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A Tauberian theorem for the Ingham summation method

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1. Introduction. Many problems in number theory involve estimating the mean values

$$\frac{1}{n}\sum_{m=1}^{n} f(m) \tag{1.1}$$

of some complex valued function $f : \mathbb{N} \to \mathbb{C}$. In many cases $f(m)$ can be naturally represented as a sum $\sum_{k|m} a_k$ where $a_k \in \mathbb{C}$. The Möbius inversion formula guarantees that for a given $f(m)$ such a_k always exist and are unique. Replacing $f(m)$ by $\sum_{k|m} a_k$, we get

$$\sum_{m=1}^{n} f(m) = \sum_{m=1}^{n}\sum_{k|m} a_k = \sum_{k=1}^{n} a_k\left[\frac{n}{k}\right],$$

where $[x]$ denotes the integer part of a real number x. Suppose we want to know under which conditions the sequence of the mean values (1.1) of $f(m)$ has a limit as $n \to \infty$. This is equivalent to the question under which conditions on a_k the sequence

$$\frac{1}{n}\sum_{k=1}^{n} a_k\left[\frac{n}{k}\right] \tag{1.2}$$

has a limit as $n \to \infty$. If, say

$$\sum_{k\ge 1}\frac{|a_k|}{k} < \infty,$$

then the theorem of Wintner (see e.g. [9]) states that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k\left[\frac{n}{k}\right] = \sum_{k=1}^{\infty}\frac{a_k}{k}. \tag{1.3}$$

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It was shown in [4] that the convergence of the series

$$\sum_{k=1}^{\infty} \frac{a_k}{k} \tag{1.4}$$

alone does not necessarily imply the existence of the limit of the sum (1.2) as $n \to \infty$. In 1910 Axer [1] (see also Chapter 3.6 of [9]) proved that if, in addition to convergence of (1.4), the condition

$$\sum_{k=1}^{n} |a_k| = O(n) \tag{1.5}$$

is satisfied, then the limit (1.3) exists.

We will show that in determining whether the sum (1.2) has a limit, an important role is played by the quantity

$$S(x) = \sum_{m \le x} \sum_{k|m} a_k \log k = \sum_{k \le x} a_k \left[\frac{x}{k}\right] \log k. \tag{1.6}$$

We will prove (see Lemma 2.2) that the condition $S(x) = O(x \log x)$ as $x \to \infty$ is enough to ensure that the Dirichlet series $\sum_{m=1}^{\infty} a_m m^{-\sigma}$ converges for all $\sigma > 1$. This means that the function

$$g(\sigma) = \sum_{m=1}^{\infty} \frac{a_m}{m^{\sigma}}$$

is then correctly defined for all $\sigma > 1$. The next theorem shows that if $S(n) = o(n \log n)$ then the value of the sum (1.2) can be approximated by $g(\sigma)$ with $\sigma = 1 + \log^{-1} n$.

Theorem 1.1. *Suppose a_n is a sequence of complex numbers such that*

$$S(n) = \sum_{k \le n} a_k \left[\frac{n}{k}\right] \log k = o(n \log n).$$

Then

$$\frac{1}{n} \sum_{k \le n} a_k \left[\frac{n}{k}\right] = g\left(1 + \frac{1}{\log n}\right) + o(1) \qquad \text{as } n \to \infty.$$

The estimate of the above theorem will allow us to prove necessary and sufficient conditions for the existence of the limit of the sum (1.2).

Theorem 1.2. *Suppose a_m is a fixed sequence of complex numbers. Then the limit*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k \le n} a_k \left[\frac{n}{k}\right] = C$$

exists if and only if the following two conditions are satisfied:

(i) $$\sum_{k\le n} a_k \left[\frac{n}{k}\right] \log k = o(n\log n) \qquad \textit{as } n\to\infty,$$

(ii) $$\lim_{\sigma\searrow 1} \sum_{m=1}^{\infty} \frac{a_m}{m^\sigma} = C.$$

Note that if condition (i) *is satisfied then the infinite series in* (ii) *converges for all* $\sigma > 1$.

The last theorem is a direct analog of the very first Tauberian theorem, proved by Tauber in 1897.

THEOREM A (Tauber, [10]). *A series*

(1.7) $$\sum_{n=0}^{\infty} a_n$$

converges and its sum is equal to A *if and only if*

(1.8) $$\sum_{k=0}^{n} k a_k = o(n)$$

and

$$\lim_{x\nearrow 1} \sum_{n=0}^{\infty} a_n x^n = A.$$

It can be shown that Tauber's condition (1.8) on the coefficients a_j of the formal series (1.7) alone is enough to provide an asymptotic estimate for partial sums

$$\sum_{k=0}^{n} a_k = \phi(e^{-1/n}) + o(1),$$

where $\phi(z) = \sum_{j=0}^{\infty} a_j z^j$. This is similar to the asymptotics given in the formulation of Theorem 1.1.

It is, however, natural to ask whether the stated theorems are really useful for analyzing the mean values of concrete arithmetical functions. The condition $\lim_{x\downarrow 1} g(x) = C$ does not cause any problem if say the Dirichlet series $g(s)$ has a closed form expression which allows us to obtain information on the behavior of $g(x)$ for real $x > 1$ close to 1. At first glance the condition $S(n) = o(n\log n)$ looks quite artificial and not much easier to check than to prove that $A(n) = \sum_{k=1}^{n} a_k[n/k] = Cn + o(n)$, since $S(n)$ is obtained by just replacing a_k by $a_k \log k$ in the expression of $A(n)$. However, this condition is quite natural for a wide class of sequences a_m such that $f(m)$ defined as $f(m) = \sum_{d|m} a_d$ is a completely multiplicative function of m, that is, a function satisfying equation

$$f(mn) = f(m)f(n) \tag{1.9}$$

for any $m, n \in \mathbb{N}$. It is easy to check that if a completely multiplicative function f is bounded, $|f(m)| \le 1$, then the condition $S(n) = o(n \log n)$ will be satisfied if

$$\sum_{p \le n} \frac{|f(p) - 1|}{p} \log p = o(\log n), \tag{1.10}$$

or

$$\sum_{m \le n} \Bigg| \sum_{p \le n/m} f(p) \log p - \frac{n}{m} \Bigg| = o(n \log n); \tag{1.11}$$

here and further we denote by $\sum_p$ and $\prod_p$ sums and products over prime numbers p. This allows us to deduce a few classical results for the mean values of multiplicative functions. For example, it can be shown that if (1.10) or (1.11) is satisfied for a fixed completely multiplicative function f such that $|f(m)| \le 1$, then Theorem 1.1 implies an estimate

$$\frac{1}{n} \sum_{m=1}^{n} f(m) = \prod_p \frac{1 - 1/p^{1+1/\log n}}{1 - f(p)/p^{1+1/\log n}} + o(1) \quad \text{as } n \to \infty.$$

Results with similar or even stronger error terms than in the next theorem can be proven by the method of Halász (see e.g. Chapter 19 of monograph [2] and papers [3], [8], [7] and [6]). We present its proof below just to demonstrate the connection between the Ingham summation method and the mean values of multiplicative functions. Its proof is an easy consequence of the same estimates that enable us to prove Theorem 1.2.

THEOREM 1.3. *Suppose $f(m)$ is a completely multiplicative function such that $|f(m)| \le 1$. Then*

$$\left| \frac{1}{n} \sum_{m=1}^{n} f(m) - \prod_{p \le n} \frac{1 - 1/p}{1 - f(p)/p} \right| \le R(\alpha) \mu_n(\alpha)$$

for any $\alpha > 1$, where $R(\alpha)$ is a positive constant, which depends on α only, and

$$\mu_n(\alpha) = \left(\frac{1}{\log n} \sum_{p \le n} \frac{|f(p) - 1|^\alpha}{p} \log p \right)^{1/\alpha}.$$

A similar result holds for general multiplicative functions, i.e. such that (1.9) is required to be satisfied only for m, n coprime. It follows from our proof of Theorem 1.3 that its modified version for general multiplicative functions holds if we weaken the condition $|f(m)| \le 1$ to $|f(1) + \cdots + f(m)| \le Dm$ for all $m \ge 1$, with some fixed D.

Unfortunately our proof of Theorems 1.1 and 1.2 is not elementary since it relies on the estimate of the number of primes in short intervals (Theorem B) that has originally been proved (see e.g. [5]) using a number of non-trivial facts about the distribution of zeroes of the Riemann zeta function.

The Tauberian theorem we prove can be reformulated in terms of the theory of summation of divergent series. Recall (see [4]) that a formal series $\sum_{m=1}^{\infty} c_m$ is called *summable in the sense of Ingham* if there exists a complex number C such that

$$\lim_{n\to\infty} \sum_{m=1}^{n} \frac{m}{n}\left[\frac{n}{m}\right] c_m = C,$$

in which case we write

$$(I)\sum_{m=1}^{\infty} c_m = C.$$

Suppose $0 < \lambda_1 < \cdots < \lambda_n < \cdots$ is a strictly increasing sequence of positive real numbers. We say that a formal series $\sum_{m=1}^{\infty} c_m$ is (A, λ_n) *summable*, and its value is C, if

$$\lim_{x\downarrow 0} \sum_{m=1}^{\infty} c_m e^{-\lambda_m x} = C,$$

in which case we write

$$(A, \lambda_n)\sum_{m=1}^{\infty} c_m = C.$$

With these notations our Tauberian theorem means that $(I)\sum_{m=1}^{\infty} c_m = C$ if and only if

$$\sum_{m=1}^{n} \frac{m}{n}\left[\frac{n}{m}\right] c_m \log m = o(\log n) \qquad \text{as } n \to \infty$$

and $(A, \log n)\sum_{m=1}^{\infty} c_m = C$.

The analogy between the classical Tauber theorem and the theorem we prove leads us to expect that a wide class of summability methods is connected to some class of (A, λ_n) summability methods in such a way that a formal series $\sum_{m=0}^{\infty} c_m$ is summable if and only if it is (A, λ_n) summable and the partial sums defining the summability method with $\lambda_m c_m$ instead of c_m are $o(\lambda_n)$. We thus prove that the Ingham summability method is connected in this sense with the $(A, \log n)$ method. It was shown in [11] that this pattern also holds for the Cesàro summability methods (C, θ) with $\theta > -1$, which are proved to be connected to the (A, n) method. In the same paper we exploited the connection of the Cesàro summation method with the multiplicative functions on permutations to obtain an analog of

Theorem 1.3 providing the asymptotic estimate of the mean value of a multiplicative function on permutations.

2. Proofs. Let us start by introducing some notation. We denote by $\Psi(x)$ the Chebyshev function

$$\Psi(x) = \sum_{m\le x} \Lambda(m),$$

where $\Lambda(m)$ is the von Mangoldt function. We also denote

$$\Delta(y,x) = \Psi(y) - \Psi(x) - (y-x).$$

We will need an upper bound on $\Delta(x,y)$. Although a much stronger estimate is known (see [5]), we formulate the weakest estimate that we know to be sufficient for our proof of Theorem 2.4.

THEOREM B ([5]). *Suppose $c > 0$ is a fixed constant. There exists a constant η with $0 < \eta < 1$ such that*

$$\Delta(x+h,x) \ll \frac{h}{\log x} \quad \textit{when} \quad h \ge cx^{\eta},$$

for $x \ge 2$, where the constant in $\ll$ depends only on c and η.

For any $t > 0$ we define a positive multiplicative function

$$f_t(m) = \sum_{d|m} \frac{\mu(d)}{d^t} = \prod_{p|m}\left(1 - \frac{1}{p^t}\right) > 0,$$

where $\mu(d)$ is the Möbius function. The Dirichlet generating series of $f_t(m)$ is

$$L_t(s) = \sum_{m=1}^{\infty} \frac{f_t(m)}{m^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{s+t}} = \frac{\zeta(s)}{\zeta(s+t)}, \tag{2.1}$$

where $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ is the Riemann zeta function. We denote the partial sums of $f_t(m)$ by

$$F_t(x) = \sum_{1\le m\le x} f_t(m) \quad \text{for } x \ge 1.$$

We will need estimates of various sums involving $f_t(m)$:

LEMMA 2.1. *For any $x > 1$ and $t > 0$ we have*

$$\sum_{m\le x} \frac{f_t(m)}{m} \ll 1 + t\log x, \tag{2.2}$$

$$F_t(x) = \frac{x}{\zeta(1+t)} + O(x^{1-t}) + O\left(\sum_{d\le x} \frac{1}{d^t}\right), \tag{2.3}$$

$$\sum_{m\le x}\frac{f_t(m)}{m}=\sum_{d\le x}\frac{\mu(d)}{d^{1+t}}\log\frac{x}{d}+O(1), \tag{2.4}$$

$$F_t(x)-F_t\Big(\frac{x}{2}\Big)\ll x\Big(\frac{1}{\log x}+t\Big), \tag{2.5}$$

and for $k\ge 2$ we have

$$\begin{aligned}\int_0^\infty F_t(x)\Big(\frac{1}{k^t}-\frac{1}{(k+1)^t}\Big)\,dt \\ = x\int_0^\infty\Big(\frac{1}{k^t}-\frac{1}{(k+1)^t}\Big)\frac{dt}{\zeta(1+t)}+O\Big(\frac{x}{k\log^2 xk}\Big).\end{aligned} \tag{2.6}$$

Proof. The estimates of the lemma are trivial if $x\le 3$, therefore we assume that $x>3$. Recalling (2.1) we obtain

$$\sum_{m\le x}\frac{f_t(m)}{m}\le e\sum_{m\le x}\frac{f_t(m)}{m^{1+1/\log x}}\le e\frac{\zeta(1+1/\log x)}{\zeta(1+1/\log x+t)}\ll 1+t\log x,$$

since $1/(u-1)<\zeta(u)<u/(u-1)$ for any $u>1$. This proves (2.2).

To prove the next two estimates we replace $f_t(m)$ by $\sum_{d|m}\mu(d)d^{-t}$ to obtain

$$\begin{aligned}F_t(x)&=\sum_{m\le x}f_t(m)=\sum_{m\le x}\sum_{d|m}\frac{\mu(d)}{d^t}=\sum_{d\le x}\frac{\mu(d)}{d^t}\Big[\frac{x}{d}\Big]\\&=\frac{x}{\zeta(1+t)}+O(x^{1-t})+O\Big(\sum_{d\le x}\frac{1}{d^t}\Big);\end{aligned}$$

here we have estimated $\sum_{d>x}\mu(d)d^{-1-t}\ll x^{-t}$ by applying partial summation and the well-known fact that

$$\Big|\sum_{d\le m}\frac{\mu(d)}{d}\Big|\le 1 \tag{2.7}$$

for all $m\ge 1$. This proves the estimate (2.3). In a similar way

$$\begin{aligned}\sum_{m\le x}\frac{f_t(m)}{m}&=\sum_{m\le x}\frac{1}{m}\sum_{d|m}\frac{\mu(d)}{d^t}=\sum_{d\le x}\frac{\mu(d)}{d^{1+t}}\sum_{k\le x/d}\frac{1}{k}\\&=\sum_{d\le x}\frac{\mu(d)}{d^{1+t}}\Big(\log\frac{x}{d}-\gamma+O\Big(\frac{d}{x}\Big)\Big)=\sum_{d\le x}\frac{\mu(d)}{d^{1+t}}\log\frac{x}{d}+O(1),\end{aligned}$$

where we have used (2.7). The estimate (2.4) is proved.

Differentiating the Dirichlet generating series of $f_t(m)$ with respect to s we get

$$\sum_{m=1}^{\infty} \frac{f_t(m)\log m}{m^s} = -\frac{d}{ds}\frac{\zeta(s)}{\zeta(s+t)} = -\frac{\zeta(s)}{\zeta(s+t)}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(s+t)}{\zeta(s+t)}\right)$$
$$= \sum_{k=1}^{\infty}\frac{f_t(k)}{k^s}\sum_{\ell=1}^{\infty}\frac{\Lambda(\ell)}{\ell^s}\left(1-\frac{1}{\ell^t}\right).$$

Equating the coefficients of $1/m^s$ in the above expression and summing over m such that $m \le x$ we get the identity

$$\sum_{m\le x} f_t(m)\log m = \sum_{k\ell\le x} f_t(k)\Lambda(\ell)\left(1-\frac{1}{\ell^t}\right).$$

Therefore

$$F_t(x) - F_t\left(\frac{x}{2}\right) \le \frac{1}{\log\frac{x}{2}}\sum_{m\le x} f_t(m)\log m \le \frac{1}{\log\frac{x}{2}}\sum_{d\ell\le x} f_t(d)\Lambda(\ell)\left(1-\frac{1}{\ell^t}\right)$$
$$\le \frac{1}{\log\frac{x}{2}}\sum_{d\le x} f_t(d)\Psi\left(\frac{x}{d}\right) \ll \frac{x}{\log\frac{x}{2}}\sum_{d\le x}\frac{f_t(d)}{d}$$
$$\ll x\left(\frac{1}{\log x} + t\right)$$

for $x \ge 3$. Here we have used (2.2) and the fact that $\Psi(x) = O(x)$. This proves (2.5).

Applying the identity $f_t(m) = \sum_{d|m}\mu(d)d^{-t}$ we obtain

$$\int_0^\infty F_t(x)\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right)dt = \sum_{m\le x}\int_0^\infty\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right)\left(\sum_{d|m}\frac{\mu(d)}{d^t}\right)dt$$
$$= \sum_{d\le x}\mu(d)\left[\frac{x}{d}\right]\int_0^\infty\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right)\frac{dt}{d^t}$$
$$= x\sum_{d\le x}\frac{\mu(d)}{d}\int_0^\infty\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right)\frac{dt}{d^t} + O\left(\sum_{d\le x}\frac{\log(1+1/k)}{\log^2 dk}\right)$$

for all $x \ge 1$. Using (2.7) and applying partial summation we can estimate the tail of the series in the last expression as

$$\left|\sum_{d>x}\frac{\mu(d)}{d}\int_0^\infty\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right)\frac{dt}{d^t}\right| \le 2\int_0^\infty\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right)\frac{dt}{x^t} \le \frac{2}{k\log^2 kx}.$$

Estimating the sum inside the symbol $O(\ldots)$ in the previous estimate by means of the inequality $\sum_{1\le d\le x} 1/\log^2 dk \ll x/\log^2 xk$ we complete the proof of the estimate (2.6). ∎

LEMMA 2.2. *Suppose the sequence a_k is such that for any $v > 1$,*

$$\lim_{k\to\infty} \frac{|S(k)|}{k^v} = 0. \tag{2.8}$$

Then the series $\sum_{m=1}^{\infty} a_m/m^v$ converges for all $v > 1$.

Proof. The condition (2.8) implies by summation by parts that the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{S(m) - S(m-1)}{m^s} \tag{2.9}$$

converges for all $s > 1$. Recalling the definition (1.6) of $S(m)$ we can write

$$S(m) - S(m-1) = \sum_{k|m} a_k \log k \quad \text{ for } m \ge 1.$$

This means that if we multiply our convergent series (2.9) by an absolutely convergent series $\sum_{m=1}^{\infty} \mu(m)/m^s = 1/\zeta(s)$ then the resulting series

$$\sum_{m=1}^{\infty} \frac{a_m \log m}{m^s}$$

is also convergent for all $s > 1$. This in its turn implies that if we integrate the above series with respect to s, then the resulting series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

is also convergent for all $s > 1$. ■

LEMMA 2.3. *Suppose the sequence a_k is such that*

$$\lim_{k\to\infty} \frac{|S(k)|}{k^v} = 0 \tag{2.10}$$

for any $v > 1$. Then by Lemma 2.2 the function $g(s) = \sum_{m=1}^{\infty} a_m/m^s$ is correctly defined for all $s > 1$ and the identity

$$\begin{aligned} \sum_{m=1}^{n} a_m \left[\frac{n}{m}\right] - ng\left(1 + \frac{1}{\log n}\right) - \frac{S(n)}{\log n} &= \sum_{k=2}^{n-1} S(k) \int_0^{\infty} \left(\frac{F_t\left(\frac{n}{k}\right)}{k^t} - \frac{F_t\left(\frac{n}{k+1}\right)}{(k+1)^t}\right) dt \\ &\quad - n \sum_{k=2}^{\infty} S(k) \int_{\sigma}^{\infty} \left(\frac{1}{k^u} - \frac{1}{(k+1)^u}\right) \frac{du}{\zeta(u)}, \end{aligned} \tag{2.11}$$

holds for all $n \ge 2$. Here we assume that $\sum_{k=2}^{1}(\ldots) = 0$.

Proof. The Möbius inversion formula yields

$$a_m \log m = \sum_{k|m} \mu\left(\frac{m}{k}\right)(S(k) - S(k-1)) \quad \text{when } m \geq 1.$$

Inserting the above expression for a_k into the left-hand side of (2.11), denoting

$$\sigma = 1 + \frac{1}{\log n}$$

and taking into account that $S(1) = S(0) = 0$ we obtain

$$\begin{aligned}\sum_{m=1}^{n} a_m \left[\frac{n}{m}\right] - ng(\sigma) &= \sum_{m=2}^{n} a_m \left[\frac{n}{m}\right] - n \sum_{m=2}^{\infty} \frac{a_m}{m^\sigma} \\ &= \sum_{m=2}^{n} \left[\frac{n}{m}\right] \frac{1}{\log m} \sum_{k|m} \mu\left(\frac{m}{k}\right)(S(k) - S(k-1)) \\ &\quad - n \sum_{m=2}^{\infty} \frac{1}{m^\sigma \log m} \sum_{k|m} \mu\left(\frac{m}{k}\right)(S(k) - S(k-1)).\end{aligned}$$

Changing the order of summation we obtain

$$\begin{aligned}\sum_{m=1}^{n} a_m \left[\frac{n}{m}\right] &- ng(\sigma) \\ &= \sum_{k=2}^{n} (S(k) - S(k-1)) \sum_{m\colon 1\leq m\leq n,\, k|m} \left[\frac{n}{m}\right] \frac{\mu(m/k)}{\log m} \\ &\quad - n \sum_{k=2}^{\infty} (S(k) - S(k-1)) \sum_{m\colon k|m} \frac{\mu(m/k)}{m^\sigma \log m}\end{aligned} \tag{2.12}$$

for $n \geq 2$. Let us show that the condition (2.10) imposed upon $|S(k)|$ guarantees that exchanging of the order of summation is justified. Indeed, Lemma 2.2 guarantees the convergence of the series $\sum_{m=1}^{\infty} a_m/m^\sigma$, which means the existence of the limit

$$\lim_{N\to\infty} \sum_{m=2}^{N} \frac{a_m}{m^\sigma} = \lim_{N\to\infty} \sum_{m=2}^{N} \frac{1}{m^\sigma \log m} \sum_{k|m} \mu\left(\frac{m}{k}\right)(S(k) - S(k-1)).$$

For any finite N we can exchange the order of summation in the expression under the limit sign and fixing an integer $M \geq 3$ we obtain

$$(2.13)\qquad \sum_{m=2}^{\infty}\frac{a_m}{m^\sigma} = \lim_{N\to\infty}\sum_{k=2}^{N}\frac{S(k)-S(k-1)}{k^\sigma}\sum_{\ell\le N/k}\frac{\mu(\ell)}{\ell^\sigma\log k\ell}$$
$$= \sum_{k=2}^{M-1}\frac{S(k)-S(k-1)}{k^\sigma}\sum_{\ell=1}^{\infty}\frac{\mu(\ell)}{\ell^\sigma\log k\ell}$$
$$+ \lim_{N\to\infty}\sum_{k=M}^{N}\frac{S(k)-S(k-1)}{k^\sigma}\sum_{\ell\le N/k}\frac{\mu(\ell)}{\ell^\sigma\log k\ell}$$
$$= \sum_{k=2}^{M-1}\frac{S(k)-S(k-1)}{k^\sigma}\sum_{\ell=1}^{\infty}\frac{\mu(\ell)}{\ell^\sigma\log k\ell} + O\Big(\frac{1}{M^{\sigma-\sigma'}}\Big),$$

where σ' is a fixed number such that $1<\sigma'<\sigma$. Indeed,

$$\sum_{k=M}^{N}(S(k)-S(k-1))\frac{1}{k^\sigma}\sum_{\ell\le N/k}\frac{\mu(\ell)}{\ell^\sigma\log k\ell} = \sum_{k=M}^{N}(S(k)-S(k-1))\alpha_k$$

with $\alpha_k=\frac{1}{k^\sigma}\sum_{\ell\le N/k}\frac{\mu(\ell)}{\ell^\sigma\log k\ell}$, which satisfy $\alpha_k\ll 1/k^\sigma$ and

$$|\alpha_k-\alpha_{k+1}|\ll\frac{1}{k^{\sigma+1}}+\frac{1}{N^\sigma}\Big(\Big[\frac{N}{k}\Big]-\Big[\frac{N}{k+1}\Big]\Big).$$

By the condition of our lemma $S(n)\ll n^{\sigma'}$. By means of summation by parts and applying the above upper bound for $|\alpha_k-\alpha_{k+1}|$, this leads to the estimate

$$\sum_{k=M}^{N}(S(k)-S(k-1))\alpha_k \ll \frac{|S(M-1)|}{M^\sigma}+\frac{|S(N)|}{N^\sigma}$$
$$+\sum_{k=M}^{N-1}|S(k)|\Big(\frac{1}{k^{\sigma+1}}+\frac{1}{N^\sigma}\Big(\Big[\frac{N}{k}\Big]-\Big[\frac{N}{k+1}\Big]\Big)\Big)$$
$$\ll\frac{1}{M^{\sigma-\sigma'}}+\frac{1}{N^{\sigma-\sigma'}},$$

whence the upper limit of the above expression as $N\to\infty$ does not exceed $O(M^{-(\sigma-\sigma')})$. This proves (2.13). Letting $M\to\infty$ in (2.13) we conclude that the change of summation in (2.12) is justified.

Let us express the quantities involving $\mu(d)$ in the identity (2.12) in terms of the function $f_t(m)$:

$$\sum_{\substack{1\le m\le n\\ k|m}}\Big[\frac{n}{m}\Big]\frac{\mu(m/k)}{\log m} = \sum_{1\le d\le n/k}\Big[\frac{n}{kd}\Big]\frac{\mu(d)}{\log kd} = \sum_{1\le m\le n/k}\sum_{d|m}\frac{\mu(d)}{\log kd}$$

$$= \sum_{1\le m\le n/k} \sum_{d|m} \int_0^\infty \frac{\mu(d)}{(dk)^t}\,dt = \sum_{1\le m\le n/k} \int_0^\infty \frac{1}{k^t} \prod_{p|m}\Big(1-\frac{1}{p^t}\Big)\,dt = \int_0^\infty \frac{F_t(n/k)}{k^t}\,dt.$$

In a similar fashion we obtain

$$\sum_{m\,:\,k|m} \frac{\mu(m/k)}{m^\sigma \log m} = \sum_{d=1}^\infty \frac{\mu(d)}{k^\sigma d^\sigma \log kd} = \int_\sigma^\infty \frac{du}{k^u \zeta(u)}.$$

Inserting the above expressions into (2.12) and using summation by parts in the resulting identities we complete the proof of the lemma. ∎

The estimate provided by the following theorem is crucial to obtaining the results stated in the introduction.

THEOREM 2.4. *Suppose the sequence a_k is such that for any $v > 1$,*

$$\lim_{k\to\infty} \frac{|S(k)|}{k^v} = 0. \tag{2.14}$$

Then the function $g(v) = \sum_{m=1}^\infty a_m/m^v$ is correctly defined for all $v > 1$, and for $n \ge 2$ we have

$$\Big|\sum_{m=1}^n a_m \Big[\frac{n}{m}\Big] - ng\Big(1+\frac{1}{\log n}\Big)\Big| \ll \sum_{k=2}^n c_{n,k}|S(k)| + \frac{n}{\log n}\sum_{k=n}^\infty \frac{|S(k)|}{k^{2+1/\log n}\log k}, \tag{2.15}$$

where $c_{n,k}$ are non-negative real constants that satisfy the condition

$$\sum_{k=2}^{n-1} c_{n,k} k (\log k)^\varepsilon \le C(\varepsilon) n (\log n)^{\varepsilon-1} \tag{2.16}$$

for any $0 < \varepsilon \le 1$, where $C(\varepsilon) > 0$ is a constant which depends on ε only. Moreover

$$c_{n,k} = o(n) \qquad \text{as } n \to \infty \tag{2.17}$$

for any fixed k.

Proof. Let us denote

$$R_n = \sum_{m=1}^n a_m \Big[\frac{n}{m}\Big] - ng\Big(1+\frac{1}{\log n}\Big) - \frac{S(n)}{\log n}.$$

We will prove the theorem by estimating the quantities involved in the right hand side of identity (2.11) expressing R_n in terms of quantities involving sums of $f_t(m)$. Throughout the proof we will denote

$$\sigma = 1 + \frac{1}{\log n}.$$

Applying the inequality $\zeta(u) > 1/(u-1)$, which is true for all $u > 1$, we obtain

$$\int_1^\sigma \left(\frac{1}{k^u} - \frac{1}{(k+1)^u}\right) \frac{du}{\zeta(u)} < \int_1^\sigma \frac{u-1}{k^u}(1 - e^{-u\log(1+1/k)})\, du < \frac{\sigma}{k^2} \int_0^{\sigma-1} u\, du = \frac{\sigma}{2k^2 \log^2 n}. \tag{2.18}$$

For $k \geq n$ we have

$$\int_\sigma^\infty \left(\frac{1}{k^u} - \frac{1}{(k+1)^u}\right) \frac{du}{\zeta(u)} \ll \frac{1}{k^{\sigma+1} \log n \log k}. \tag{2.19}$$

Putting $x = n/(k+1)$ in (2.6) we obtain

$$\int_0^\infty F_t\left(\frac{n}{k+1}\right)\left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right) dt = \frac{n}{k+1} \int_0^\infty \left(\frac{1}{k^t} - \frac{1}{(k+1)^t}\right) \frac{dt}{\zeta(1+t)} + O\left(\frac{n}{k^2 \log^2 n}\right).$$

Let us now use the above estimate together with (2.18) and (2.19) to further simplify the expression of R_n:

$$R_n = \sum_{k=2}^{n-1} S(k) \left[\int_0^\infty \frac{F_t\left(\frac{n}{k}\right) - F_t\left(\frac{n}{k+1}\right)}{k^t}\, dt - \frac{n}{k(k+1)} \int_0^\infty \frac{dt}{k^t \zeta(1+t)}\right] + O\left(\frac{n}{\log n} \sum_{k=n}^\infty \frac{|S(k)|}{k^{\sigma+1}\log k} + \frac{n}{\log^2 n} \sum_{k=2}^n \frac{|S(k)|}{k^2}\right).$$

Suppose $\sqrt{n} \leq k \leq n-1$; then $n/k - n/(k+1) = n/(k(k+1)) < 1$. This means that there can be only one natural number, at most, between n/k and $n/(k+1)$. In that case, if there exists m such that $n/k \geq m > n/(k+1)$, then $k \leq n/m$ and $k+1 > n/m$. This means that $[n/m] \geq k > [n/m] - 1$, which implies that $k = [n/m]$. And conversely, for $k = [n/m]$, we have $n/k \geq m > n/(k+1)$. Thus the only natural numbers k in the interval $\sqrt{n} \leq k \leq n-1$ such that the interval $[n/k, n/(k+1))$ contains some natural number m and subsequently $F_t(n/k) - F_t(n/(k+1)) = f_t(m)$ are of the form $k = [n/m]$. This observation allows us to further simplify the estimate of the sum over $k > \sqrt{n}$ in the estimate of R_n and obtain

$$|R_n| \leq \sum_{2 \leq k < \sqrt{n}} |S(k)| \left|\int_0^\infty \frac{F_t\left(\frac{n}{k}\right) - F_t\left(\frac{n}{k+1}\right)}{k^t}\, dt - \frac{n}{k(k+1)} \int_0^\infty \frac{dt}{k^t \zeta(1+t)}\right| + \sum_{2 \leq m \leq \sqrt{n}} \left|S\left(\left[\frac{n}{m}\right]\right)\right| \int_0^\infty \frac{f_t(m)}{[n/m]^t}\, dt + O\left(\frac{n}{\log n} \sum_{k=n}^\infty \frac{|S(k)|}{k^{\sigma+1}\log k} + \frac{n}{\log^2 n} \sum_{k=2}^n \frac{|S(k)|}{k^2}\right).$$

Thus the inequality (2.15) holds if for $k \le \sqrt{n}$ we put

$$(2.20)\quad c_{n,k} = \left| \int_0^\infty \frac{F_t\left(\frac{n}{k}\right) - F_t\left(\frac{n}{k+1}\right)}{k^t}\, dt - \frac{n}{k(k+1)} \int_0^\infty \frac{dt}{k^t \zeta(1+t)} \right| + \frac{n}{k^2 \log^2 n}$$

and for $k > \sqrt{n}$ define

(2.21)

$$c_{n,k} = \begin{cases} \dfrac{n}{k^2 \log^2 n} & \text{if } \sqrt{n} < k \le n-1 \text{ and } k \ne [n/m] \text{ for any } m \le \sqrt{n}, \\ \displaystyle\int_0^\infty \frac{f_t(m)}{[n/m]^t}\, dt + \frac{n}{k^2 \log^2 n} & \\ \qquad\qquad \text{if } \sqrt{n} < k \le n-1 \text{ and } k = [n/m] \text{ for some } m \le \sqrt{n}, & \\ \dfrac{n}{\log n} & \text{if } k = n. \end{cases}$$

Plugging the estimate (2.3) of $F_t(x)$ into our definition of $c_{n,k}$ in (2.20), after some easy calculations we conclude that for fixed k we have $c_{n,k} = o(n)$.

It remains to check that the $c_{n,k}$ satisfy (2.16) for any fixed $0 < \varepsilon \le 1$. We will do this by splitting the sum involving $c_{n,k}$ into three parts,

$$(2.22)\quad \begin{aligned} \sum_{2 \le k \le n-1} c_{n,k} k (\log k)^\varepsilon &= \sum_{k \le n^\alpha} c_{n,k} k (\log k)^\varepsilon + \sum_{n^\alpha < k < \sqrt{n}} c_{n,k} k (\log k)^\varepsilon \\ &\quad + \sum_{\sqrt{n} \le k \le n-1} c_{n,k} k (\log k)^\varepsilon \\ &=: K_1 + K_2 + K_3. \end{aligned}$$

Here and further, $0 < \alpha < 1/2$ will be fixed arbitrarily; depending on it we will later impose additional upper bound conditions.

Estimating K_3 is the easiest. By (2.21) we have

$$(2.23)\quad \begin{aligned} K_3 &= \sum_{\sqrt{n} \le k \le n-1} c_{n,k} k (\log k)^\varepsilon \\ &\ll \sum_{m \le \sqrt{n}} \frac{n}{m} \left(\log \frac{n}{m} \right)^\varepsilon \int_0^\infty \frac{f_t(m)}{[n/m]^t}\, dt + \sum_{\sqrt{n} \le k \le n-1} \frac{n}{k^2 \log^2 n} k (\log k)^\varepsilon \\ &\ll n (\log n)^\varepsilon \int_0^\infty \frac{1}{n^{t/2}} \sum_{m \le \sqrt{n}} \frac{f_t(m)}{m}\, dt + n (\log n)^{\varepsilon - 1} \\ &\ll n (\log n)^\varepsilon \int_0^\infty \frac{1 + t \log n}{n^{t/2}}\, dt + n (\log n)^{\varepsilon-1} \ll n (\log n)^{\varepsilon - 1}. \end{aligned}$$

Here we have used the upper bound for $\sum_{m \le x} f_t(m)/m$ provided by (2.2).

Let us now estimate K_2. We have

$$K_2 = \sum_{n^\alpha<k<\sqrt{n}} c_{n,k} k(\log k)^\varepsilon \tag{2.24}$$
$$\ll (\log n)^\varepsilon \sum_{n^\alpha<k<\sqrt{n}} k \int_0^\infty \frac{F_t\big(\frac{n}{k}\big) - F_t\big(\frac{n}{k+1}\big)}{k^t}\, dt$$
$$+ (\log n)^\varepsilon \sum_{n^\alpha<k<\sqrt{n}} \frac{n}{k} \int_0^\infty \frac{dt}{k^t \zeta(1+t)} + \sum_{n^\alpha<k<\sqrt{n}} \frac{n}{k^2 \log^2 n} k(\log k)^\varepsilon.$$

The second and third sums on the right hand side are clearly $O(n(\log n)^{\varepsilon-1})$. The first sum can be estimated as

$$\sum_{n^\alpha<k<\sqrt{n}} k \int_0^\infty \frac{F_t\big(\frac{n}{k}\big) - F_t\big(\frac{n}{k+1}\big)}{k^t}\, dt$$
$$\leq \int_0^\infty \frac{1}{n^{\alpha t}} \sum_{n^\alpha<k<\sqrt{n}} k\left(F_t\left(\frac{n}{k}\right) - F_t\left(\frac{n}{k+1}\right)\right) dt$$
$$\leq n \int_0^\infty \frac{1}{n^{\alpha t}} \sum_{n^\alpha<k<\sqrt{n}} \frac{k}{n} \sum_{n/(k+1)<m\leq n/k} f_t(m)\, dt$$
$$\leq n \int_0^\infty \frac{1}{n^{\alpha t}} \sum_{n^\alpha<k<\sqrt{n}} \sum_{n/(k+1)<m\leq n/k} \frac{f_t(m)}{m}\, dt$$
$$\leq n \int_0^\infty \frac{1}{n^{\alpha t}} \sum_{m=1}^{n} \frac{f_t(m)}{m}\, dt.$$

We can use the upper bound for $\sum_{m=1}^n f_t(m)/m$ provided in Lemma 2.1 to further estimate

$$\sum_{n^\alpha<k<\sqrt{n}} k \int_0^\infty \frac{F_t\big(\frac{n}{k}\big) - F_t\big(\frac{n}{k+1}\big)}{k^t}\, dt \ll n \int_0^\infty \frac{1 + t\log n}{n^{\alpha t}}\, dt \ll \frac{n}{\log n}.$$

Inserting this estimate into (2.24) we get

$$K_2 = \sum_{n^\alpha<k<\sqrt{n}} c_{n,k} k(\log k)^\varepsilon \ll n(\log n)^{\varepsilon-1}. \tag{2.25}$$

The case of K_1 is more complicated. We will prove that also $K_1 = O(n(\log n)^{\varepsilon-1})$. The reason for considering $k \geq n^\alpha$ separately is that when $k \leq n^\alpha$ the gap between n/k and $n/(k+1)$ is large enough to apply Theorem B to estimate $F_t(n/k) - F_t(n/(k+1))$. We have

$$-\frac{d}{ds}L_t(s) = L_t(s)\biggl(-\frac{\zeta'(s)}{\zeta(s)}\biggr) - \frac{d}{dt}L_t(s)$$

(see (2.1)), which means that

$$f_t(m)\log m = \sum_{dl=m} f_t(d)\Lambda(l) + \frac{d}{dt}f_t(m).$$

Hence for $k \le \sqrt{n}$,

$$\begin{aligned}
F_t\biggl(\frac{n}{k}\biggr) - F_t\biggl(\frac{n}{k+1}\biggr) &= \sum_{n/(k+1)<m\le n/k} f_t(m)\\
&= \frac{1}{\log\frac{n}{k}}\sum_{n/(k+1)<m\le n/k} f_t(m)\log m + \frac{1}{\log\frac{n}{k}}\sum_{n/(k+1)<m\le n/k} f_t(m)\log\frac{n}{km}\\
&= \frac{1}{\log\frac{n}{k}}\sum_{n/(k+1)<m\le n/k} f_t(m)\log m + O\biggl(\frac{1}{k\log n}\biggl(F_t\biggl(\frac{n}{k}\biggr) - F_t\biggl(\frac{n}{k+1}\biggr)\biggr)\biggr)\\
&= \frac{1}{\log\frac{n}{k}}\sum_{m\le n/k} f_t(m)\biggl(\Psi\biggl(\frac{n}{km}\biggr) - \Psi\biggl(\frac{n}{(k+1)m}\biggr)\biggr)\\
&\quad + O\biggl(\frac{1}{\log n}\frac{d}{dt}\biggl(F_t\biggl(\frac{n}{k}\biggr) - F_t\biggl(\frac{n}{k+1}\biggr)\biggr)\biggr) + O\biggl(\frac{n}{k^3\log n}\biggr).
\end{aligned}$$

Plugging this estimate into (2.20) we obtain

$$\begin{aligned}
(2.26)\quad K_1 &= \sum_{2\le k\le n^\alpha} k(\log k)^\varepsilon\\
&\quad\times\biggl|\int_0^\infty \frac{F_t\bigl(\frac{n}{k}\bigr) - F_t\bigl(\frac{n}{k+1}\bigr)}{k^t}\,dt - \frac{n}{k(k+1)}\int_0^\infty \frac{dt}{k^t\zeta(1+t)}\biggr|\\
&\quad + O(n(\log n)^{\varepsilon-1})\\
&\ll \sum_{k\le n^\alpha} k(\log k)^\varepsilon\\
&\quad\times\biggl|\int_0^\infty \frac{1}{\log\frac{n}{k}}\sum_{m\le n/k} f_t(m)\biggl(\Psi\biggl(\frac{n}{km}\biggr) - \Psi\biggl(\frac{n}{(k+1)m}\biggr)\biggr)\frac{dt}{k^t}\\
&\quad - \frac{n}{k(k+1)}\int_0^\infty \frac{dt}{k^t\zeta(1+t)}\biggr|\\
&\quad + \frac{1}{\log n}\sum_{2\le k<\sqrt{n}} k(\log k)^\varepsilon\int_0^\infty \frac{1}{k^t}\frac{d}{dt}\biggl(F_t\biggl(\frac{n}{k}\biggr) - F_t\biggl(\frac{n}{k+1}\biggr)\biggr)dt\\
&\quad + O(n(\log n)^{\varepsilon-1}).
\end{aligned}$$

Note that $df_t(m)/dt > 0$ for all $t > 0$. Therefore, by partial integration,

$$\begin{aligned}
\frac{1}{\log n} & \sum_{2\le k<\sqrt{n}} k(\log k)^{\varepsilon} \int_0^\infty \frac{1}{k^t}\frac{d}{dt}\Big(F_t\Big(\frac{n}{k}\Big) - F_t\Big(\frac{n}{k+1}\Big)\Big)\,dt \\
&\ll \frac{1}{\log n} \sum_{2\le k<\sqrt{n}} k(\log k)^{\varepsilon+1} \int_0^\infty \Big(F_t\Big(\frac{n}{k}\Big) - F_t\Big(\frac{n}{k+1}\Big)\Big)\frac{dt}{k^t} \\
&\ll \frac{1}{\log n} \sum_{1\le s\le \frac{\log n}{2\log 2}} \sum_{2^s\le k<2^{s+1}} k(\log k)^{\varepsilon+1} \int_0^\infty \Big(F_t\Big(\frac{n}{k}\Big) - F_t\Big(\frac{n}{k+1}\Big)\Big)\frac{dt}{k^t} \\
&\ll \frac{1}{\log n} \sum_{1\le s\le \frac{\log n}{2\log 2}} 2^s s^{\varepsilon+1} \int_0^\infty \Big(F_t\Big(\frac{n}{2^s}\Big) - F_t\Big(\frac{n}{2^{s+1}}\Big)\Big)\frac{dt}{2^{st}}.
\end{aligned}$$

Applying now (2.5) to estimate $F_t(n/2^s) - F_t(n/2^{s+1})$ we get

$$\begin{aligned}
\frac{1}{\log n} & \sum_{2\le k<\sqrt{n}} k(\log k)^{\varepsilon} \int_0^\infty \frac{1}{k^t}\frac{d}{dt}\Big(F_t\Big(\frac{n}{k}\Big) - F_t\Big(\frac{n}{k+1}\Big)\Big)\,dt \\
&\ll \frac{n}{\log n} \sum_{1\le s\le \frac{\log n}{2\log 2}} s^{\varepsilon+1} \int_0^\infty \Big(t + \frac{1}{\log n}\Big)\frac{dt}{2^{st}} \ll n(\log n)^{\varepsilon-1}.
\end{aligned}$$

Applying this in (2.26) we obtain

$$\begin{aligned}
K_1 \ll & \sum_{k\le n^\alpha} k(\log k)^{\varepsilon} \Big| \int_0^\infty \frac{1}{\log\frac{n}{k}} \sum_{m\le n/k} f_t(m)\varDelta\Big(\frac{n}{km}, \frac{n}{(k+1)m}\Big)\frac{dt}{k^t}\Big| \\
&+ n \sum_{k\le n^\alpha} \frac{(\log k)^{\varepsilon}}{k+1} \Big| \int_0^\infty \Big(\frac{1}{\zeta(1+t)} - \frac{1}{\log\frac{n}{k}} \sum_{m\le n/k} \frac{f_t(m)}{m}\Big)\frac{dt}{k^t}\Big| \\
&+ O(n(\log n)^{\varepsilon-1}).
\end{aligned}$$

By (2.4),

$$\int_0^\infty \Big(\frac{1}{\zeta(1+t)} - \frac{1}{\log\frac{n}{k}} \sum_{m\le n/k} \frac{f_t(m)}{m}\Big)\frac{dt}{k^t} \ll \frac{1}{\log n\log k}$$

for $k \le \sqrt{n}$. This gives us

$$K_1 = \sum_{2\le k\le n^\alpha} c_{n,k} k(\log k)^{\varepsilon} \ll D + O(n(\log n)^{\varepsilon-1}).$$

Here

$$D=\sum_{k\le n^\alpha} k(\log k)^\varepsilon\left|\int\limits_0^\infty \frac{1}{\log\frac{n}{k}}\sum_{m\le n/k} f_t(m)\Delta\left(\frac{n}{km},\frac{n}{(k+1)m}\right)\frac{dt}{k^t}\right|\le D_1+D_2$$

where

$$D_1=\sum_{k\le n^\alpha} k(\log k)^\varepsilon\left|\int\limits_0^\infty \frac{1}{\log\frac{n}{k}}\sum_{n/k^{1+\delta}\le m\le n/k} f_t(m)\Delta\left(\frac{n}{km},\frac{n}{(k+1)m}\right)\frac{dt}{k^t}\right|,$$

$$D_2=\sum_{k\le n^\alpha} k(\log k)^\varepsilon\int\limits_0^\infty \frac{1}{\log\frac{n}{k}}\sum_{m\le n/k^{1+\delta}} f_t(m)\left|\Delta\left(\frac{n}{km},\frac{n}{(k+1)m}\right)\right|\frac{dt}{k^t},$$

for some fixed $\delta>0$ such that $\alpha(1+\delta)<1$. Then

$$\begin{aligned}D_1&\ll\frac{1}{\log n}\sum_{k\le n^\alpha}k(\log k)^\varepsilon\\&\quad\times\int\limits_0^\infty\sum_{n/k^{1+\delta}\le m\le n/k}f_t(m)\left(\Psi\left(\frac{n}{km}\right)-\Psi\left(\frac{n}{(k+1)m}\right)\right)\frac{dt}{k^t}\\&\quad+\frac{n}{\log n}\sum_{k\le n^\alpha}\frac{(\log k)^\varepsilon}{k}\int\limits_0^\infty\sum_{n/k^{1+\delta}\le m\le n/k}\frac{f_t(m)}{m}\frac{dt}{k^t}=:J_1+J_2.\end{aligned}$$

Changing the order of summation in J_1 we get

$$\begin{aligned}J_1&=\frac{1}{\log n}\sum_{k\le n^\alpha}k(\log k)^\varepsilon\\&\quad\times\int\limits_0^\infty\sum_{n/k^{1+\delta}\le m\le n/k}f_t(m)\left(\Psi\left(\frac{n}{km}\right)-\Psi\left(\frac{n}{(k+1)m}\right)\right)\frac{dt}{k^t}\\&=\frac{1}{\log n}\sum_{n^{1-\alpha(1+\delta)}\le m\le n/2}\int\limits_0^\infty f_t(m)\\&\quad\times\sum_{(n/m)^{1/(1+\delta)}\le k\le n/m}k(\log k)^\varepsilon\left(\Psi\left(\frac{n}{km}\right)-\Psi\left(\frac{n}{(k+1)m}\right)\right)\frac{dt}{k^t}\\&\le\frac{1}{\log n}\sum_{n^{1-\alpha(1+\delta)}\le m\le n/2}\left(\log\frac{n}{m}\right)^\varepsilon\\&\quad\times\int\limits_0^\infty f_t(m)\sum_{k\le n/m}k\left(\Psi\left(\frac{n}{km}\right)-\Psi\left(\frac{n}{(k+1)m}\right)\right)\left(\frac{m}{n}\right)^{\frac{t}{1+\delta}}dt.\end{aligned}$$

Since for any $x \geq 1$,

$$\sum_{k=1}^{\infty} k\left(\Psi\left(\frac{x}{k}\right) - \Psi\left(\frac{x}{k+1}\right)\right) = \sum_{k=1}^{\infty} \Psi\left(\frac{x}{k}\right) \ll x \sum_{k=1}^{[x]} \frac{1}{k} \ll x \log x$$

we get

$$J_1 \ll \frac{n}{\log n} \sum_{n^{1-\alpha(1+\delta)} \leq m \leq n/2} \left(\log \frac{n}{m}\right)^{\varepsilon+1} \int_0^{\infty} \frac{f_t(m)}{m} \left(\frac{m}{n}\right)^{\frac{t}{1+\delta}} dt \tag{2.27}$$

$$\ll \frac{n}{\log n} \sum_{1 \leq s \leq (1-\alpha(1+\delta)) \log_2 n} \sum_{\frac{n}{2^{s+1}} < m \leq \frac{n}{2^s}} \left(\log \frac{n}{m}\right)^{\varepsilon+1} \int_0^{\infty} \frac{f_t(m)}{m} \left(\frac{m}{n}\right)^{\frac{t}{1+\delta}} dt$$

$$\ll \frac{1}{\log n} \sum_{1 \leq s \leq (1-\alpha(1+\delta)) \log_2 n} s^{\varepsilon+1} 2^s \int_0^{\infty} \left(F_t\left(\frac{n}{2^s}\right) - F_t\left(\frac{n}{2^{s+1}}\right)\right) 2^{-\frac{st}{1+\delta}} \, dt$$

$$\ll \frac{n}{\log n} \sum_{1 \leq s \leq (1-\alpha(1+\delta)) \log_2 n} s^{\varepsilon+1} \int_0^{\infty} \left(\frac{1}{\log n} + t\right) 2^{-\frac{st}{1+\delta}} \, dt \ll n (\log n)^{\varepsilon-1}.$$

We estimate J_2 in a similar way. First changing the order of summation we get

$$J_2 = \frac{n}{\log n} \sum_{n^{1-\alpha(1+\delta)} \leq m \leq n/2} \int_0^{\infty} f_t(m) \sum_{(n/m)^{1/(1+\delta)} \leq k \leq n/m} \frac{(\log k)^{\varepsilon}}{k^{1+t}} \, dt$$

$$\leq \frac{n}{\log n} \sum_{n^{1-\alpha(1+\delta)} \leq m \leq n/2} \left(\log \frac{n}{m}\right)^{\varepsilon+1} \int_0^{\infty} \frac{f_t(m)}{m} \left(\frac{m}{n}\right)^{\frac{t}{1+\delta}} dt.$$

The last sum has already been estimated in (2.27), thus we finally get

$$J_2 \ll n (\log n)^{\varepsilon-1}.$$

Our estimates of J_1 and J_2 imply that

$$D_1 \ll n (\log n)^{\varepsilon-1}.$$

Let us now turn to estimating the sum D_2. Let us choose $\delta = 1/(1-\eta)$ where η is as in Theorem B. Then

$$\frac{n}{mk} - \frac{n}{m(k+1)} \geq \left(\frac{2}{3}\right)^{1-\eta} \left(\frac{n}{m(k+1)}\right)^{\eta}$$

for $m \leq n/k^{1+\delta}$. Additionally let us assume that $\alpha > 0$ is small enough to ensure that $\alpha(\delta+1) < 1$. Then we can make use of Theorem B to estimate

$$\Delta\left(\frac{n}{km}, \frac{n}{(k+1)m}\right) \ll \frac{n}{k^2} \frac{1}{\log \frac{n}{km}}.$$

Hence we obtain

$$D_2 \ll \frac{n}{\log n} \sum_{k\le n^\alpha} \frac{(\log k)^\varepsilon}{k} \int_0^\infty \sum_{m\le n/k^{1+\delta}} \frac{f_t(m)}{m\log\frac{n}{mk}}\,\frac{dt}{k^t}.$$

Changing the order of summation we get

$$\begin{aligned} D_2 &\ll \frac{n}{\log n} \sum_{1\le m\le n/2} \Bigl(\log\frac{n}{m}\Bigr)^{\varepsilon-1} \int_0^\infty \frac{f_t(m)}{m} \sum_{2\le k\le (n/m)^{1/(1+\delta)}} \frac{1}{k^{t+1}}\,dt \\ &\ll \frac{n}{\log n} \sum_{1\le s\le \log_2(n/2)} \sum_{n/2^{s+1}<m\le n/2^s} \Bigl(\log\frac{n}{m}\Bigr)^{\varepsilon-1} \\ &\quad\times \int_0^\infty \frac{f_t(m)}{m} \sum_{2\le k\le (n/m)^{1/(1+\delta)}} \frac{1}{k^{t+1}}\,dt \\ &\ll \frac{1}{\log n} \sum_{1\le s\le \log_2(n/2)} s^{\varepsilon-1}2^s \\ &\quad\times \int_0^\infty \Bigl(F_t\Bigl(\frac{n}{2^s}\Bigr) - F_t\Bigl(\frac{n}{2^{s+1}}\Bigr)\Bigr) \sum_{2\le k\le 2^{s+1}} \frac{1}{k^{t+1}}\,dt. \end{aligned}$$

Applying (2.5) with $x = n2^{-s}$ we further estimate

$$\begin{aligned} D_2 &\ll \frac{n}{\log n} \sum_{1\le s\le \log_2(n/2)} s^{\varepsilon-1} \int_0^\infty \Bigl(\frac{1}{\log\frac{n}{2^s}} + t\Bigr) \sum_{2\le k\le 2^{s+1}} \frac{1}{k^{t+1}}\,dt \\ &\ll \frac{n}{\log n} \sum_{1\le s\le \log_2(n/2)} s^{\varepsilon-1} \Bigl(\frac{1}{\log\frac{n}{2^s}} \sum_{2\le k\le 2^{s+1}} \frac{1}{k\log k} + \sum_{2\le k\le 2^{s+1}} \frac{1}{k(\log k)^2}\Bigr) \\ &\ll n(\log n)^{\varepsilon-1}. \end{aligned}$$

Thus we have proved that

$$K_1 = \sum_{k\le n^\alpha} c_{n,k} k(\log k)^\varepsilon \ll D_1 + D_2 + n(\log n)^{\varepsilon-1} \ll n(\log n)^{\varepsilon-1}. \tag{2.28}$$

The estimates (2.28), (2.25) and (2.23) allow us to estimate the sum (2.22) as $O(n(\log n)^{\varepsilon-1})$, which completes the proof of the theorem. ∎

Proof of Theorem 1.1. Plugging $S(n) = o(n\log n)$ into (2.15) and making use of (2.16) and (2.17) we conclude that the right hand side of (2.15) is $o(n)$. Dividing both sides of the resulting inequality by n we complete the proof. ∎

Lemma 2.5. *Suppose a_k is a sequence of complex numbers such that*

$$A(n) = \sum_{k\le n} a_k \Bigl[\frac{n}{k}\Bigr] = Cn + o(n) \qquad \text{as } n\to\infty,$$

with some constant $C \in \mathbb{C}$. Then

$$S(n) = \sum_{k \le n} a_k \left[\frac{n}{k}\right] \log k = o(n \log n) \qquad \text{as } n \to \infty.$$

Proof. The equality $f(m) = \sum_{d|m} a_d$ is equivalent to the identity

$$U(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{m=1}^{\infty} \frac{a_m}{m^s} = \zeta(s)g(s).$$

Therefore

$$\zeta(s)g'(s) = (\zeta(s)g(s))' - \zeta'(s)g(s) = U'(s) - \frac{\zeta'(s)}{\zeta(s)}U(s);$$

this identity corresponds to the equality of the coefficients of m^{-s} of the corresponding Dirichlet series

$$\sum_{k|m} a_k \log k = f(m) \log m - \sum_{k\ell = m} \Lambda(k) f(\ell)$$

for all $m \ge 1$. Summing the above identity over all m such that $m \le n$ and recalling that $f(1) + \cdots + f(k) = A(k)$ we get

$$\begin{aligned} S(n) &= \sum_{k=1}^{n} f(k) \log k - \sum_{k\ell \le n} \Lambda(k) f(\ell) \\ &= \sum_{k=1}^{n} (A(k) - A(k-1)) \log k - \sum_{k \le n} \Lambda(k) A\left(\frac{n}{k}\right) \\ &= A(n) \log n - \sum_{k=1}^{n} A(k) \log\left(1 + \frac{1}{k}\right) - \sum_{k \le n} \Lambda(k) A\left(\frac{n}{k}\right). \end{aligned}$$

By assumption, $A(n) = Cn + o(n)$. Inserting this into the above expression we get

$$\begin{aligned} S(n) &= A(n) \log n - \sum_{k \le n} \Lambda(k) A\left(\frac{n}{k}\right) + O(n) \\ &= Cn \log n - Cn \sum_{k \le n} \frac{\Lambda(k)}{k} + o(n \log n) = o(n \log n), \end{aligned}$$

where we have used the fact that $\sum_{k \le n} \Lambda(k)/k = \log n + O(n)$. ∎

Proof of Theorem 1.2. The sufficiency of the two conditions follows immediately from Theorem 1.1.

The necessity of (i) follows from Lemma 2.5. The necessity of (ii) will follow if we note that the function $g(s)$ can be represented as a fraction

$$g(s) = \frac{\zeta(s)g(s)}{\zeta(s)} = \frac{\sum_{m=1}^{\infty} f(m)/m^s}{\sum_{m=1}^{\infty} 1/m^s},$$

where as before $f(m) = \sum_{d|m} a_d$. By assumption,

$$f(1) + \cdots + f(n) = \sum_{k=1}^{n} a_k [n/k] = Cn + o(n).$$

Thus letting $s \downarrow 1$ we conclude that $\lim_{s\downarrow 1} g(s) = C$. ∎

Proof of Theorem 1.3. The values of f on prime numbers p such that $p \le n$ determine the value of f on any integer m such that $m \le n$. The numbers $f(p)$ with $p > n$ do not influence the value of

$$\frac{1}{n} \sum_{m=1}^{n} f(m),$$

therefore we will assume that $f(p) = 1$ for $p > n$.

We have already noted that if $f(m) = \sum_{d|m} a_d$ then the Dirichlet generating function $U(s)$ of $f(m)$ can be represented as a product

$$U(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \zeta(s)g(s).$$

On the other hand, as $f(m)$ is multiplicative, its generating function can be represented as the Euler product

$$U(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

Comparing the above two expressions we conclude that

$$g(s) = \sum_{m=0}^{\infty} \frac{a_m}{m^s} = \frac{1}{\zeta(s)} \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1} = \prod_p \frac{1 - 1/p^s}{1 - f(p)/p^s}$$
$$= \exp\left\{\sum_p \sum_{k\ge 1} \frac{f(p^k) - 1}{kp^{ks}}\right\}.$$

Differentiating this expression we obtain the differential equation $g'(s) = -g(s) \sum_{m=1}^{\infty} \frac{f(m)-1}{m^s} \Lambda(m)$. Multiplying both sides by $\zeta(s)$ and using the fact that $U(s) = \zeta(s)g(s)$ we obtain

$$\zeta(s)g'(s) = -U(s) \sum_{m=1}^{\infty} \frac{f(m) - 1}{m^s} \Lambda(m),$$

or equivalently

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{k=1}^{\infty} \frac{a_k \log k}{k^s} = \sum_{k=1}^{\infty} \frac{f(k)}{k^s} \sum_{m=1}^{\infty} \frac{f(m) - 1}{m^s} \Lambda(m).$$

Equating the coefficients of d^{-s} on both sides and summing over all d such that $d \le m$ we obtain

$$S(m) = \sum_{d\le m}\sum_{k|d} a_k \log k = \sum_{d\le m}\sum_{k|d}(f(k)-1)\Lambda(k) f\left(\frac{d}{k}\right)$$
$$= \sum_{k\le m}(f(k)-1)\Lambda(k)\sum_{\ell\le m/k} f(\ell).$$

Therefore, recalling that $|f(m)| \le 1$ and $f(p) = 1$ for $p > n$, we can estimate

$$|S(m)| \le \sum_{k\le m}|f(k)-1|\Lambda(k)\Big|\sum_{\ell\le m/k} f(\ell)\Big| = \sum_{k\le m}|f(k)-1|\left[\frac{m}{k}\right]\Lambda(k)$$
$$\ll m\sum_{p\le m}\frac{|f(p)-1|}{p}\log p \ll m(\log m)^{1/\beta}\left(\sum_{p\le n}\frac{|f(p)-1|^\alpha}{p}\log p\right)^{1/\alpha}$$
$$\ll m(\log m)^{1/\beta}(\log n)^{1/\alpha}\mu_n(\alpha)$$

for $m \ge 2$. Here we have applied the Cauchy inequality with parameters $1/\alpha + 1/\beta = 1$. Inserting this estimate into the inequality of Theorem 2.4 with $\varepsilon = 1/\beta$ we get

$$\frac{1}{n}\sum_{m=1}^{n} f(m) = g\left(1+\frac{1}{\log n}\right) + O(\mu_n(\alpha)).$$

An easy calculation yields

$$g\left(1+\frac{1}{\log n}\right) = \prod_{p\le n}\frac{1-1/p^{1+1/\log n}}{1-f(p)/p^{1+1/\log n}} = \prod_{p\le n}\frac{1-1/p}{1-f(p)/p}(1+O(\mu_n(\alpha))).$$

Hence follows the assertion of the theorem. ∎

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