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p-adic valuations of some sums of multinomial coefficients

by

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1. Introduction. Let p be a prime. In 2006 Pan and Sun [PS] obtained various congruences modulo p involving central binomial coefficients and Catalan numbers. Later Sun and Tauraso [ST1, ST2] made some further refinements; for example, they proved that for any $a \in \mathbb{Z}^+ = \{1, 2, \ldots\}$ we have

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2}.$$

Recently the author [S10] managed to determine $\sum_{k=0}^{p^a-1} \binom{2k}{k}/m^k \pmod{p^2}$ for any integer m not divisible by p.

Motivated by the above work, Guo and Zeng [GZ] obtained some congruences involving central q-binomial coefficients and raised several conjectures on p-adic valuations of some sums of binomial coefficients.

Throughout the paper, for a prime p, the *p-adic valuation* (or p-adic order) of an integer m is given by

$$\nu_p(m) = \sup\{a \in \mathbb{Z} : p^a \mid m\},$$

and we define $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$ for any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. For example,

$$\nu_2\left(\frac{2}{3}\right) = \nu_2(2) - \nu_2(3) = 1 \quad \text{and} \quad \nu_3\left(\frac{4}{9}\right) = \nu_3(4) - \nu_3(9) = -2.$$

For an assertion A we adopt the Iverson notation:

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $[m = n]$ coincides with the Kronecker symbol $\delta_{m,n}$.

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The following result implies several conjectures of Guo and Zeng [GZ, Section 5].

THEOREM 1.1. *Let $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Suppose that p is an odd prime dividing $m-4$. Then*

$$\nu_p\bigg(\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k}\bigg) \ge \nu_p(n) \quad \text{and} \quad \nu_p\bigg(\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{\binom{2k}{k}}{m^k}\bigg) \ge \nu_p(n). \tag{1.1}$$

Furthermore,

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k} \equiv \frac{\binom{2n-1}{n-1}}{4^{n-1}} + \delta_{p,3}[3\mid n]\frac{m-4}{3}\binom{2n/3^{\nu_3(n)}-1}{n/3^{\nu_3(n)}-1} \pmod{p^{\nu_p(m-4)}} \tag{1.2}$$

and also

$$\frac{1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{\binom{2k}{k}}{m^k} \equiv \frac{C_{n-1}}{4^{n-1}} \pmod{p^{\nu_p(m-4)-\delta_{p,3}}}, \tag{1.3}$$

where C_k denotes the Catalan number $\frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}$. Thus, for $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}}{m^k} \equiv 1 + \delta_{p,3}\frac{m-4}{3} \equiv \frac{m-1}{3} \pmod{p} \tag{1.4}$$

and also

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\binom{p^a-1}{k}(-1)^k\frac{\binom{2k}{k}}{m^k} \equiv -1 \pmod{p} \quad \text{provided } p \ne 3. \tag{1.5}$$

Now we give various consequences of Theorem 1.1.

COROLLARY 1.1 ([GZ, Conjecture 5.1]). *Let p be a prime divisor of $4m-1$ with $m \in \mathbb{Z}$. Then*

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{2k}{k}m^k\bigg) \ge \nu_p(n) \tag{1.6}$$

for all $n \in \mathbb{Z}^+$.

Proof. As $p \nmid m$, there exists an integer m_* with $m_*m \equiv 1 \pmod{p^{\nu_p(n)}}$ and hence $m_* \equiv 4 \pmod{p}$. By Theorem 1.1, for any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}\binom{2k}{k}m^k \equiv \sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m_*^k} \equiv 0 \pmod{p^{\nu_p(n)}}.$$

This concludes the proof. ∎

COROLLARY 1.2 ([GZ, Conjecture 5.2]). *Let $n = |4m - 1|$ with $m \in \mathbb{Z}$. Then*

$$\sum_{k=0}^{n-1} \binom{2k}{k} m^k \equiv 0 \pmod{n}. \tag{1.7}$$

Proof. By Corollary 1.1, (1.6) holds for any prime p dividing n. So (1.7) is valid. ∎

COROLLARY 1.3 ([GZ, Conjecture 5.4]). *Let $p > 3$ be a prime and $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \left(\frac{1 - (-1)^{(p-1)/2} p}{4} \right)^k \equiv p^a \pmod{p^{a+1}}. \tag{1.8}$$

Proof. Let $m = (1-(-1)^{(p-1)/2}p)/4$. Then $m \in \mathbb{Z}$ and $p \mid 4m-1$. Choose an integer m_* such that $mm_* \equiv 1 \pmod{p^{a+1}}$. Clearly $m_* \equiv 4 \pmod{p}$. Applying Theorem 1.1 we get

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \binom{2k}{k} m^k \equiv \frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}}{m_*^k} \equiv 1 \pmod{p}.$$

So (1.8) holds. ∎

Note that (1.8) in the case $p = 5$ yields

$$\sum_{k=0}^{5^a-1} (-1)^k \binom{2k}{k} \equiv 5^a \pmod{5^{a+1}}, \tag{1.9}$$

which is the second congruence in [GZ, Conjecture 3.5].

COROLLARY 1.4 ([GZ, Conjecture 5.3]). *For $a \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{3^a-1} (-2)^k \binom{2k}{k} \equiv 3^a \pmod{3^{a+1}}, \tag{1.10}$$

$$\sum_{k=0}^{3^a-1} (-5)^k \binom{2k}{k} \equiv -3^a \pmod{3^{a+1}}, \tag{1.11}$$

$$\sum_{k=0}^{7^a-1} (-5)^k \binom{2k}{k} \equiv 7^a \pmod{7^{a+1}}. \tag{1.12}$$

Proof. Choose integers m_1, m_2, m_3 such that

$$m_1 \equiv -\frac{1}{2} \pmod{3^{a+1}}, \quad m_2 \equiv -\frac{1}{5} \pmod{3^{a+1}}, \quad m_3 \equiv -\frac{1}{5} \pmod{7^{a+1}}.$$

Then

$$m_1 \equiv 4 \pmod{3^2}, \quad m_2 \equiv 4 \pmod{3}, \quad m_3 \equiv 4 \pmod{7}.$$

So it suffices to apply (1.4). ∎

Formula (1.4) in the case $p = 3$, together with our computation via `Mathematica`, leads us to raise the following conjecture.

Conjecture 1.1. *Let $m \in \mathbb{Z}$ with $m \equiv 1 \pmod 3$. Then*

$$\nu_3\bigg(\frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k}\bigg) \geq \min\{\nu_3(n), \nu_3(m-1)-1\} \tag{1.13}$$

and

$$\nu_3\bigg(\frac{1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{\binom{2k}{k}}{m^k}\bigg) \geq \min\{\nu_3(n), \nu_3(m-1)\}-1 \tag{1.14}$$

for every $n \in \mathbb{Z}^+$. Furthermore,

$$\frac{1}{3^a}\sum_{k=0}^{3^a-1}\frac{\binom{2k}{k}}{m^k} \equiv \frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}$$

for any integer $a \geq \nu_3(m-1)$, and

$$\frac{1}{3^a}\sum_{k=0}^{3^a-1}\binom{3^a-1}{k}(-1)^k\frac{\binom{2k}{k}}{m^k} \equiv -\frac{m-1}{3} \pmod{3^{\nu_3(m-1)}}$$

for each integer $a > \nu_3(m-1)$. Also,

$$\sum_{k=0}^{3^a-1}\binom{3^a-1}{k}(-1)^k\binom{2k}{k} \equiv -3^{2a-1} \pmod{3^{2a}} \quad \textit{for every } a = 2, 3, \ldots.$$

We remark that Strauss, Shallit and Zagier [SSZ] used a special technique to show that for any $n \in \mathbb{Z}^+$ we have

$$\nu_3\bigg(\sum_{k=0}^{n-1}\binom{2k}{k}\bigg) = 2\nu_3(n) + \nu_3\bigg(\binom{2n}{n}\bigg).$$

For any $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the central binomial coefficient $\binom{2k}{k}$ coincides with the multinomial coefficient $\binom{2k}{k,k}$. In general, the multinomial coefficient

$$\binom{k_1+\cdots+k_n}{k_1,\ldots,k_n} = \frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}$$

in the case $k_1, \ldots, k_n = k \in \mathbb{N}$ gives

$$\binom{nk}{k,\ldots,k} = \frac{(nk)!}{(k!)^n}.$$

Now we pose one more conjecture which involves multinomial coefficients.

Conjecture 1.2. *For any prime p and positive integer n we have*

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\bigg)\ge \nu_p(n) \tag{1.15}$$

and

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\binom{(p-1)k}{k,\ldots,k}\bigg)\ge \nu_p(n). \tag{1.16}$$

Furthermore, $\nu_p(n)$ in (1.15) *can be replaced by $\nu_p\big(n\binom{2n}{n}\big)$ if $p\neq 2$.*

Observe that

$$\frac{(4k)!}{(k!)^4}=\binom{4k}{2k}\binom{2k}{k}^2$$

and hence (1.15) in the case $p=5$ yields the first congruence in [GZ, Conjecture 5.6].

Concerning Conjecture 1.2 we can prove the following result.

Theorem 1.2. *Let p be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k}\equiv pB_{p-1}+(-1)^{p-1}-2p \pmod{p^2}, \tag{1.17}$$

where B_n denotes the nth Bernoulli number. Also, an integer $m>1$ is a prime if and only if

$$\sum_{k=0}^{m-1}\binom{(m-1)k}{k,\ldots,k}\equiv 0 \pmod{m}. \tag{1.18}$$

(ii) *Let $n\in\mathbb{Z}^+$. If $n\not\equiv 1 \pmod p$ or there is a digit greater than 1 in the representation of n in base p, then*

$$\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\equiv 0 \pmod{p}, \tag{1.19}$$

otherwise we have

$$\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\equiv (-1)^{\psi_p(n)-1} \pmod{p}, \tag{1.20}$$

where $\psi_p(n)$ denotes the sum of all the digits in the representation of n in base p.

(iii) (1.15) *holds for all $n\in\mathbb{Z}^+$ if and only if so does* (1.16)*.*

A basic problem in number theory is to characterize primes. However, besides the well-known Wilson theorem, no other simple congruence characterization of primes has been proved before. Thus our characterization of primes via (1.18) is particularly interesting.

It is of interest to know what odd primes p satisfy the congruence

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\dots,k}\equiv 0 \pmod{p^2}\quad \text{(i.e., } pB_{p-1}\equiv 2p-1 \pmod{p^2}).$$

Using `Mathematica` we only found four such primes (they are 3, 11, 107, 4931) among the first 15 000 primes. It seems that all such primes are congruent to 3 modulo 8. From the proof of (1.17) we see that such odd primes are exactly those odd primes p satisfying $(p-2)!\equiv 1 \pmod{p^2}$, which were investigated by P. Saridis [S] who also found the above four primes. (The author thanks Prof. N. J. A. Sloane for informing him about the reference [S].)

In the next section we are going to provide some lemmas. Theorems 1.1 and 1.2 will be proved in Sections 3 and 4 respectively.

2. Some lemmas

LEMMA 2.1 ([ST1, Theorem 2.1]). *For any $n\in\mathbb{Z}^+$ and $d\in\mathbb{Z}$, we have*

$$\sum_{0\le k<n}\binom{2k}{k+d}x^{n-1-k}+[d>0]x^n u_d(x-2) = \sum_{0\le k<n+d}\binom{2n}{k}u_{n+d-k}(x-2), \tag{2.1}$$

where the polynomial sequence $\{u_k(x)\}_{k\ge 0}$ is defined as follows:

$$u_0(x)=0,\quad u_1(x)=1,\quad u_{k+1}(x)=xu_k(x)-u_{k-1}(x)\quad (k=1,2,3,\dots).$$

Let $A,B\in\mathbb{Z}$. The *Lucas sequence* $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ is defined by

$$u_0=0,\quad u_1=1,\quad u_{n+1}=Au_n-Bu_{n-1}\quad (n=1,2,3,\dots).$$

The characteristic equation $x^2-Ax+B=0$ has two roots:

$$\alpha=\frac{A+\sqrt{\Delta}}{2}\quad\text{and}\quad \beta=\frac{A-\sqrt{\Delta}}{2},$$

where $\Delta=A^2-4B$. It is well known that for any $n\in\mathbb{N}$ we have

$$u_n=\sum_{0\le k<n}\alpha^k\beta^{n-1-k}\quad\text{and hence}\quad (\alpha-\beta)u_n=\alpha^n-\beta^n.$$

The reader may consult [S06] for connections between Lucas sequences and quadratic fields.

LEMMA 2.2. *Let $A,B\in\mathbb{Z}$ and let $d\in\mathbb{Z}^+$ be an odd divisor of $\Delta=A^2-4B$. Then, for any $n\in\mathbb{Z}^+$, we have*

$$\frac{u_n(A,B)}{n}\equiv\left(\frac{A}{2}\right)^{n-1}+\begin{cases}(A/2)^{n-3}\Delta/3 \pmod d & \text{if } 3\mid d \text{ and } 3\mid n,\\ 0 \pmod d & \text{otherwise.}\end{cases} \tag{2.2}$$

Proof. When $\Delta = 0$, by induction $u_k(A,B) = k(A/2)^{k-1}$ for all $k \in \mathbb{Z}^+$, and hence the desired result follows.

Now we assume that $\Delta \neq 0$. Then

$$\begin{aligned} u_n(A,B) &= \frac{1}{\sqrt{\Delta}}\left(\left(\frac{A+\sqrt{\Delta}}{2}\right)^n - \left(\frac{A-\sqrt{\Delta}}{2}\right)^n\right) \\ &= \frac{2}{2^n}\sum_{\substack{0\le k\le n\\ 2\nmid k}}\binom{n}{k}A^{n-k}\Delta^{(k-1)/2} \\ &= \frac{1}{2^{n-1}}\sum_{\substack{1\le k\le n\\ 2\nmid k}}\frac{n}{k}\binom{n-1}{k-1}A^{n-k}\Delta^{(k-1)/2} \end{aligned}$$

and hence

$$\frac{u_n(A,B)}{n} - \left(\frac{A}{2}\right)^{n-1} = \sum_{\substack{1<k\le n\\ 2\nmid k}}\binom{n-1}{k-1}\left(\frac{A}{2}\right)^{n-k}\frac{\Delta^{(k-1)/2}}{k2^{k-1}}. \tag{2.3}$$

For $k = 5,7,9,\ldots$, clearly $k < 3^{(k-1)/2}$ and hence $\nu_p(k) \le (k-3)/2$ for any prime divisor p of d, thus $\Delta\Delta^{(k-3)/2}/k \equiv 0 \pmod{d}$. Note also that

$$\begin{aligned} \binom{n-1}{3-1}\left(\frac{A}{2}\right)^{n-3}\frac{\Delta^{(3-1)/2}}{3\cdot 2^{3-1}} &= \frac{(n-1)(n-2)}{2}\left(\frac{A}{2}\right)^{n-3}\frac{\Delta}{3\cdot 4} \\ &\equiv \begin{cases} (A/2)^{n-3}\Delta/3 \pmod{d} & \text{if } 3\mid d \text{ and } 3\mid n, \\ 0 \pmod{d} & \text{otherwise.} \end{cases} \end{aligned}$$

So (2.2) follows from (2.3).

The proof of Lemma 2.2 is now complete. ∎

Lemma 2.3. *If p is a prime, and*

$$a = \sum_{i=0}^{k} a_i p^i \quad \text{and} \quad b = \sum_{i=0}^{k} b_i p^i \qquad (a_i, b_i \in \{0,\ldots,p-1\}),$$

then we have the Lucas congruence

$$\binom{a}{b} \equiv \prod_{i=0}^{k}\binom{a_i}{b_i} \pmod{p}.$$

This lemma is a well-known result due to Lucas (see, e.g., [St, p. 44]).

Lemma 2.4. *Let p be a prime and let $h \in \mathbb{Z}^+$ and $m \in \mathbb{Z}\setminus\{0\}$. Then we have*

$$\min_{1\le k\le n}\nu_p\left(\frac{1}{k}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^l\frac{\binom{hl}{l,\ldots,l}}{m^l}\right) = \min_{1\le k\le n}\nu_p\left(\frac{1}{k}\sum_{l=0}^{k-1}\frac{\binom{hl}{l,\ldots,l}}{m^l}\right) \tag{2.4}$$

for every $n = 1,2,3,\ldots$.

Proof. By a confirmed conjecture of Dyson (cf. [D, Go, Z] or [St, p. 44]), for any $k \in \mathbb{N}$ the constant term of the Laurent polynomial

$$\prod_{\substack{1\le i,j\le h\\ i\ne j}}\left(1-\frac{x_i}{x_j}\right)^k$$

coincides with the multinomial coefficient $\binom{hk}{k,\ldots,k}$.

Let $n \in \mathbb{Z}^+$. Then

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{1}{m^k}&\prod_{\substack{1\le i,j\le h\\ i\ne j}}\left(1-\frac{x_i}{x_j}\right)^k\\ &=\frac{(m^{-1}\prod_{1\le i,j\le h,\,i\ne j}(1-x_i/x_j))^n-1}{m^{-1}\prod_{1\le i,j\le h,\,i\ne j}(1-x_i/x_j)-1}\\ &=\sum_{k=1}^{n}\binom{n}{k}\Bigg(\frac{1}{m}\prod_{\substack{1\le i,j\le h\\ i\ne j}}\left(1-\frac{x_i}{x_j}\right)-1\Bigg)^{k-1}\\ &=\sum_{k=1}^{n}\frac{n}{k}\binom{n-1}{k-1}\sum_{l=0}^{k-1}\binom{k-1}{l}\frac{(-1)^{k-1-l}}{m^l}\prod_{\substack{1\le i,j\le h\\ i\ne j}}\left(1-\frac{x_i}{x_j}\right)^l.\end{aligned}$$

Comparing the constant terms of both sides we get

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{hk}{k,\ldots,k}}{m^k}=\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(-1)^{k-1}}{k}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^l\frac{\binom{hl}{l,\ldots,l}}{m^l}. \tag{2.5}$$

Recall that for any sequences $\{a_n\}_{n\ge0}$ and $\{b_n\}_{n\ge0}$ of complex numbers we have

$$\begin{aligned} & a_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k b_k \quad \text{for all } n=0,1,2,\ldots\\ \Leftrightarrow\quad & b_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k \quad \text{for all } n=0,1,2,\ldots.\end{aligned}$$

(See, e.g., [R, p. 43].) So (2.5) holds for all $n \in \mathbb{Z}^+$ if and only if for each $n \in \mathbb{Z}^+$ we have

$$\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(-1)^{k-1}}{k}\sum_{l=0}^{k-1}\frac{\binom{hl}{l,\ldots,l}}{m^l}=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{\binom{hl}{l,\ldots,l}}{m^l}. \tag{2.6}$$

Since both (2.5) and (2.6) are valid for all $n \in \mathbb{Z}^+$, (2.4) holds for any $n \in \mathbb{Z}^+$. This concludes the proof. ∎

3. Proof of Theorem 1.1. Observe that $p \nmid m$ since $p \mid m-4$ and $p \neq 2$. Applying Lemma 2.1 with $x = m$ and $d = 0$, we get

$$\begin{aligned}\frac{m^{n-1}}{n}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k} &= \frac{1}{n}\sum_{k=0}^{n-1}\binom{2n}{k}u_{n-k}(m-2,1)\\ &= \sum_{k=0}^{n-1}\left(2\binom{2n-1}{k}-\binom{2n}{k}\right)\frac{u_{n-k}(m-2,1)}{n-k}.\end{aligned}$$

Since $m-2 \equiv 2 \pmod{p^{\nu_p(m-4)}}$, we have

$$\sum_{k=0}^{n-1}\left(2\binom{2n-1}{k}-\binom{2n}{k}\right)\left(\frac{m-2}{2}\right)^{n-k-1} \equiv \Sigma \pmod{p^{\nu_p(m-4)}}$$

where

$$\Sigma := \sum_{k=0}^{n-1}\left(2\binom{2n-1}{k}-\binom{2n}{k}\right) = \binom{2n-1}{n-1}.$$

Thus, by Lemma 2.2 and the above,

$$\begin{aligned}&\frac{m^{n-1}}{n}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k} - \binom{2n-1}{n-1}\\ &\equiv \delta_{p,3}\sum_{\substack{k=0\\ 3\mid n-k}}^{n-1}\left(2\binom{2n-1}{k}-\binom{2n}{k}\right)\left(\frac{m-2}{2}\right)^{(n-k)-3}\frac{m(m-4)}{3}\\ &\equiv \delta_{p,3}\frac{m-4}{3}S_n \pmod{p^{\nu_p(m-4)}} \quad (\text{since } m \equiv 4 \pmod{p^{\nu_p(m-4)}}),\end{aligned}$$

where

$$S_n = \sum_{\substack{k=0\\ 3\mid n-k}}^{n-1}\left(2\binom{2n-1}{k}-\binom{2n}{k}\right).$$

In the case $3 \nmid n$, for any $k \in \{0,\ldots,n-1\}$ with $k \equiv n \pmod 3$ we have

$$2\binom{2n-1}{k}-\binom{2n}{k} = \frac{n-k}{n}\binom{2n}{k} \equiv 0 \pmod 3.$$

So $3 \mid S_n$ if $3 \nmid n$.

In the case $3 \mid n$, by Lemma 2.3, for $k \in \mathbb{N}$ we have

$$\binom{2n}{3k} \equiv \binom{2n/3}{k} \pmod 3$$

and

$$\begin{aligned}\binom{2n-1}{3k} &= \frac{(2n-1)(2n-2)}{(2n-3k-1)(2n-3k-2)}\binom{2n-3}{3k} \\ &\equiv \binom{2n-3}{3k} \equiv \binom{2n/3-1}{k} \pmod 3,\end{aligned}$$

thus

$$\begin{aligned}S_n &= \sum_{k=0}^{n/3-1}\left(2\binom{2n-1}{3k} - \binom{2n}{3k}\right) \\ &\equiv -\sum_{k=0}^{n/3-1}\left(\binom{2n/3-1}{k} + \binom{2n/3}{k}\right) \pmod 3\end{aligned}$$

and hence

$$S_n \equiv -2^{2n/3-2} - 2^{2n/3-1} + \frac{1}{2}\binom{2n/3}{n/3} \equiv \frac{1}{2}\binom{2q}{q} = \binom{2q-1}{q-1} \pmod 3$$

with $q = n/3^{\nu_3(n)}$.

Combining the above we get

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k} &\equiv \frac{\binom{2n-1}{n-1} + \delta_{p,3}[3\,|\,n]\frac{m-4}{3}\binom{2q-1}{q-1}}{m^{n-1}} \\ &\equiv \frac{\binom{2n-1}{n-1}}{4^{n-1}} + \delta_{p,3}[3\,|\,n]\frac{m-4}{3}\binom{2q-1}{q-1} \pmod{p^{\nu_p(m-4)}}.\end{aligned}$$

This, together with (2.6) in the case $h = 2$, yields

$$\frac{1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{\binom{2k}{k}}{m^k} = \sigma \pmod{p^{\nu_p(m-4)-\delta_{p,3}}},$$

where

$$\begin{aligned}\sigma &:= \sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(-1)^{k-1}}{4^{k-1}}\binom{2k-1}{k-1} \\ &= -2\sum_{k=0}^{n}\binom{n-1}{n-k}\binom{-1/2}{k} = -2\binom{n-3/2}{n} = \frac{C_{n-1}}{4^{n-1}}\end{aligned}$$

with the help of the Chu–Vandermonde identity (see (5.22) of [GKP, p. 169]).

Clearly, if $n = p^a$ for some $a \in \mathbb{Z}^+$ then

$$\frac{\binom{2n-1}{n-1}}{4^{n-1}} \equiv \binom{2p^a-1}{p^a-1} = \prod_{k=1}^{p^a-1}\left(1+\frac{p^a}{k}\right) \equiv 1 \pmod p$$

and

$$\frac{C_{n-1}}{4^{n-1}} \equiv \frac{1}{p^a}\binom{2p^a-2}{p^a-1} = \frac{1}{2p^a-1}\binom{2p^a-1}{p^a} \equiv -1 \pmod{p}.$$

This concludes our proof of Theorem 1.1. ■

4. Proof of Theorem 1.2

LEMMA 4.1. *Let p be a prime and let $n \in \mathbb{Z}^+$. If all the digits in the representation of n in base p belong to $\{0,1\}$, then*

$$\prod_{j=1}^{p-1}\binom{jn}{n} \equiv (-1)^{\psi_p(n)} \pmod{p}$$

(where $\psi_p(n)$ is defined as in Theorem 1.2), otherwise we have

$$\prod_{j=1}^{p-1}\binom{jn}{n} \equiv 0 \pmod{p}.$$

Proof. Suppose that $n = \sum_{i=0}^{k} a_i p^i$ with $a_0, \ldots, a_k \in \{0, \ldots, p-1\}$.

If $a_0, \ldots, a_k \in \{0,1\}$ then $ja_i \le j < p$ for all $i = 0, \ldots, k$ and $j = 1, \ldots, p-1$, thus

$$\begin{aligned}
\prod_{j=1}^{p-1}\binom{jn}{n} &= \prod_{j=1}^{p-1}\binom{\sum_{i=0}^{k}(ja_i)p^i}{\sum_{i=0}^{k}a_i p^i} \\
&\equiv \prod_{j=1}^{p-1}\prod_{i=0}^{k}\binom{ja_i}{a_i} = \prod_{i=0}^{k}\prod_{j=1}^{p-1}\binom{ja_i}{a_i} \qquad \text{(by Lemma 2.3)} \\
&\equiv ((p-1)!)^{|\{0\le i\le k:\, a_i=1\}|} \equiv (-1)^{\psi_p(n)} \pmod{p} \qquad \text{(by Wilson's theorem).}
\end{aligned}$$

Now assume that $\{a_0, \ldots, a_k\} \not\subseteq \{0,1\}$. We want to show that $p \mid \binom{jn}{n}$ for some $j \in \{1, \ldots, p-1\}$. Set $s = \min\{0 \le i \le k : a_i > 1\}$. As $1 < a_s < p$, we may choose $j \in \{1, \ldots, p-1\}$ such that $ja_s \equiv 1 \pmod{p}$. Thus

$$jn = \sum_{s<i\le k}(ja_i)p^i + (ja_s-1)p^s + p^s + \sum_{0\le t<s}(ja_t)p^t.$$

Write

$$\sum_{s<i\le k}(ja_i)p^i + (ja_s-1)p^s = \sum_{s<i\le k} b_i p^i + bp^{k+1}$$

with $b_i \in \{0, \ldots, p-1\}$ and $b \in \mathbb{N}$. Then, with the help of Lemma 2.3, we have

$$\binom{jn}{n} = \binom{bp^{k+1} + \sum_{s<i\le k} b_i p^i + p^s + \sum_{0\le t<s}(ja_t)p^t}{\sum_{i=0}^k a_i p^i}$$
$$\equiv \prod_{s<i\le k}\binom{b_i}{a_i} \times \binom{1}{a_s} \times \prod_{0\le t<s}\binom{ja_t}{a_t} = 0 \pmod p.$$

Combining the above we obtain the desired result. ∎

Proof of Theorem 1.2. (i) If n is an integer greater than 1, then $(pn-1)! \equiv 0 \pmod p$ and hence

$$\sum_{k=0}^{pn-1}\binom{(pn-1)k}{k,\ldots,k} = \sum_{k=0}^{pn-1}\prod_{j=1}^{pn-1}\binom{jk}{k} = 1 + \sum_{k=1}^{pn-1}\prod_{j=1}^{pn-1}\left(j\binom{jk-1}{k-1}\right)$$
$$= 1 + (pn-1)!\sum_{k=1}^{pn-1}\prod_{j=1}^{pn-1}\binom{jk-1}{k-1} \equiv 1 \pmod p.$$

So (1.18) fails for any composite number $m > 1$.

If $1 < k \le p-1$, then $(p-1)k \ge 2(p-1) \ge p$ and hence

$$\binom{(p-1)k}{k,\ldots,k} = \frac{((p-1)k)!}{(k!)^{p-1}} \equiv 0 \pmod p.$$

Thus

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k} \equiv \sum_{k=0}^{1}\binom{(p-1)k}{k,\ldots,k} = 1 + (p-1)! \equiv 0 \pmod p$$

with the help of Wilson's theorem.

Now we determine $\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k}$ modulo p^2.

In the case $p = 2$, as $B_1 = -1/2$ we have

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k} = 1 + (p-1)! = 2 \equiv 2B_{p-1} + (-1)^{p-1} - 2p \pmod{p^2}.$$

Now let p be an odd prime. If $2 < k \le p-1$, then there exist $j_1, j_2 \in \{1,\ldots,p-1\}$ such that $j_1 k \equiv 1 \pmod p$ and $j_2 k \equiv 2 \pmod p$, hence $\binom{j_1 k}{k} \equiv \binom{j_2 k}{k} \equiv 0 \pmod p$ by Lemma 2.3, and thus

$$\binom{(p-1)k}{k,\ldots,k} = \prod_{j=1}^{p-1}\binom{jk}{k} \equiv 0 \pmod{p^2}.$$

Note also that

$$\binom{(p-1)2}{2,\ldots,2} = \prod_{j=1}^{p-1}\binom{2j}{2} = \prod_{j=1}^{p-1}(j(2j-1)) \equiv p!(p-2)! \equiv -p \pmod{p^2}.$$

Therefore

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k}\equiv\sum_{k=0}^{1}\binom{(p-1)k}{k,\ldots,k}-p\equiv 1+(p-1)!-p \pmod{p^2}$$

and hence we have (1.17) with the help of Glaisher's result $(p-1)!\equiv pB_{p-1}-p \pmod{p^2}$ (cf. [Gl]).

(ii) Write $n=pm+r$ with $m\in\mathbb{N}$ and $r\in\{0,\ldots,p-1\}$. If $m>0$ then

$$\begin{aligned}\sum_{k=0}^{pm-1}\binom{(p-1)k}{k,\ldots,k}&=\sum_{k=0}^{pm-1}\prod_{j=1}^{p-1}\binom{jk}{k}=\sum_{k=0}^{m-1}\sum_{t=0}^{p-1}\prod_{j=1}^{p-1}\binom{pjk+jt}{pk+t}\\&\equiv\sum_{k=0}^{m-1}\sum_{t=0}^{1}\prod_{j=1}^{p-1}\binom{pjk+jt}{pk+t}\quad\text{(by Lemma 4.1)}\\&\equiv\sum_{k=0}^{m-1}\sum_{t=0}^{1}\prod_{j=1}^{p-1}\left(\binom{jt}{t}\binom{jk}{k}\right)\quad\text{(by Lemma 2.3)}\\&\equiv\sum_{k=0}^{m-1}(1+(p-1)!)\prod_{j=1}^{p-1}\binom{jk}{k}\equiv 0 \pmod{p}.\end{aligned}$$

Similarly,

$$\sum_{pm\le k<pm+r}\binom{(p-1)k}{k,\ldots,k}=\sum_{0\le s<r}\prod_{j=1}^{p-1}\binom{j(pm+s)}{pm+s}\equiv S \pmod{p},$$

where

$$S:=\sum_{0\le s<\min\{r,2\}}\prod_{j=1}^{p-1}\left(\binom{js}{s}\binom{jm}{m}\right).$$

Clearly $S=0$ when $r=0$. If $r\ge 2$, then

$$S=(1+(p-1)!)\prod_{j=1}^{p-1}\binom{jm}{m}\equiv 0 \pmod{p}.$$

In the case $r=1$ (i.e., $n\equiv 1 \pmod{p}$), if all the digits in the representation of $n=pm+1$ in base p belong to $\{0,1\}$, then

$$S=\prod_{j=1}^{p-1}\binom{jm}{m}\equiv(-1)^{\psi_p(n)-1} \pmod{p}$$

by Lemma 4.1, otherwise $S\equiv 0 \pmod{p}$ in view of Lemma 4.1. This ends the proof of part (ii).

(iii) This part follows immediately from Lemma 2.4.

By the above we have completed the proof of Theorem 1.2. ∎

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