligned}\mathrm{fp}_\beta(p) &\le \max\Big\{\mathrm{fp}_\beta(2^j), \mathrm{fp}_\beta\Big(\sum_{i=0}^{j-1} a_i 2^i\Big)\Big\} + L_\oplus(\beta)\\ &\le \max\{jL_\oplus(\beta), jL_\oplus(\beta)\} + L_\oplus(\beta) = (j+1)L_\oplus(\beta).\end{aligned}$$

The statement of the lemma follows easily from the fact that $j \le \log_2 p$. ∎

The following theorem is a simple consequence of Corollary 5.2 and Lemma 5.3.

THEOREM 5.4. *Let $\beta > 1$ be a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \ge n$. Then*

$$L_\odot(\beta) \le 4L_\oplus(\beta)\log_2(m+2).$$

*Proof.* Let $x, y \in \mathbb{Z}_\beta$. Using Corollary 5.2, we have $\mathrm{fp}_\beta(xy) = \mathrm{fp}_\beta(z+p)$ for some $z \in \mathbb{Z}_\beta$ and $p \in \mathbb{N}$, $p \le \frac{1}{4}(m+2)^4$. Now, due to (7),

$$\begin{aligned}\mathrm{fp}_\beta(z+p) &\le \mathrm{fp}_\beta(p) + L_\oplus(\beta) \le (2+\log_2 p)L_\oplus(\beta)\\ &\le \Big(2 + \log_2\frac{(m+2)^4}{4}\Big)L_\oplus(\beta).\end{aligned}$$

The statement of the theorem follows easily. ∎

**6. $L_\oplus$ for quadratic $\beta$.** In this section we obtain an upper bound on $L_\oplus(\beta)$. This is done in two steps: first we find an upper bound on $\mathrm{fp}(x+y)$ where $x$ is an arbitrary $\beta$-integer and $y$ is a $\beta$-integer of a specific form. Then we show that any $\beta$-integer can be written as a finite sum of numbers of this specific form. An upper bound on $L_\oplus(\beta)$ is obtained by combining both results.

Let $\beta > 1$ be a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \ge n$. Let $(x)_\beta = x_k x_{k-1} \dots x_1 x_0 \bullet x_{-1} x_{-2} \dots x_{-p}$ be a $\beta$-representation of $x$, i.e. $0 \le x_i \le m$. The $\beta$-representation $(x)_\beta$ is a $\beta$-expansion of $x$ if and only if $x_i x_{i-1}$ is lexicographically smaller than $mn = d(1,\beta)$ for every $i$.

The following lemma is an easy consequence of the result of Frougny and Solomyak in [4]. It is mentioned here in order to make the article self-contained.

LEMMA 6.1. *Let $(x)_\beta = x_k x_{k-1} \dots x_1 x_0 \bullet x_{-1} x_{-2} \dots x_{-p}$ be a $\beta$-representation of $x$. Then $\mathrm{fp}_\beta(x) \le p$.*

*Proof.* If the representation is already in the form of a $\beta$-expansion, then $\mathrm{fp}_\beta(x) = p$. Otherwise we can find the largest $j$ such that $x_j x_{j-1}$ is lexicographically greater than or equal to $mn$. Since $x_i \le m$ for all $i$, necessarily $x_j = m$ and $x_{j-1} \ge n$. Since $j$ was the largest index with this property, $x_{j+1} < m$. Therefore we can define a new representation of $x$ as

$$(x)_\beta = \widetilde{x}_k \widetilde{x}_{k-1} \dots \widetilde{x}_1 \widetilde{x}_0 \bullet \widetilde{x}_{-1} \widetilde{x}_{-2} \dots \widetilde{x}_{-p},$$

where $\widetilde{x}_j := x_j - m$, $\widetilde{x}_{j-1} := x_{j-1} - n$, $\widetilde{x}_{j+1} := x_{j+1} + 1$, and $\widetilde{x}_i = x_i$ otherwise. In the new representation the sum of digits is strictly smaller than in the previous one. This procedure can be repeated and in finitely many steps we obtain the $\beta$-expansion of $x$. The result follows easily, since in each step the number of digits in the fractional part of the representation does not increase. ∎

Let us first determine a lower bound on $L_\oplus(\beta)$. It suffices to find a single example of addition with specified fractional part length. We use the following example.

EXAMPLE 6.2. Consider $x = m \sum_{i=0}^{k-1} \beta^{2i}$. Then it can be shown by induction on $k$ that

$$x + x = \sum_{i=0}^{k-1} (A_{k-i}\beta + B_{k-i})\beta^{2i} + \sum_{i=0}^{k-1} \left( \frac{a_{k-i}}{\beta} + \frac{b_{k-i}}{\beta^2} \right) \beta^{-2i},$$

where the coefficients $A_i$, $B_i$, $a_i$ and $b_i$, $i \in \mathbb{N}$, are defined by

$$\begin{aligned} A_i &= i(m-n+1) - m + n, \\ B_i &= 2m - n - i(m-n+1), \\ a_i &= i(m-n+1) - 1, \\ b_i &= m + 1 - i(m-n+1). \end{aligned}$$

Formally, we have

$$x + x = A_1 B_1 A_2 B_2 \dots A_k B_k \bullet a_k b_k \dots a_2 b_2 a_1 b_1.$$

The above expression is a $\beta$-expansion if and only if all the coefficients $A_i$, $B_i$, $a_i$ and $b_i$ take values in $\{0, 1, \dots, m\}$ for $i = 1, \dots, k$. This implies the following conditions on $k$:

$$\begin{aligned} k(m-n+1) &\le m+1, \\ (k-1)(m-n+1) &\le m-1. \end{aligned}$$

For $m = n$ the latter condition is stronger and the maximal $k$ satisfying it is $k = m$. If on the other hand $m > n$, the first condition is stronger and the maximal $k \in \mathbb{N}$ satisfying it is

$$k_0 := \left[ \frac{m+1}{m-n+1} \right].$$

Corollary 6.3. *Let $\beta$ be the larger solution of $x^2 = mx + n$, $m, n \in \mathbb{Z}$, $m \geq n > 0$. Then*

$$L_\oplus(\beta) \geq \begin{cases} 2m & \textit{if } m = n, \\ 2k_0 & \textit{if } m > n. \end{cases}$$

From now on we focus on determining the upper bound for $L_\oplus(\beta)$.

Lemma 6.4. *Let $x, y \in \mathbb{Z}_\beta$, $x, y \geq 0$, with $\beta$-expansions*

$$(x)_\beta = x_l x_{l-1} \dots x_1 x_0 \bullet,$$
$$(y)_\beta = y_k y_{k-1} \dots y_1 y_0 \bullet,$$

*where $y_i \leq m - n + 1$ for $i = 0, 1, \dots, k-2, k-1$. Then the $\beta$-expansion of $x + y$ is*

$$(x+y)_\beta = z_r z_{r-1} \dots z_1 z_0 \bullet z_{-1} z_{-2},$$

*where*

$$\frac{z_{-1}}{\beta} + \frac{z_{-2}}{\beta^2} \in \left\{0, \frac{n}{\beta}, \frac{m-n}{\beta} + \frac{n}{\beta^2}\left(= 1 - \frac{n}{\beta}\right)\right\}.$$

*Proof.* We make use of the relation $m+p = \beta + p - 1 + (m-n)\beta^{-1} + n\beta^{-2}$ for $p \leq m$, i.e. $(m+p)_\beta = 1(p-1) \bullet (m-n)n$. Symbolically it may be rewritten as

$$\begin{array}{ccc|cc} & m & & & \\ + & p & & & \\ \hline 1 & (p-1) & & (m-n) & n \end{array} \tag{8}$$

We proceed by induction on the values of $y$. Let $y = y_0 \leq m - n + 1$. Then, according to (8), the $\beta$-representation of $x + y$ is

$$(x+y)_\beta = \begin{cases} x_l \dots x_1 (x_0 + y_0) \bullet & \text{if } x_0 + y_0 \leq m, \\ x_l \dots (x_1 + 1)(x_0 + y_0 - m - 1) \bullet (m-n)n & \text{if } x_0 + y_0 > m. \end{cases}$$

Note that $x_1 + 1 \leq m$ in the second case, since $x_1 = m$ implies $x_0 \leq n - 1$, and thus $x_0 + y_0 \leq n - 1 + m - n + 1 = m$, which is a contradiction.

Now assume that the statement holds for all $\widetilde{y} < y$ satisfying the conditions of the lemma. Suppose that there exists an index $i$ such that $y_i > 0$ and $x_i < m$. Then $x + y = \widetilde{x} + \widetilde{y}$, where according to Lemma 6.1, $\widetilde{x} = x + \beta^i \in \mathbb{Z}_\beta$ and $\widetilde{y} = y - \beta^i$ satisfies the conditions of the lemma. We may thus use the induction hypothesis.

Suppose that $y_i > 0$ implies $x_i = m$ for all $i \leq k$. Since $x_l x_{l-1} \dots x_1 x_0$ is an expansion, $x_i = m$ implies $x_{i-1} \leq n - 1 < m$. Thus $y_i > 0$ implies $y_{i-1} = 0$. Since $y_k > 0$, we have $x_k = m$ and $x_{k+1} < m$. Without loss of generality we can consider only the case when $l \leq k + 1$. Therefore we have the following situation:

$$\begin{array}{ccccccc|}
x_{k+1} & m & x_{k-1} & x_{k-2} & \dots & x_1 & x_0 \\
 & y_k & 0 & y_{k-2} & \dots & y_1 & y_0 \\ \hline
\end{array}$$

Let $j$ be the smallest integer among $\{1,2,\dots,[k/2]\}$ such that $y_{k-2j} < m-n+1$. Then

$$\begin{array}{cccccccccccc|}
x_{k+1} & m & x_{k-1} & m & \dots & x_{k-2j+3} & m & x_{k-2j+1} & x_{k-2j} & x_{k-2j-1} & \dots & x_0 \\
 & y_k & 0 & m-n+1 & \dots & 0 & m-n+1 & 0 & y_{k-2j} & y_{k-2j-1} & \dots & y_0 \\ \hline
\end{array}$$

We may check by elementary algebra using the relation $\beta^2 = m\beta + n$ that

$$\begin{aligned}
(9)\qquad & m\beta^k + (m-n+1)\sum_{i=1}^{j-1}\beta^{k-2i} \\
&= \beta^{k+1} - \beta^k + (m-n+1)\beta\sum_{i=1}^{j-1}\beta^{k-2i} + (m-n)\beta^{k-2j+1} + n\beta^{k-2j}.
\end{aligned}$$

Using this relation, we may write the sum $x+y = \widetilde{x}+\widetilde{y}$ as

$$\begin{array}{cccccccccc|}
(x_{k+1}+1) & (y_k-1) & \widetilde{x}_{k-1} & m & \dots & \widetilde{x}_{k-2j+3} & m & \widetilde{x}_{k-2j+1} & x_{k-2j} & x_{k-2j-1} \ \dots \ x_0 \\
 & & & & & & & & (y_{k-2j}+n) & y_{k-2j-1} \ \dots \ y_0 \\ \hline
\end{array}$$

where $\widetilde{x}_{k-2i+1} = x_{k-2i+1} + m - n + 1$ for $i = 1,\dots,j-1$ and $\widetilde{x}_{k-2j+1} = x_{k-2j+1} + m - n$. The first row represents the summand $\widetilde{x}$, the second row the summand $\widetilde{y}$. Due to (9) we have $x+y = \widetilde{x}+\widetilde{y}$. Obviously $\widetilde{x},\widetilde{y} \in \mathbb{Z}_\beta$, the digits of $\widetilde{y}$ are $\le m-n+1$, except its first non-zero digit from the left. We have $\widetilde{y} < y$ and thus we may use the induction hypothesis.

It remains to solve the case where $y_{k-2i} = m-n+1$ for all $i \in \{1,2,\dots,[k/2]\}$. Then either

$$y = y_k\ 0\ (m-n+1)\ 0\ (m-n+1)\dots\ 0\ (m-n+1),$$

or

$$y = y_k\ 0\ (m-n+1)\ 0\ (m-n+1)\ \dots\ 0\ (m-n+1)\ 0,$$

i.e.

$$y = y_k\beta^k + (m-n+1)\sum_{i=1}^{[k/2]}\beta^{k-2i}$$

for $k$ even or odd. We may deduce from (9) that the results of the addition $x+y$ have fractional parts $1-n/\beta$ and $n/\beta$ respectively. This completes the proof. ∎

LEMMA 6.5. *Let $x, y \in \mathbb{Z}_\beta$, $x > y \geq 0$. Then*

$$x - y = \begin{cases} z \\ z + (n-1)\displaystyle\sum_{i=0}^{k} \beta^{2i} + 1 \\ z + (n-1)\displaystyle\sum_{i=1}^{k} \beta^{2i-1} + \dfrac{n}{\beta} \end{cases} \qquad \textit{with } z \in \mathbb{Z}_\beta,\ z \geq 0,\ k \geq 0.$$

*Proof.* First note that for every $x \in \mathbb{Z}_\beta$ there exists a $\beta$-representation $(x)_\beta = x_l \dots x_1 x_0 \bullet$ such that $x_i + x_{i-1} > 0$ for all $0 < i \leq l$, i.e. the $\beta$-representation is "dense". The dense form can be found by the following procedure: Find the first pair of zeros from the left, say $x_i = x_{i-1} = 0$, $x_{i+1} > 0$. Put $\widetilde{x}_{i+1} = x_{i+1} - 1$, $\widetilde{x}_i = m$, $\widetilde{x}_{i-1} = n$, and $\widetilde{x}_j = x_j$ for all other $0 \leq j \leq l$. The new $\beta$-representation $(x)_\beta = \widetilde{x}_l \dots \widetilde{x}_1 \widetilde{x}_0 \bullet$ has strictly lower number of vanishing coefficients. Thus the procedure is finite.

The proof of the lemma is by induction on the value of $y$. Without loss of generality we may assume that both $(x)_\beta = x_l \dots x_1 x_0 \bullet$ and $(y)_\beta = y_k \dots y_1 y_0 \bullet$ are written in their dense form.

Assume that there is an index $i$ such that both $x_i$ and $y_i$ are non-zero. Then $x - y = \widetilde{x} - \widetilde{y}$, where $\widetilde{x} = x - \beta^i$ and $\widetilde{y} = y - \beta^i$. Clearly, $\widetilde{x}, \widetilde{y} \in \mathbb{Z}_\beta$ and $\widetilde{y} < y$, thus we may use the induction hypothesis.

Assume that $y_i > 0$ implies $x_i = 0$ for all indices $i$. Since $x_i + x_{i-1} > 0$, we have $y_{i-1} = 0$. Since $y_k > 0$, we have $x_k = 0$ and $x_{k+1} > 0$. Without loss of generality we consider $l = k + 1$ and $x_{k+1} = 1$. Since both $x$ and $y$ are in their dense form, the remaining cases are as follows. First assume that the maximal index $k$ such that $y_k$ is non-zero, is even. We have $x - y$ equal to

$$\begin{array}{cccccccccc|c}
 & 1 & 0 & x_{k-1} & 0 & x_{k-3} & \dots & x_1 & 0 & \\
- & & y_k & 0 & y_{k-2} & 0 & \dots & 0 & y_0 & \\
\hline
 & 1 & 0 & x_{k-1} & 0 & x_{k-3} & \dots & x_1 & 0 & \\
- & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \\
+ & & m & (n-1) & m & (n-1) & \dots & (n-1) & m & n \\
- & & y_k & 0 & y_{k-2} & 0 & \dots & 0 & y_0 & \\
\hline
 & & (m-y_k) & x_{k-1} & (m-y_{k-2}) & x_{k-3} & \dots & x_1 & (m-y_0) & \\
+ & & & (n-1) & 0 & (n-1) & \dots & (n-1) & 0 & n \\
\hline\hline
\end{array}$$

which corresponds to the statement of the lemma. For $k$ odd we may write similarly that $x-y$ equals

$$\begin{array}{rcccccccc|}
1 & 0 & x_{k-1} & 0 & x_{k-3} & \dots & x_2 & 0 & x_0 \\
- & y_k & 0 & y_{k-2} & 0 & \dots & 0 & y_1 & 0 \\
\hline
 & (m-y_k) & x_{k-1} & (m-y_{k-2}) & x_{k-3} & \dots & x_2 & (m-y_1) & x_0 \\
+ & & (n-1) & 0 & (n-1) & \dots & (n-1) & 0 & n \\
\hline\hline
\end{array}$$

which is of the desired form. ■

THEOREM 6.6. *Let $\beta$ be the larger solution of $x^2 = mx+n$, $m,n \in \mathbb{Z}$, $m \geq n > 0$. Then*

$$L_\oplus(\beta) = 2m \quad \text{if } m = n$$

*and*

$$2\left\lfloor \frac{m+1}{m-n+1} \right\rfloor \leq L_\oplus(\beta) \leq 2\left\lceil \frac{m}{m-n+1} \right\rceil \quad \text{if } m > n.$$

*Proof.* Let $x, y \in \mathbb{Z}_\beta$, $xy > 0$. Every $y$ splits as $y = y_{(1)} + \ldots + y_{(s)}$, for some $s$, where the summands $y_{(i)}$ have digits $\leq m-n+1$, and thus satisfy the assumptions of Lemma 6.4. We can always choose $y_{(i)}$ in such a way that the sum has at most

$$s_0 := \left\lceil \frac{m}{m-n+1} \right\rceil$$

non-vanishing summands. Lemma 6.4 then implies that

$$\mathrm{fp}_\beta(x+y) \leq 2s_0.$$

Now let $xy < 0$, without loss of generality $x > -y$. Then, according to Lemma 6.5, $x+y$ can be written either as $z+w$ for some $0 \leq z, w \in \mathbb{Z}_\beta$, or

$$x + y = z + (n-1)\sum_{i=1}^{k} \beta^{2i-1} + n/\beta \quad \text{for } 0 \leq z \in \mathbb{Z}_\beta.$$

The sum $(n-1)\sum_{i=1}^{k} \beta^{2i-1}$ can be written as addition of $\lceil \frac{n-1}{m-n+1} \rceil = s_0 - 1$ summands with digits $\leq m-n+1$. Therefore

$$\mathrm{fp}_\beta\left(z + (n-1)\sum_{i=1}^{k} \beta^{2i-1}\right) \leq 2(s_0 - 1).$$

Adding $n/\beta$ to the result may yield only two more fractional digits (cf. Lemma 6.4).

Thus the proof for the upper bound on $L_\oplus(\beta)$ is finished. The lower bound of $L_\oplus(\beta)$ is given by Corollary 6.3. ■

The last two sections were devoted to the study of arithmetic of $\beta$-expansions for $\beta > 1$ a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. This is the case where Theorem 3.3 does not provide us with any results, since $K = 0$. Let us comment on the results obtained in Sections 5 and 6:

1. The lower and upper bounds for $L_{\oplus}(\beta)$ found in Theorem 6.6 differ at most by 2. They coincide if and only if

$$m - n + 1 \quad \text{divides} \quad m \text{ or } m + 1.$$

Based on observation, we conjecture that for $m > n$ we actually have $L_{\oplus}(\beta) = 2k_0$. We also note that for $m > n$ the results of subtraction $x - y$, where $x, y > 0$, have lower numbers of fractional digits than addition, more precisely, $\mathrm{fp}_{\beta}(x - y) \leq 2k_0 - 1$.

2. According to Theorem 5.4 we may use the bound on $L_{\oplus}(\beta)$ to derive an upper estimate on $L_{\odot}(\beta)$. For example for $m = n$ this gives

$$L_{\odot}(\beta) \leq 8m(\log_2(m + 2)).$$

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Department of Mathematics and Statistics
McGill University
Burnside Hall, 805 Sherbrooke Street West
Montréal (Québec) H3A 2K6, Canada
E-mail: guimond@math.mcgill.ca

Department of Mathematics
Faculty of Nuclear Sciences
and Physical Engineering
Czech Technical University
Trojanova 13
120 00 Praha 2, Czech Republic
E-mail: masakova@km1.fjfi.cvut.cz
pelantova@km1.fjfi.cvut.cz

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