

Mordell–Weil ranks of families of elliptic curves associated to Pythagorean triples

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1. Introduction. Consider a triple of integers a, b and c that satisfy the Pythagorean equation

$$a^2 + b^2 = c^2.$$

We intend to study the family of elliptic curves $E_{(a,b,c)}$

$$(1.1) \quad y^2 = x(x - a^2)(x - b^2)$$

parametrized by such triples. The family (1.1) is similar to another family of curves,

$$(1.2) \quad y^2 = x(x - a^2)(x + b^2)$$

with $a^2 + b^2 = c^2$, which is a special case of the well-known Frey family. For low conductors there are many curves of high Mordell–Weil rank (up to rank 6) in the family (1.1). This is, however, usually not the case for the family (1.2), since generically it is of rank 0.

The family (1.1) is equivalent to the family of curves in Legendre form

$$y^2 = x(x - 1)(x - \lambda)$$

with the parameter λ limited to rational numbers of the form

$$\lambda = \left(\frac{2t}{t^2 - 1} \right)^2,$$

for t rational, not equal to 0 or ± 1 . The Mordell–Weil rank of the family (1.1) was considered for the first time in [5], where it was proven that the group $E_{(a,b,c)}(\mathbb{Q})$ of rational points contains a point (c^2, abc) of infinite order.

In order to state our results, we need some extra notation. Consider the set

$$\mathcal{T} = \{(a, b, c) \in \mathbb{Z}^3 : a^2 + b^2 = c^2, ab \neq 0\}$$

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of triples of integers that satisfy the Pythagorean equation, and consider a smooth curve in the family (1.1). For any triple $(a, b, c) \in \mathcal{T}$, the rank of the Mordell–Weil group of rational points on $y^2 = x(x - a^2)(x - b^2)$ is at least one by [5, Lemma 6.8]. We now define an infinite subset \mathcal{S} of \mathcal{T} . A triple (a, b, c) belongs to \mathcal{S} if and only if its coordinates can be written in the form

$$a = P^2 - Q^2, \quad b = 2PQ, \quad c = P^2 + Q^2,$$

where

$$\frac{P}{Q} = \frac{2pq}{p^2 + 5q^2} \quad \text{for some } p, q \in \mathbb{Z}.$$

Our first main result is the following statement.

THEOREM 1.1. *For infinitely many $(a, b, c) \in \mathcal{S}$ the curve*

$$y^2 = x(x - a^2)(x - b^2)$$

has Mordell–Weil group of rank at least two. There are two linearly independent points

$$Q_1 = \left(\frac{1}{2}(a + b - c)^2, \frac{1}{2}(a + b)(a + b - c)^2 \right),$$

$$Q_2 = \left(\frac{1}{2}a(a - c), \frac{1}{2}ab \frac{1}{k^2}(p^4 - 25q^4) \right),$$

where $k = \text{GCD}(2pq, p^2 + 5q^2)$ and p and q are as above.

REMARK 1.2. The set \mathcal{S} splits into countably many nonempty subsets $\{\mathcal{C}_i\}_{i=1}^{\infty}$ that correspond to isomorphism classes of curves over \mathbb{Q} (cf. Proposition 2.1). For all but finitely many i the rank of the Mordell–Weil group of \mathbb{Q} -rational points of the curve $y^2 = x(x - a^2)(x - b^2)$, where $a^2 + b^2 = c^2$ and $(a, b, c) \in \mathcal{C}_i$, is at least two.

Concerning the generic rank of (1.1) we have the following result.

THEOREM 1.3. *The group of $\mathbb{Q}(t)$ -rational points on the curve*

$$(1.3) \quad y^2 = x(x - 1) \left(x - \left(\frac{2t}{t^2 - 1} \right)^2 \right)$$

is of rank one.

We prove Theorem 1.3 by applying the Shioda–Tate formula. In fact, a stronger result holds. The rank of the group of $\overline{\mathbb{Q}}(t)$ -rational points of the curve from Theorem 1.1 is equal to 2 but only a subgroup of rank one is defined over $\mathbb{Q}(t)$. A similar investigation of the generic rank of (1.2) shows that the rank of the Mordell–Weil group of the corresponding model over $\overline{\mathbb{Q}}(t)$ is equal to 0.

The result in Theorem 1.1 displays the generic rank but the corresponding geometric result is more involved.

THEOREM 1.4. *Let*

$$E : y^2 = x \left(x - \left(\left(\frac{2t}{t^2 + 5} \right)^2 - 1 \right)^2 \right) \left(x - 4 \left(\frac{2t}{t^2 + 5} \right)^2 \right)$$

be the elliptic curve over $\overline{\mathbb{Q}}(t)$ which is obtained from (1.3) by a suitable change of the parameter t and a linear change of coordinates (cf. (6.1)). The geometric Mordell–Weil group $E(\overline{\mathbb{Q}}(t))$ is isomorphic to $\mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. We put $u = \frac{2t}{t^2+5}$. The free part of the group $E(\overline{\mathbb{Q}}(t))$ is generated by the points

$$\begin{aligned} P_1 &= (2(1 + \sqrt{2})(-1 + u)^2 u, 2\sqrt{-1}(1 + \sqrt{2})(-1 + (\sqrt{2} - u)^2)(-1 + u)^2 u), \\ P_2 &= (2(u - 1)^2, 2(-1 + u)^2(-1 + 2u + u^2)), \\ P_3 &= \left(1 - u^2, \frac{(-5 + t^2)u(-1 + u^2)}{5 + t^2} \right). \end{aligned}$$

The torsion subgroup of $E(\overline{\mathbb{Q}}(t))$ is generated by the points

$$\begin{aligned} T_1 &= (-4u^2, 0), \\ T_2 &= (2(-u + u^3), 2\sqrt{-1}(u^2 - 1)u(-1 - 2u + u^2)). \end{aligned}$$

Moreover the group of $\mathbb{Q}(t)$ -rational points on E is generated by the points P_2, P_3, T_1 and $2T_2$.

The proof of Theorem 1.4 requires more involved methods. Note that the geometric approach of Shioda [13] only implies that the rank is at most 6. We base the proof of Theorem 1.4 on the approach of van Luijk [19] and Kloosterman [7].

To the best of our knowledge, the method of van Luijk and Kloosterman was used in the past exclusively for rational or $K3$ surfaces (cf. [12], [17], [18] and [4]). If an elliptic surface is of high geometric genus, then the method described below becomes very ineffective and it is computationally difficult to determine the zeta function of the surface. In our case, we perform calculations on elliptic surfaces which are rational or $K3$. In particular, we attach to the elliptic curve E over $\overline{\mathbb{Q}}(t)$ an elliptic surface over \mathbb{P}^1 (cf. [2]). We find its integral model S as a scheme over A , where A is a discrete valuation ring of a number field with a residue field isomorphic to \mathbb{F}_q . If the scheme $S \rightarrow A$ is smooth of relative dimension 2, we obtain an elliptic surface $\tilde{S} = S_{\overline{\mathbb{F}}_q}$ over the field $\overline{\mathbb{F}}_q$. The action of the Frobenius automorphism on the second ℓ -adic cohomology group $H_{\text{et}}^2(\tilde{S}, \mathbb{Q}_\ell)$, where $\ell \neq q$, gives rise to the characteristic polynomial of the automorphism. The computation of the characteristic polynomial involves counting of \mathbb{F}_{q^r} -rational points on the surface $S_{\overline{\mathbb{F}}_q}$ up to some r . The Lefschetz fixed point formula allows us to compute the traces and the characteristic polynomials of the Frobenii. We apply [19, Proposition 6.2] to estimate the number of eigenvalues of the form $q\zeta$, for a root of unity ζ , which gives a sharp upper bound on the rank of the Néron–Severi

group $\text{NS}(S_{\overline{\mathbb{F}}_q})$. To conclude the computations, we apply the Shioda–Tate formula to obtain the rank of the group $E(\overline{\mathbb{Q}}(t))$. The rank of $E(\mathbb{Q}(t))$ is equal to $\text{rank } E(\overline{\mathbb{Q}}(t)) - 1$, because only one generator of the free part of $E(\overline{\mathbb{Q}}(t))$ is not defined over $\mathbb{Q}(t)$.

2. Notation and preliminaries. Let S be the set of Pythagorean triples,

$$(2.1) \quad S = \{(a, b, c) \in \mathbb{Z}^3 : a^2 + b^2 = c^2\}.$$

For each $s = (a, b, c) \in S$ we consider a curve over \mathbb{Q} ,

$$(2.2) \quad E_s : y^2 = x(x - a^2)(x - b^2).$$

When $ab \neq 0$ the equation defines a nonsingular curve of genus one, hence an elliptic curve. The discriminant of the equation E_s and its j -invariant are

$$(2.3) \quad \Delta(s) = \Delta(a, b, c) = 16(a - b)^2(a + b)^2b^4a^4,$$

$$(2.4) \quad j(s) = j(a, b, c) = 256 \frac{(a^4 - a^2b^2 + b^4)^3}{b^4a^4(a - b)^2(a + b)^2}.$$

Observe that $\{s \in S : \Delta(s) \neq 0\} = \mathcal{T}$.

Let us now introduce a notion of equivalence of $s_1, s_2 \in \mathcal{T}$. We call two such triples equivalent if the smooth curves E_{s_1} and E_{s_2} are equivalent via a linear change of coordinates defined over \mathbb{Q} , transforming one Weierstrass equation into the other, i.e. we assume that the curves are \mathbb{Q} -isomorphic. We will then write $s_1 \sim s_2$. It is easy to check that $(a, b, c) \sim (A, B, C)$ if and only if there exists $u \in \mathbb{Q}^\times$ such that either $(a, b, c) = (\pm uA, \pm uB, \pm uC)$ or $(a, b, c) = (\pm uB, \pm uA, \pm uC)$. The relation \sim is an equivalence relation. Hence, if s_1 and s_2 do not lie in the same equivalence class, the associated elliptic curves E_{s_1} and E_{s_2} are non-isomorphic over \mathbb{Q} .

For any $s = (a, b, c) \in \mathcal{T}$ we introduce a new parameter

$$t = t(s) = \frac{b}{c - a}.$$

It is well-defined because a triple with $a = c$ cannot lie in \mathcal{T} . We have the following equalities:

$$(2.5) \quad \frac{t^2 - 1}{t^2 + 1} = \frac{a}{c},$$

$$(2.6) \quad \frac{2t}{t^2 + 1} = \frac{b}{c}.$$

Define an elliptic curve over $\mathbb{Q}(t)$:

$$(2.7) \quad E_t : y^2 = x(x - (t^2 - 1)^2)(x - 4t^2).$$

The linear change of variables

$$x \mapsto x \frac{4}{(a - c)^2}, \quad y \mapsto y \frac{8}{(c - a)^3}$$

defines a \mathbb{Q} -isomorphism between the elliptic curves $E_{(a,b,c)}$ and $E_{b/(c-a)}$ for any $(a, b, c) \in \mathcal{T}$.

The discriminant and j -invariant of E_t are

$$\Delta(t) = 256t^4(-1 + t^2)^4(1 - 6t^2 + t^4)^2,$$

$$j(t) = \frac{16(1 - 8t^2 + 30t^4 - 8t^6 + t^8)^3}{t^4(-1 + t^2)^4(1 - 6t^2 + t^4)^2}.$$

The set $P = \{t \in \mathbb{Q} : \Delta(t) \neq 0\} = \mathbb{Q} \setminus \{0, \pm 1\}$ consists of all parameters for which E_t is nonsingular. If $t \in P$, then E_t is \mathbb{Q} -isomorphic to a curve in Legendre form

$$y^2 = x(x - 1)\left(x - \left(\frac{2t}{t^2 - 1}\right)^2\right).$$

It is easy to check that E_t and $E_{t'}$ are \mathbb{Q} -isomorphic if and only if

$$t' \in \left\{t, -t, \frac{1}{t}, -\frac{1}{t}, \frac{1+t}{1-t}, \frac{1-t}{1+t}, -\frac{1-t}{1+t}, -\frac{1+t}{1-t}\right\}.$$

The rational functions in t , appearing above, form a group under composition. It is the dihedral group of eight elements generated by $f(t) = -t$ and $g(t) = \frac{1+t}{1-t}$. For $t, t' \in P$ we define $t \sim t'$ if and only if the curves E_t and $E_{t'}$ are \mathbb{Q} -isomorphic. Then each equivalence class contains exactly eight different elements.

Consequently, we obtain the following

PROPOSITION 2.1. *There is a bijection of sets of equivalence classes*

$$(2.8) \quad \mathcal{T}/\sim \rightarrow P/\sim$$

given by $(a, b, c) \mapsto b/(c - a)$ on representatives. The inverse is given by

$$p/q \mapsto (p^2 - q^2, 2pq, p^2 + q^2).$$

It follows from Proposition 2.1 that elements in \mathcal{S}/\sim map bijectively to elements in the set $\{u \in P : u = \frac{2t}{5+t^2} \text{ for some } t \in \mathbb{Q}\}/\sim$. The latter set is infinite, hence so is the former.

3. Elliptic surfaces and Picard numbers. We start this section by recalling the necessary theorems and definitions relating to elliptic surfaces. Then we compute the Picard numbers of several elliptic surfaces and deduce the generic rank of the Mordell–Weil group of elliptic curves related to family (1.1).

DEFINITION 3.1. Let k be an algebraically closed field. Let C be a smooth, irreducible, projective curve over k . An *elliptic surface over C* is a smooth, irreducible, projective surface S over k together with a relatively minimal elliptic fibration $f : S \rightarrow C$ with a singular fiber and a zero section.

To an elliptic curve E over the function field $k(C)$ of the curve C we can associate an elliptic surface $f : \mathcal{E} \rightarrow C$ with generic fiber E . It follows from the work of Kodaira and Néron that f always exists and is unique.

Below we define three different elliptic surfaces, starting from three distinguished elliptic curves over the function field of \mathbb{P}^1 over $\overline{\mathbb{Q}}$. Let $\mathcal{E}_1 \rightarrow \mathbb{P}^1$ be the elliptic surface over \mathbb{P}^1 associated to

$$y^2 = x(x - (t - 1)^2)(x - 4t).$$

Let $\mathcal{E}_2 \rightarrow \mathbb{P}^1$ be the elliptic surface over \mathbb{P}^1 associated to

$$y^2 = x(x - (t^2 - 1)^2)(x - 4t^2).$$

Finally, let $\mathcal{E}_3 \rightarrow \mathbb{P}^1$ be the elliptic surface over \mathbb{P}^1 associated to

$$y^2 = x(x - (u^2 - 1)^2)(x - 4u^2), \quad u = \frac{2t}{5 + t^2}.$$

For any smooth, projective, geometrically integral variety V over a field K we denote by $\text{NS}(V_{\overline{K}})$ the Néron–Severi group, i.e. the group of divisors on V modulo algebraic equivalence.

THEOREM 3.2 ([13, Corollary 2.2]). *Let $S \rightarrow C$ be an elliptic surface. The Néron–Severi group $\text{NS}(S)$ is finitely generated and torsion-free.*

DEFINITION 3.3. Let $S \rightarrow C$ be an elliptic surface. The *Picard number* $\rho(S)$ of the surface S is the rank of the Néron–Severi group $\text{NS}(S)$.

We recall the classical Shioda–Tate formula.

THEOREM 3.4 ([13, Corollary 5.3]). *Let $S \rightarrow C$ be an elliptic surface. Let $\Sigma \subset C$ be the set of points under singular fibers. For each $v \in \Sigma$ let m_v denote the number of components of the singular fiber above v . Let E denote the generic fiber of S , and K the function field of C . Then*

$$\rho(S) = 2 + \sum_{v \in \Sigma} (m_v - 1) + \text{rank}(E(K)).$$

LEMMA 3.5. *Let E be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Let $\Sigma \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be the set of points of bad reduction of E . Let F_v denote the fiber at $v \in \Sigma$. Denote by $G(F_v)$ the group generated by all simple components of F_v . Then there exists an injective homomorphism*

$$\phi : E(\overline{\mathbb{Q}}(t))_{\text{tors}} \rightarrow \prod_{v \in \Sigma} G(F_v).$$

If F_v is of multiplicative type I_n in Kodaira notation (cf. [16, Theorem IV.8.2]), the group $G(F_v)$ is $\mathbb{Z}/n\mathbb{Z}$. If F_v is of additive type I_{2n}^ , then $G(F_v)$ is $(\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. Let ϕ be the map taking a section in $E(\overline{\mathbb{Q}}(t))$ to the respective fiber component of F_v that it meets. The map ϕ is a group homomorphism

by [16, Corollary IV.9.2]. It is injective on the torsion subgroup $E(\overline{\mathbb{Q}}(t))_{\text{tors}}$ by [14, Corollary 7.5]. ■

A multiplicative fiber of type I_n has exactly n components. An additive fiber of type I_{2n}^* has $5 + 2n$ components.

We gather the information about surfaces in Tables 1–3. We apply the Shioda–Tate formula to the surfaces $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{E}_3 .

LEMMA 3.6.

- (1) \mathcal{E}_1 has Kodaira dimension $-\infty$.
- (2) \mathcal{E}_2 has Kodaira dimension 0.
- (3) \mathcal{E}_3 has Kodaira dimension 1.

Proof. The Euler number $e(S)$ of an elliptic surface $S \rightarrow C$ (over the base field of characteristic different from 2 and 3) equals

$$e(S) = \sum_{v \in \Sigma} e(F_v),$$

where Σ is the set of points over which there are singular fibers. The local Euler number $e(F_v)$ is equal to the number of components m_v if the fiber has multiplicative reduction, or to $m_v + 1$ if the reduction is additive (cf. [3, Proposition 5.1.6]). An easy computation with the Tate algorithm (cf. Tables 1–3) shows that $e(\mathcal{E}_1) = 12$, $e(\mathcal{E}_2) = 24$ and $e(\mathcal{E}_3) = 48$. From [2, Corollary V.12.3] it follows that the Kodaira dimensions are: $\kappa(\mathcal{E}_1) = -\infty$, $\kappa(\mathcal{E}_2) = 0$ and $\kappa(\mathcal{E}_3) = 1$, respectively. ■

Table 1. Singular fibers, $E_1 : y^2 = x(x - (t - 1)^2)(x - 4t)$

Place	Type of singular fiber	Automorphism group
$t = 1$	I_4	$\mathbb{Z}/4\mathbb{Z}$
$t = 0$	I_2	$\mathbb{Z}/2\mathbb{Z}$
roots of $1 - 6t + t^2 = 0$	I_2	$\mathbb{Z}/2\mathbb{Z}$
$t = \infty$	I_2	$\mathbb{Z}/2\mathbb{Z}$

Table 2. Singular fibers, $E_2 : y^2 = x(x - (t^2 - 1)^2)(x - 4t^2)$

Place	Type of singular fiber	Automorphism group
$t = 1$	I_4	$\mathbb{Z}/4\mathbb{Z}$
$t = 0$	I_4	$\mathbb{Z}/4\mathbb{Z}$
$t = -1$	I_4	$\mathbb{Z}/4\mathbb{Z}$
roots of $-1 - 2t + t^2 = 0$	I_2	$\mathbb{Z}/2\mathbb{Z}$
roots of $-1 + 2t + t^2 = 0$	I_2	$\mathbb{Z}/2\mathbb{Z}$
$t = \infty$	I_4	$\mathbb{Z}/4\mathbb{Z}$

Table 3. Singular fibers, $E_3 : y^2 = x(x - (u^2 - 1)^2)(x - 4u^2)$, $u = \frac{2t}{5+t^2}$

Place	Type of singular fiber	Automorphism group
$t = 0$	I_4	$\mathbb{Z}/4\mathbb{Z}$
roots of $5 + t^2 = 0$	I_4	$\mathbb{Z}/4\mathbb{Z}$
roots of $5 - 2t + t^2 = 0$	I_4	$\mathbb{Z}/4\mathbb{Z}$
roots of $5 + 2t + t^2 = 0$	I_4	$\mathbb{Z}/4\mathbb{Z}$
roots of $(5 - 2t + t^2)^2 = 8t^2$	I_2	$\mathbb{Z}/2\mathbb{Z}$
roots of $(5 + 2t + t^2)^2 = 8t^2$	I_2	$\mathbb{Z}/2\mathbb{Z}$
$t = \infty$	I_4	$\mathbb{Z}/4\mathbb{Z}$

The information gathered in Tables 1–3 allows us to prove the following results.

LEMMA 3.7. *The generic fiber of \mathcal{E}_1 has rank 1 over $\overline{\mathbb{Q}}(t)$.*

Proof. The surface \mathcal{E}_1 is rational by Lemma 3.6, hence $\rho(\mathcal{E}_1) = 10$. A section of \mathcal{E}_1 corresponds to a point on the generic fiber

$$(3.1) \quad P = (-4t, 4\sqrt{-2}t(t + 1)).$$

As $2P$ and $4P$ are not zero, the point is non-torsion by Lemma 3.5. The group $E_1(\overline{\mathbb{Q}}(t))$ is at least of rank 1. By applying the Shioda–Tate formula we get

$$\begin{aligned} \rho(\mathcal{E}_1) - 2 - \sum_{v \in R} (m_v - 1) &= \text{rank}(E_1(\overline{\mathbb{Q}}(t))), \\ \rho(\mathcal{E}_1) - 2 - \sum_{v \in R} (m_v - 1) &= 10 - 2 - (4 - 1 + 4(2 - 1)) = 10 - 2 - 7 = 1. \end{aligned}$$

Hence the rank equals 1. ■

LEMMA 3.8. *The generic fiber of \mathcal{E}_2 has rank 2 over $\overline{\mathbb{Q}}(t)$.*

Proof. The surface is K3 so $\rho(\mathcal{E}_2) \leq 20$. The points

$$\begin{aligned} P &= (-4t^2, 4\sqrt{-2}t^2(t^2 + 1)), \\ Q &= (2(t - 1)^2, 2(-1 + t)^2(-1 + 2t + t^2)) \end{aligned}$$

are of infinite order. We compute the height pairing $\langle P, P \rangle = 2$, $\langle Q, Q \rangle = 1$ and $\langle P, Q \rangle = 0$. Application of the Shioda–Tate formula shows that

$$20 \geq \rho(\mathcal{E}_2) = 2 + (4(4 - 1) + 4(2 - 1)) + \text{rank}(E_2(\overline{\mathbb{Q}}(t))) = 18 + \text{rank}(E_2(\overline{\mathbb{Q}}(t))).$$

Hence the rank is equal to two. ■

An application of the Shioda–Tate formula allows us to conclude that the Mordell–Weil group of $\overline{\mathbb{Q}}(t)$ -rational points of the generic fiber of \mathcal{E}_3 has

rank at most 6. More precisely, $\rho(\mathcal{E}_3) \leq 40$ (since $\chi(\mathcal{O}_{\mathcal{E}_3}) = 4$) and

$$2 + \sum_{v \in R} (m_v - 1) = 2 + 8(4 - 1) + 8(2 - 1) = 2 + 24 + 8 = 34.$$

There are only three sections of infinite order which are linearly independent.

Let X be any scheme over a finite field \mathbb{F}_q of characteristic p . Let $\ell \neq p$ be a prime. Let us consider the étale ℓ -adic cohomology groups

$$H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = \varprojlim H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

and the groups with Tate twist

$$H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)(1) = H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}_\ell} (\varprojlim \mu_{\ell^n}),$$

where $\mu_{\ell^n} \subset \overline{\mathbb{F}}_q$ is the group of ℓ^n th roots of unity. For simplicity we denote them by $H^i(X, \mathbb{Q}_\ell)$ and $H^i(X, \mathbb{Q}_\ell)(1)$.

THEOREM 3.9 ([19, Proposition 6.2]). *Let A be a discrete valuation ring of a number field L with residue field $k \cong \mathbb{F}_q$. Let S be an integral scheme with a morphism $S \rightarrow \text{Spec } A$ that is projective and smooth of relative dimension 2. Assume that the surfaces $\overline{S} = S_{\overline{L}}$ and $\tilde{S} = S_{\overline{k}}$ are integral. Let $l \nmid q$ be a prime number. Then there are natural injective homomorphisms*

$$(3.2) \quad \text{NS}(\overline{S}) \otimes \mathbb{Q}_l \hookrightarrow \text{NS}(\tilde{S}) \otimes \mathbb{Q}_l \hookrightarrow H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_l)(1)$$

of finite-dimensional inner product spaces over \mathbb{Q}_l . The first injection is induced by the natural injection $\text{NS}(\overline{S}) \otimes \mathbb{Q} \hookrightarrow \text{NS}(\tilde{S}) \otimes \mathbb{Q}$. The second injection respects the Galois action of $G(\overline{k}/k)$.

For any prime p , any positive integer r and a variety X over \mathbb{F}_{p^r} , we denote by $F_X : X \rightarrow X$ the absolute Frobenius morphism which acts as the identity on points and as $f \mapsto f^p$ on the structure sheaf. Let $\Phi_X = (F_X)^r$ and $\overline{X} = X_{\overline{\mathbb{F}}_{p^r}}$ and denote by $\Phi_X \times 1$ the morphism which acts on $X \times \text{Spec } \overline{\mathbb{F}}_{p^r}$. This induces an automorphism Φ_X^* of $H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_l)$.

THEOREM 3.10 ([19, Corollary 2.3]). *With notation as in the previous theorem, the ranks of $\text{NS}(\overline{S})$ and $\text{NS}(\tilde{S})$ are bounded from above by the number of eigenvalues of the linear map $\Phi_{\tilde{S}}^*$ for which λ/q is a root of unity, counted with multiplicity.*

In order to use the above theorem effectively, we recall the Lefschetz trace formula (cf. [9, VI, Theorem 12.3]).

THEOREM 3.11. *Let X be a smooth projective variety over \mathbb{F}_q of dimension n . For any prime $l \nmid q$ and any integer m , we have*

$$\#X(\mathbb{F}_{q^m}) = \sum_{i=0}^n (-1)^i \text{Tr}((\Phi_X^*)^m \mid H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)).$$

We explain the use of the Lefschetz trace formula in numerical computations of the characteristic polynomial of the Frobenius automorphism, which we apply in the proofs of Lemmas 4.6 and 5.3 below. We proceed with $X = S_{\mathbb{F}_q}$, an elliptic surface fibered over \mathbb{P}^1 . Note that $\dim H^1(X, \mathbb{Q}_l) = \dim H^3(X, \mathbb{Q}_l)$ by [6, Corollary 2A10] and $\dim H^1(X, \mathbb{Q}_l) = 0$ by [3, Corollary 5.2.2]. The automorphism Φ_X^* acts on $H^4(X, \mathbb{Q}_l) \cong \mathbb{Q}_l$ by multiplication by q^2 . By the Lefschetz trace formula we obtain

$$\text{Tr}((\Phi_X^*)^m | H^2(X, \mathbb{Q}_l)) = \#X(\mathbb{F}_{q^m}) - 1 - q^{2m}.$$

Let V be the linear subspace of $H^2(X, \mathbb{Q}_l)$ generated by the components of singular fibers and sections. Let $W = H^2(X, \mathbb{Q}_l)/V$. By the multiplicativity of the characteristic polynomial $\text{char}(\Phi_X^*)$ we have

$$\text{char}(\Phi_X^*) = \text{char}(\Phi_X^*|V) \cdot \text{char}(\Phi_{X,W}^*)$$

where the operator $\Phi_{X,W}^* : W \rightarrow W$ is induced by Φ_X^* . Moreover,

$$\text{Tr}((\Phi_X^*)^m) = \text{Tr}((\Phi_X^*|V)^m) + \text{Tr}((\Phi_{X,W}^*)^m)$$

for any m . For T a linear operator acting on a finite-dimensional vector space U , the characteristic polynomial $p(x) = \det(I \cdot x - T)$ can be computed if the traces $t_n = \text{Tr}(T^n)$ are known for $0 \leq n \leq \dim U$. In order to do that, we expand

$$(3.3) \quad p(x) = \frac{x^{\dim U}}{\exp(\sum_{r=1}^{\infty} t_r \frac{x^{-r}}{r})}$$

as a series in $\frac{1}{x}$ and truncate the series to the polynomial part. In numerical computations below, we put $T = \Phi_X^*$.

4. Twisted elliptic surfaces. Now we prove that the rank of the Mordell–Weil group over $\overline{\mathbb{Q}}(t)$ of the elliptic curve

$$y^2 = x(x - (u^2 - 1)^2)(x - 4u^2)$$

for $u = \frac{2t}{5+t^2}$ is equal to 3. For this purpose the notion of a twist of an elliptic curve is needed. In this section we use Kloosterman’s approach [7]. We assume that the base curve C of the elliptic fibration $\mathcal{E} \rightarrow C$ is defined over a field of characteristic not equal to 2 or 3.

Let C be a smooth curve over $k = \overline{k}$, and let $k(C)$ be the function field of C . Let E be an elliptic curve over $k(C)$ given by the Weierstrass equation

$$(4.1) \quad E : y^2 = x^3 + Ax + B$$

for $A, B \in k(C)$. Fix $u \in k(C)^*$. We consider the quadratic twist

$$E^{(u)} : uy^2 = x^3 + Ax + B$$

of the curve E by the element u .

PROPOSITION 4.1. *With the above notation,*

$$\text{rank } E(k(C)) + \text{rank } E^{(u)}(k(C)) = \text{rank } E(k(C)(\sqrt{u})).$$

Proof. See [15, Exercise 10.16]. ■

DEFINITION 4.1. Let $f : \mathcal{E} \rightarrow C$ be an elliptic surface. Fix $P, Q \in C(k)$. Let $E/k(C)$ be the generic fiber of f given by the Weierstrass equation (4.1). An elliptic surface $g : \mathcal{E}' \rightarrow C$ is the *twist* of f by points P and Q if the generic fiber of g is isomorphic over $k(C)$ to $E^{(u)}$, where $u \in k(C)^*$ and

$$\text{ord}_P(u) \equiv 1 \pmod{2}, \quad \text{ord}_Q(u) \equiv 1 \pmod{2}.$$

In addition, we require that for all $R \neq P, Q$,

$$\text{ord}_R(u) \equiv 0 \pmod{2}.$$

REMARK 4.2. For any pair of points P and Q we form the divisor $(P) - (Q) \in \text{Div}^0(C)$. The group of k -rational points of the Jacobian $\text{Jac}(C)$ of the curve C equals $\text{Pic}^0(C)$. Since k is algebraically closed, the group $\text{Pic}^0(C)$ is 2-divisible. We find that $(P) - (Q) = 2D + \text{div}(f)$ for a function $f \in k(C)^*$ and $D \in \text{Div}^0(C)$. We put $u := f$. Let u' be another function such that $\text{div}(u') \equiv (P) - (Q) \pmod{2\text{Div}^0(C)}$. The twists $E^{(u)}$ and $E^{(u')}$ may not be isomorphic over $k(C)$. We have $\text{div}(u/u') = 2T$ for some divisor $T \in \text{Div}^0(C)$. If the genus g of C is greater than 0, then $T \in \text{Jac}(C)(k)$ is a 2-torsion point. There are 2^{2g} distinct torsion points in $\text{Jac}(C)(k)$, hence there are 2^{2g} distinct twists by points P, Q , up to a $k(C)$ -isomorphism. However, for $C = \mathbb{P}^1$, a pair of points P, Q determines a twist uniquely.

LEMMA 4.3. *Let $f : \mathcal{E} \rightarrow C$ be an elliptic surface and let $f^{(P,Q)} : \mathcal{E}^{(P,Q)} \rightarrow C$ be the twist by $P, Q \in C(k)$. Then there exists a double cover $\phi : C' \rightarrow C$ ramified at P and Q such that the relatively minimal nonsingular models of $\mathcal{E} \times_C C' \rightarrow C'$ and $\mathcal{E}^{(P,Q)} \times_C C' \rightarrow C'$ are isomorphic as fibered surfaces.*

Proof. Let E denote the generic fiber of the elliptic fibration f . Let u be the function in $k(C)$ satisfying the conditions of Definition 4.1 for the points P and Q in $C(k)$. The generic fiber of $f^{(P,Q)}$ is the twist $E^{(u)}$ of the curve E .

There exists a projective curve C' and a surjective morphism $\phi : C' \rightarrow C$ such that $u \circ \phi = v^2$ for some $v \in k(C')$. We denote by $e_\phi(R)$ the ramification index of the morphism ϕ at the point R in the fiber above $\phi(R) \in C(k)$. By definition, the function u has $\text{div } u = (P) + (Q) + 2D$ for some $D \in \text{Div}(C)$. Hence

$$\begin{aligned} (4.2) \quad \text{div}(u \circ \phi) &= \phi^*(\text{div } u) \\ &= \sum_{R \in \phi^{-1}(P)} e_\phi(R)(R) + \sum_{R' \in \phi^{-1}(Q)} e_\phi(R')(R') + \phi^* D \\ &= 2 \text{div } v, \end{aligned}$$

where $\phi^* : \text{Div}(C) \rightarrow \text{Div}(C')$ denotes the induced map. The extension $k(C')$ of $k(C)$ is of degree 2, so

$$(4.3) \quad 2 = \deg \phi = \sum_{R \in \phi^{-1}(P)} e_\phi(R) = \sum_{R' \in \phi^{-1}(Q)} e_\phi(R').$$

Identities (4.2) and (4.3) imply that ϕ is ramified at P and Q and the preimages $\phi^{-1}(P)$ and $\phi^{-1}(Q)$ are singletons.

Let $S_1 = \mathcal{E} \times_C C'$ and $S_2 = \mathcal{E}^{(P,Q)} \times_C C'$ denote the surfaces obtained from \mathcal{E} and $\mathcal{E}^{(P,Q)}$ by the base change $\phi : C' \rightarrow C$. The morphisms f and $f^{(P,Q)}$ are projective, hence $S_1 \rightarrow C'$ and $S_2 \rightarrow C'$ are projective. As the base field k is algebraically closed, all but finitely many fibers of $S_1 \rightarrow C'$ and $S_2 \rightarrow C'$ are nonsingular elliptic. Let \tilde{S}_1 denote a relatively minimal nonsingular model of S_1 , respecting the fibration over C' . Similarly, let \tilde{S}_2 denote the relatively minimal nonsingular model of S_2 . By a linear change of coordinates, the generic fibers E and $E^{(u)}$ are isomorphic over $k(C')$. This implies that there is a birational map $\psi : \tilde{S}_1 \dashrightarrow \tilde{S}_2$. Each such map is a composition of smooth blow-ups and blow-downs. The desingularizations \tilde{S}_1 and \tilde{S}_2 are isomorphic outside the singular fibers. The surfaces \tilde{S}_1 and \tilde{S}_2 are relatively minimal with respect to C' , so the fibers do not contain (-1) -curves. Hence, the map ψ is a trivial composition, hence extends to an isomorphism. ■

We introduce the following elliptic surfaces which will be used in the computation of ranks of the families associated to Pythagorean triples (cf. proofs of Theorems 1.1, 1.4).

DEFINITION 4.4. (1) Let $\mathcal{E}_1 \rightarrow \mathbb{P}^1$ be the elliptic surface with the generic fiber

$$E_1 : y^2 = x(x - (t - 1)^2)(x - 4t).$$

(2) We denote by $\mathcal{E}'_1 = \mathcal{E}_1^{(1/5, \infty)} \rightarrow \mathbb{P}^1$ the twist of \mathcal{E}_1 by the points $1/5$ and ∞ , which has the generic fiber

$$E'_1 : -(-1 + 5t)y^2 = x(x - (t - 1)^2)(x - 4t).$$

(3) We denote by $\mathcal{E}''_1 = (\mathcal{E}'_1)^{(0, \infty)} \rightarrow \mathbb{P}^1$ the twist of \mathcal{E}'_1 by the points 0 and ∞ , which has the generic fiber

$$E''_1 : -t(-1 + 5t)y^2 = x(x - (t - 1)^2)(x - 4t).$$

(4) Let $\mathcal{E}_2 \rightarrow \mathbb{P}^1$ be the elliptic surface with the generic fiber

$$E_2 : y^2 = x(x - (t^2 - 1)^2)(x - 4t^2).$$

(5) We denote by $\mathcal{E}'_2 = \mathcal{E}_2^{(-1/\sqrt{5}, 1/\sqrt{5})} \rightarrow \mathbb{P}^1$ the twist of \mathcal{E}_2 by the points $-1/\sqrt{5}$ and $1/\sqrt{5}$, which has the generic fiber

$$E'_2 : -(-1 + 5t^2)y^2 = x(x - (t^2 - 1)^2)(x - 4t^2).$$

(6) Let $\mathcal{E}_3 \rightarrow \mathbb{P}^1$ be the elliptic surface which has the generic fiber

$$E_3 : y^2 = x \left(x - \left(\left(\frac{2t}{5+t^2} \right)^2 - 1 \right)^2 \right) \left(x - 4 \left(\frac{2t}{5+t^2} \right)^2 \right).$$

Proposition 4.1 implies the following statement.

COROLLARY 4.5.

$$\begin{aligned} \text{rank } E_3(\overline{\mathbb{Q}}(t)) &= \text{rank } E_2(\overline{\mathbb{Q}}(t)) + \text{rank } E'_2(\overline{\mathbb{Q}}(t)), \\ \text{rank } E'_2(\overline{\mathbb{Q}}(t)) &= \text{rank } E'_1(\overline{\mathbb{Q}}(t)) + \text{rank } E''_1(\overline{\mathbb{Q}}(t)). \end{aligned}$$

LEMMA 4.6. *The rank of $E'_1(\overline{\mathbb{Q}}(t))$ is equal to 0.*

Proof. First we perform the Tate algorithm to compute the types of singular fibers on our elliptic surface $\mathcal{E}'_1 \rightarrow \mathbb{P}^1$ associated with the curve E'_1 .

A computation in MAGMA reveals that we have one fiber over the point $t = 1$ of multiplicative type I_4 , split over \mathbb{Q} . One singular fiber lies above $t = 0$ and is nonsplit multiplicative of type I_2 , but the equations are defined over \mathbb{Q} . We have a fiber over $t = 1/5$, additive of type I_0^* and again by the Tate algorithm and MAGMA the defining equations of the fiber have coefficients in \mathbb{Q} . The singular fiber over $t = \infty$ is additive of type I_2^* given by equations with coefficients in \mathbb{Q} . Finally, we have two singular fibers of nonsplit multiplicative type I_2 above $t = 3 + 2\sqrt{2}$ and $t = 3 - 2\sqrt{2}$.

The equations of the fibers are defined over $\mathbb{Q}(\sqrt{2})$ by the Tate algorithm. However, the surface \mathcal{E}'_1 is defined over \mathbb{Q} , since we have started with the Weierstrass equation of the elliptic curve E'_1 defined over \mathbb{Q} and the singular locus defines an ideal where the generators have \mathbb{Q} -coefficients. In fact, \mathcal{E}'_1 is defined over \mathbb{Z} . We check that the elliptic surface associated with E'_1 over \mathbb{F}_{17} has the types of singular fibers above the reductions of the points $t = 1, 0, \infty, 3 \pm 2\sqrt{2}$ the same as in characteristic zero.

Put $\mathfrak{p} = (17) \in \text{Spec } \mathbb{Z}$ and $A = \mathbb{Z}_{(\mathfrak{p})}$. The surface \mathcal{E}'_1 defines an integral scheme $S \rightarrow \text{Spec } A$ that is projective and smooth of relative dimension 2. The smoothness comes from the fact that we have good reduction at 17. The residue field $k = A/\mathfrak{p}$ is equal to \mathbb{F}_{17} . Hence, the special fiber of $S \rightarrow \text{Spec } A$ is a surface defined over \mathbb{F}_{17} . It determines the elliptic surface $\tilde{S} = S_{\mathbb{F}_{17}} \rightarrow \mathbb{P}^1$ which is the reduction of our elliptic surface $\mathcal{E}'_1 \rightarrow \mathbb{P}^1$.

By Theorem 3.9 the rank of the Néron–Severi group of \mathcal{E}'_1 is bounded from above by the rank of the Néron–Severi group of \tilde{S} . The components of the singular fibers and the zero section generate a rank 18 subgroup in $\text{NS}(\tilde{S})$. The Euler number $e(\mathcal{E}'_1)$ equals 24, as follows by an argument based on the proof of Lemma 3.6. Hence, the surface \mathcal{E}'_1 is $K3$.

Good reduction at prime 17 implies that \tilde{S} is also a $K3$ surface, so the subspace $\text{NS}(\tilde{S}) \otimes \mathbb{Q}_\ell \hookrightarrow H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_\ell)(1)$ is at most of dimension 22,

because $\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_\ell)(1) = \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^2(\tilde{S}, \mathbb{Q}_\ell) = 22$ by [11, Theorem 4, Part III].

On the subspace V generated by the components of the singular fibers and by the zero section the Frobenius automorphism $\Phi_{\tilde{S}}^*$ acts by multiplication by 17. This follows from the analysis of the singular fibers, i.e. by the Tate algorithm. The characteristic polynomial of the Frobenius automorphism $\Phi_{\tilde{S}}^*$ splits as

$$\text{char}(\Phi_{\tilde{S}}^*) = \text{char}(\Phi_{\tilde{S}}^* | V) \cdot \text{char}(\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*).$$

Further, $\text{char}(\Phi_{\tilde{S}}^* | V) = \det(\text{Id} \cdot x - \Phi_{\tilde{S}}^* | V) = (x - 17)^{18}$. For any natural m ,

$$\text{Tr}((\Phi_{\tilde{S}}^*)^m) = \text{Tr}((\Phi_{\tilde{S}}^* | V)^m) + \text{Tr}((\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)^m).$$

But $\text{Tr}((\Phi_{\tilde{S}}^* | V)^m) = 18 \cdot 17^m$ and $\text{Tr}((\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)^m) = \#\tilde{S}(\mathbb{F}_{17^m}) - 1 - 17^{2m}$ by the Lefschetz trace formula (Theorem 3.11). Combining those facts we obtain

$$\text{Tr}((\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)^m) = \#\tilde{S}(\mathbb{F}_{17^m}) - 1 - 17^{2m} - 18 \cdot 17^m.$$

The characteristic polynomial $\text{char}(\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)$ is of the form $x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$. We present explicit formulas for c_m in terms of $t_m = \text{Tr}((\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*)^m)$ (cf. (3.3)):

$$\begin{aligned} c_1 &= -t_1, \\ c_2 &= \frac{1}{2}(t_1^2 - t_2), \\ c_3 &= \frac{1}{6}(-t_1^3 + 3t_1t_2 - 2t_3), \\ c_4 &= \frac{1}{24}(t_1^4 - 6t_1^2t_2 + 3t_2^2 + 8t_1t_3 - 6t_4). \end{aligned}$$

We compute the number of \mathbb{F}_{17^m} -rational points on \tilde{S} up to $m = 4$:

m	1	2	3	4
$\#\tilde{S}(\mathbb{F}_{17^m})$	604	88312	24227740	6977057176

We obtain the characteristic polynomial

$$\text{char}(\Phi_{\tilde{S}, H_{\text{ét}}^2/V}^*) = x^4 - 8x^3 + 238x^2 - 2312x + 83521.$$

Suppose a root of this polynomial is $x = 17\zeta$ for a root of unity ζ . Then

$$4913(17\zeta^4 - 8\zeta^3 + 14\zeta^2 - 8\zeta + 17) = 0.$$

But ζ is an algebraic integer and the polynomial $17x^4 - 8x^3 + 14x^2 - 8x + 17$ is irreducible over \mathbb{Q} , hence its roots are not algebraic integers, which leads to a contradiction.

Hence, the characteristic polynomial

$$\text{char}(\Phi_{\tilde{S}}^*) = (x - 17)^{18}(x^4 - 8x^3 + 238x^2 - 2312x + 83521)$$

has only 18 roots of the shape 17 times a root of unity. By Theorem 3.10 the rank of $\text{NS}(\tilde{S})$ is at most 18. Then by Theorem 3.9 the rank of $\text{NS}(\mathcal{E}'_1)$ is 18, since we have an explicit rank 18 subgroup generated by the singular fiber components and the zero section. By the Shioda–Tate formula the rank of $E'_1(\overline{\mathbb{Q}}(t))$ equals zero. ■

5. Computing ranks by reductions. Let p be a prime of good reduction for an elliptic surface $\mathcal{E} \rightarrow C$ defined over a number field K . Let S be an integral model of \mathcal{E} over \mathcal{O}_K with special fiber defined over \mathbb{F}_{p^r} . We know by Theorem 3.9 that

$$\text{NS}(S_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \hookrightarrow \text{NS}(S_{\mathbb{F}_{p^r}}) \otimes \mathbb{Q}.$$

Assume for a moment this the map is an isomorphism. Then by classical lattice theory, the determinants of the Gram matrices of the intersection pairings on $\text{NS}(S_{\overline{\mathbb{Q}}})$ and $\text{NS}(S_{\mathbb{F}_{p^r}})$ differ by a square. In the following, we denote the determinant of the Gram matrix of a lattice Λ by $\Delta(\Lambda)$.

We will compute discriminants modulo squares using the Tate conjecture and the Artin–Tate conjecture for $K3$ surfaces, which we recall for the reader’s convenience.

THEOREM 5.1. *Let Y be a $K3$ surface over \mathbb{F}_q . Let Φ_Y^* be the Frobenius automorphism acting on the cohomology group $H^2(Y, \mathbb{Q}_l)$, $l \nmid q$. Then the number of roots of the characteristic polynomial of Φ_Y^* of the form $q\zeta$, where ζ is a root of unity, is equal to the Picard number $\rho(Y) = \text{rank NS}(Y_{\mathbb{F}_q})$.*

THEOREM 5.2. *Let Y be a $K3$ surface over \mathbb{F}_q . Let Φ_Y^* be the Frobenius automorphism and $P(T) = \det(1 - T\Phi_Y^* \mid H^2(Y, \mathbb{Q}_l))$. Then*

$$\lim_{s \rightarrow 1} \frac{P(q^{-s})}{(1 - q^{1-s})^{\rho'(Y)}} = \frac{(-1)^{\rho'(Y)-1} \#\text{Br}(Y) \Delta(\text{NS}(Y_{\mathbb{F}_q}))}{q^{\alpha(Y)} (\#\text{NS}(Y_{\mathbb{F}_q})_{\text{tor}})^2},$$

where $\alpha(Y) = \chi(Y, \mathcal{O}_Y) - 1 + \dim \text{Pic}^0(Y)$ and $\text{Br}(Y)$ is the Brauer group of Y . Moreover $\rho'(Y) = \text{rank NS}(Y_{\mathbb{F}_q})$. The group $\text{NS}(Y_{\mathbb{F}_q})$ is the subgroup of the Néron–Severi group $\text{NS}(Y_{\overline{\mathbb{F}_q}})$ generated by \mathbb{F}_q -rational divisors.

Tate conjectures for elliptic $K3$ surfaces are proven in [1, Theorem 5.2]. J. S. Milne proved that the Tate conjectures imply the Artin–Tate conjectures for characteristic different from 2 (cf. [8, Theorem 6.1]). Finally, in [10, Theorem 0.4b] the assumption on the characteristic was dropped.

PROPOSITION 5.1 ([7, Proposition 4.7]). *Suppose q is a prime power. Let $Y \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface, defined over \mathbb{F}_q . Assume that q is a square and that $\rho(Y) = \rho'(Y)$. Then*

$$\Delta(\text{NS}(Y_{\mathbb{F}_q})) \equiv - \lim_{s \rightarrow 1} \frac{P(q^{-s})}{(1 - q^{1-s})^{\rho(Y)}} \pmod{(\mathbb{Q}^*)^2}.$$

LEMMA 5.3. *The rank of $E_1''(\overline{\mathbb{Q}}(t))$ is equal to 1.*

Proof. It is easy to check that the point $Q = (1-t, 1-t)$ lies in $E_1''(\overline{\mathbb{Q}}(t))$ and that it is a point of infinite order. The configuration of singular fibers is given in Table 4.

Table 4. Singular fibers, $E_1'' : -t(-1 + 5t)y^2 = x(x - (t - 1)^2)(x - 4t)$

Place	Type of singular fiber	Automorphism group
$t = 1$	I_4	$\mathbb{Z}/4\mathbb{Z}$
$t = \infty$	I_2	$\mathbb{Z}/2\mathbb{Z}$
$t = 0$	I_2^*	$(\mathbb{Z}/2\mathbb{Z})^2$
$t = 1/5$	I_0^*	$(\mathbb{Z}/2\mathbb{Z})^2$
$t = 3 + \sqrt{2}$	I_2	$(\mathbb{Z}/2\mathbb{Z})$
$t = 3 - \sqrt{2}$	I_2	$(\mathbb{Z}/2\mathbb{Z})$

It follows by Lemma 3.5 that it is enough to check that $2Q$ and $4Q$ are nonzero. The Euler number $e(\mathcal{E}_1'')$ is 24, which shows that \mathcal{E}_1'' is a $K3$ surface (cf. Table 4). The surface \mathcal{E}_1'' is defined over \mathbb{Z} (cf. proof of Lemma 4.6). We have two primes of good reduction 11 and 17. Consider the reduction of \mathcal{E}_1'' at 11, which we denote by S_{11} . It is a $K3$ surface defined over \mathbb{F}_{11} . We also have a $K3$ surface obtained by reduction at 17. We denote it by S_{17} . Note that it is defined over \mathbb{F}_{17} . Since we are only interested in surfaces defined over \mathbb{F}_{11^2} and \mathbb{F}_{17^2} , we will denote by S_{11} and S_{17} the base changes of the original surfaces to \mathbb{F}_{11^2} and \mathbb{F}_{17^2} , respectively.

By an argument similar to the proof of Lemma 4.6 we compute the characteristic polynomials of the Frobenius automorphism acting on the second ℓ -adic cohomology group for some auxiliary prime $\ell \neq 11, 17$.

For $p = 11$ using MAGMA we get

$$\text{char}(\Phi_{S_{11}}^*) = (x - 11^2)^{20}(x^2 - 158x + 14641).$$

Roots of the polynomial $x^2 - 158x + 14641$ are not of the form $11^2\zeta$, for a root of unity ζ . The rank of $\text{NS}((S_{11})_{\overline{\mathbb{F}}_{11^2}})$ equals 20 by the Tate conjectures for $K3$ surfaces (cf. Theorem 5.1) For $p = 17$ we get

$$\text{char}(\Phi_{S_{17}}^*) = (x - 17^2)^{20}(x^2 + 94x + 83521).$$

Roots of the polynomial $x^2 + 94x + 83521$ are not of the form $17^2\zeta$ for a root of unity ζ . Hence, the rank of $\text{NS}((S_{17})_{\overline{\mathbb{F}}_{17^2}})$ equals 20 by the Tate conjectures for $K3$ surfaces. The rank of $\text{NS}((\mathcal{E}_1'')_{\overline{\mathbb{Q}}})$ does not exceed the rank of the corresponding Néron–Severi group after reduction (cf. Theorem 3.9). Assume for a moment that it is maximal possible, hence equal to 20. This implies

that the discriminants of the lattices $\text{NS}((S_{11})_{\overline{\mathbb{F}}_{11^2}})$ and $\text{NS}((S_{17})_{\overline{\mathbb{F}}_{17^2}})$ should differ by a square. We apply Theorem 5.2 to compute the discriminants of $\text{NS}(\tilde{S}_{\overline{\mathbb{F}}_{17^2}})$ and $\text{NS}((S_{11})_{\overline{\mathbb{F}}_{11^2}})$. They are not equal modulo squares:

$$\begin{aligned} \Delta(\text{NS}((S_{11})_{\overline{\mathbb{F}}_{11^2}})) &\equiv -3 \cdot 7 \pmod{(\mathbb{Q}^*)^2}, \\ \Delta(\text{NS}((S_{17})_{\overline{\mathbb{F}}_{17^2}})) &\equiv -2 \cdot 3 \cdot 7 \pmod{(\mathbb{Q}^*)^2}. \end{aligned}$$

So the rank of $\text{NS}((\mathcal{E}'_1)_{\overline{\mathbb{Q}}})$ is ≤ 19 . Note that the trivial sublattice generated by the components of the singular fibers and the zero section is of rank 18. We also have the point Q of infinite order, so $19 \leq \text{rank NS}((\mathcal{E}'_1)_{\overline{\mathbb{Q}}})$. Hence the rank equals 19. Now an application of the Shioda–Tate formula reveals that $\text{rank } E''_1(\overline{\mathbb{Q}}(t)) = 1$. ■

COROLLARY 5.4. *The rank of $E_3(\overline{\mathbb{Q}}(t))$ is equal to 3.*

Proof. We apply Lemma 4.5 to the ranks obtained in Lemmas 4.6, 5.3 and 3.8. ■

REMARK 5.5. One could give a more direct proof of Corollary 5.4 using brute force and more powerful numerical computations. The statement of Corollary 5.4 is equivalent to $\rho((\mathcal{E}_3)_{\overline{\mathbb{Q}}}) = 37$ by the Shioda–Tate formula (see Table 3 for the number of components in singular fibers). Suppose to the contrary that $\rho((\mathcal{E}_3)_{\overline{\mathbb{Q}}}) \geq 38$. This lower bound holds for the Néron–Severi group of the reduced elliptic surface at primes of good reduction. Suppose we have two such primes, p_1 and p_2 . Here 17 is a good candidate, with the characteristic polynomial of the Frobenius automorphism (acting on the second cohomology group) equal to

$$(t+17)^8(t-17)^{30}(289-22t+t^2)(289-2t+t^2)(83521-2312t+238t^2-8t^3+t^4).$$

To compute the degree 8 factor we need to work with surfaces with points in the field \mathbb{F}_{17^8} or \mathbb{F}_{17^4} , which follows by Poincaré duality. None of the roots of

$$(289-22t+t^2)(289-2t+t^2)(83521-2312t+238t^2-8t^3+t^4)$$

are of the shape 17ζ for ζ a root of unity. Note that the Tate conjecture holds automatically for such a prime. By the results of J. S. Milne [8, Theorem 6.1] the Artin–Tate conjecture holds as well. Put $p_1 = 17$ and assume we have another such prime p_2 . This means that we can compare the discriminants of the lattices modulo squares and arrive at a contradiction, which proves that $\rho((\mathcal{E}_3)_{\overline{\mathbb{Q}}}) = 37$. Using the method of twists and further computation other good primes can be found, namely 73 and 97, but no other up to 140. However, a direct computation of the points on surfaces over \mathbb{F}_{73^4} or \mathbb{F}_{97^4} is beyond the range of our computational resources.

6. Proofs of main results

LEMMA 6.1. *The torsion subgroup of $E_3(\overline{\mathbb{Q}}(t))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It is generated by the points*

$$\begin{aligned} T_1 &= (-4u^2, 0), \\ T_2 &= (2(-u + u^3), 2\sqrt{-1}(u^2 - 1)u(-1 - 2u + u^2)), \end{aligned}$$

where $u = \frac{2t}{5+t^2}$.

Proof. Let $K = \overline{\mathbb{Q}}(t)$. The elliptic surface associated to E_3 has singular fibers of types I_2 and I_4 (cf. Table 3), hence

$$E_3(K)_{\text{tors}} \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^a \oplus (\mathbb{Z}/4\mathbb{Z})^b$$

for some natural numbers a and b by Lemma 3.5. The 2-torsion subgroup is generated by T_1 and $(0, 0)$. We will check that $T_1 \notin 2E_3(K)$ and $T_1 + (0, 0) \notin 2E_3(K)$, but $(0, 0) \in 2E_3(K)$.

Let $P = (x, y)$ be any point in $E_3(K)$. Then the x -coordinate of $2P$ is

$$x(2P) = \frac{(4u^2 - 8u^4 + 4u^6 - x^2)^2}{4(4u^2 - x)(1 - 2u^2 + u^4 - x)x},$$

where $u = \frac{2t}{5+t^2}$. If T_1 were in $2E_3(K)$, then

$$x(2P) = 4u^2$$

and so the equation

$$16t^2(25 + 6t^2 + t^4)^2 - 32t^2(5 + t^2)^4x + (5 + t^2)^6x^2 = 0$$

would have a solution $x \in K$. The discriminant of the above quadratic polynomial is equal to

$$-64t^2(5 + t^2)^6(625 - 100t^2 - 74t^4 - 4t^6 + t^8)$$

and it is not a square in K , hence we get a contradiction. Similarly one can show that $T_1 + (0, 0) = ((u^2 - 1)^2, 0)$ is not in $2E_3(K)$. Finally, it is easy to check that $2T_2 = (0, 0)$. The claim follows from that. ■

LEMMA 6.2. *The group $E_3(\overline{\mathbb{Q}}(t))/E_3(\overline{\mathbb{Q}}(t))_{\text{tors}}$ is free abelian of rank 3. It is generated by the points*

$$\begin{aligned} P_1 &= (2(1 + \sqrt{2})(-1 + u)^2u, 2\sqrt{-1}(1 + \sqrt{2})(-1 + (\sqrt{2} - u)^2)(-1 + u)^2u), \\ P_2 &= (2(u - 1)^2, 2(-1 + u)^2(-1 + 2u + u^2)), \\ P_3 &= \left(1 - u^2, \frac{(-5 + t^2)u(-1 + u^2)}{5 + t^2} \right), \end{aligned}$$

where $u = \frac{2t}{5+t^2}$.

Proof. We follow the proof of [18, Proposition 4.2]. We put $K = \overline{\mathbb{Q}}(t)$. Let $(E_3(K)/E_3(K)_{\text{tors}}, \langle \cdot, \cdot \rangle_{E_3})$ denote the Mordell–Weil lattice with the height pairing $\langle \cdot, \cdot \rangle_{E_3}$. From the type of singular fibers, I_2 and I_4 (Table 3), we know that for each $P, Q \in E_3(K)/E_3(K)_{\text{tors}}$ we have $\langle P, Q \rangle_{E_3} \in \frac{1}{4}\mathbb{Z}$.

Consider the lattice $\Lambda = E_3(K)/E_3(K)_{\text{tors}}$ with the pairing $\langle \cdot, \cdot \rangle = 4\langle \cdot, \cdot \rangle_{E_3}$. Let Λ' be generated by P_1, P_2 and P_3 . It is a sublattice of Λ of a finite index $n = [\Lambda : \Lambda']$. In Λ we have $\langle P_i, P_i \rangle = 4i$ for $i = 1, 2, 3$ and $\langle P_i, P_j \rangle = 0$ for $i \neq j$. Hence, the following equality holds for the discriminants of Λ and Λ' with the pairing $\langle \cdot, \cdot \rangle$:

$$6 \cdot 4^3 = \Delta(\Lambda') = n^2 \Delta(\Lambda).$$

Therefore, n divides 8. We want to show that $n = 1$. Consider the 2-descent homomorphism

$$\psi : E_3(K)/2E_3(K) \hookrightarrow K^*/(K^*)^2 \times K^*/(K^*)^2.$$

For (x, y) in $E_3(K) \setminus E_3(K)[2]$, it is defined by

$$\psi(x, y) = (x - e_1, x - e_2),$$

where $e_1 = 0$ and $e_2 = 4u^2$.

Let H denote the group generated by P_1, P_2, P_3, T_1, T_2 , and let G denote $E_3(K)$. The index n equals $[G : H]$. There exist $R_1, R_2, R_3 \in G$ such that G is generated by R_1, R_2, R_3, T_1, T_2 and H is generated by $aR_1, bR_2, cR_3, T_1, T_2$, where $n = abc$ and $a | b | c$.

For $n = 8$, it follows that $(a, b, c) \in \{(1, 1, 8), (1, 2, 4), (2, 2, 2)\}$.

For $n = 4$, we have $(a, b, c) \in \{(1, 1, 4), (1, 2, 2)\}$ and for $n = 2$ there is only one tuple $(a, b, c) = (1, 1, 2)$.

Consider the modulo 2 map $\phi : G \rightarrow G/2G$ and $\eta = \psi \circ \phi$. Then $\eta(G)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. If $n = 8$, then $\eta(H) \cong (\mathbb{Z}/2\mathbb{Z})^i$, where $1 \leq i \leq 3$. If $n = 4$, then $\eta(H) \cong (\mathbb{Z}/2\mathbb{Z})^i$, where $2 \leq i \leq 3$. If $n = 2$, then $\eta(H) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Hence, to show that $H = G$ it is sufficient to prove that $\eta(H) \cong (\mathbb{Z}/2\mathbb{Z})^4$. We easily compute

$$\eta(P_1) = (t(t^2 + 5), (t^2 + 5)t(-5 + (-2 + 2\sqrt{2})t - t^2)(-5 + (2 + 2\sqrt{2})t - t^2)),$$

$$\eta(P_2) = (1, t^4 - 4t^3 + 6t^2 - 20t + 25),$$

$$\eta(P_3) = ((t^2 - 2t + 5)(t^2 + 2t + 5), 1),$$

$$\eta(T_2) = (t(t^2 - 2t + 5)(t^2 + 2t + 5)(t^2 + 5), t(t^4 + 4t^3 + 6t^2 + 20t + 25)(t^2 + 5)),$$

and prove that $|\eta(H)| = 16$, which proves the theorem. ■

COROLLARY 6.3. *The group $E_3(\mathbb{Q}(t))$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The free part is generated by P_2, P_3 . The torsion part is generated by $T_1, 2T_2 = (0, 0)$.*

Proof. First we prove that the rank of the group $E_3(\mathbb{Q}(t))$ equals 2. From Corollary 5.4 we know that $\text{rank } E_3(\mathbb{Q}(t)) \leq 3$. Since the points P_2 and P_3 are linearly independent it is enough to show that $\text{rank } E_3(\mathbb{Q}(t)) < 3$.

Suppose to the contrary that the rank of $H = E_3(\mathbb{Q}(t))$ equals 3. Then H is a finite index subgroup of the group $G = E_3(\overline{\mathbb{Q}}(t))$ generated by P_1, P_2, P_3, T_1 and T_2 . Consider the 3-dimensional \mathbb{Q} -vector space $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a natural Galois representation

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(G_{\mathbb{Q}}).$$

For $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $P \in G$ we define $\sigma(P \otimes 1) = \sigma(P) \otimes 1$, where $\sigma(P)$ denotes the element in G such that σ acts on the coefficients of rational functions in the coordinates of P . If $\sigma(\sqrt{-1}) = -\sqrt{-1}$ and $\sigma(\sqrt{2}) = \sqrt{2}$, then $\sigma(P_1 \otimes 1) = -(P_1 \otimes 1)$. In the basis $\{P_1 \otimes 1, P_2 \otimes 1, P_3 \otimes 1\}$ of $G_{\mathbb{Q}}$, the matrix of the automorphism $\rho(\sigma)$ is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, the representation ρ is nontrivial. However, $G_{\mathbb{Q}} = H_{\mathbb{Q}}$, since we assumed that H is of finite index in G . The representation ρ acts trivially on $H_{\mathbb{Q}}$, which leads to a contradiction. Hence, H is not of finite index in G , which implies that $\text{rank } H = 2$.

If the Mordell–Weil lattice $E_3(\mathbb{Q}(t))/E_3(\mathbb{Q}(t))_{\text{tors}}$ were not generated by P_2 and P_3 , then the lattice generated by those points would be of finite index greater than 1 in the full Mordell–Weil lattice. Then the lattice generated by P_1, P_2 and P_3 would be of index greater than 1 in the full Mordell–Weil lattice $E_3(\overline{\mathbb{Q}}(t))/E_3(\overline{\mathbb{Q}}(t))_{\text{tors}}$, which contradicts Lemma 6.2.

To conclude the proof, we compute the torsion part. Lemma 6.1 shows that the torsion defined over $\mathbb{Q}(t)$ is generated by $T_1 = (4u^2, 0)$ and $(0, 0) = 2T_2$. It is the full torsion subgroup of $E_3(\mathbb{Q}(t))$. ■

Proof of Theorem 1.4. From Corollary 5.4 it follows that $\text{rank } E_3(\overline{\mathbb{Q}}(t)) = 3$. Lemmas 6.1 and 6.2 give explicit generators over $\overline{\mathbb{Q}}(t)$. Finally, Corollary 6.3 shows that the rank over $\mathbb{Q}(t)$ is 2 and it gives explicit generators. ■

Proof of Theorem 1.1. We apply the specialization theorem (cf. [16, Theorem 11.4]) to the family

$$(6.1) \quad y^2 = x(x - (u^2 - 1)^2)(x - 4u^2)$$

with a rational parameter t and $u = \frac{2t}{5+t^2}$. The curve is nonsingular for any $t \neq 0$. Let $p/q = t$ denote a rational number where p and $q \neq 0$ are integers. Let

$$\frac{P}{Q} = u = \frac{2pq}{p^2 + 5q^2}$$

where P and $Q \neq 0$ are integers. We claim that the triple $(a, b, c) = (P^2 - Q^2, 2PQ, P^2 + Q^2)$ is an element of \mathcal{S} .

Suppose to the contrary that $a = 0$ or $b = 0$. If $a = 0$, then $P^2 = Q^2$, hence $u = \pm 1$ and $(5 + t^2) = \pm 2t$, which does not have solutions $t \in \mathbb{Q}$, a contradiction. If $b = 0$, then $P = 0$ and $q = 0$, so $t = 0$, which is not possible, because the curve is nonsingular.

The triple (a, b, c) determines an associated elliptic curve

$$y^2 = x(x - a^2)(x - b^2)$$

which has the following two points:

$$Q_1 = \left(\frac{1}{2}(a + b - c)^2, \frac{1}{2}(a + b)(a + b - c)^2 \right),$$

$$Q_2 = \left(\frac{1}{2}a(a - c), \frac{1}{2}ab \frac{1}{k^2}(p^4 - 25q^4) \right),$$

where $k = \text{GCD}(2pq, p^2 + 5q^2)$. The points Q_1 and Q_2 are obtained from

$$P_2 = (2(u - 1)^2, 2(-1 + u)^2(-1 + 2u + u^2)),$$

$$P_3 = \left(1 - u^2, \frac{(-5 + t^2)u(-1 + u^2)}{5 + t^2} \right)$$

by the map $(x, y) \mapsto (x(a - c)^2/4, y(c - a)^3/8)$. The specialization theorem shows that Q_1 and Q_2 are linearly independent for almost all t . By Proposition 2.1 for all but finitely many elements of \mathcal{S}/\sim , the rank of the group of \mathbb{Q} -rational points on the curve $E_{(a,b,c)}$, $(a, b, c) \in \mathcal{S}/\sim$ is at least two. Hence for infinitely many $(a, b, c) \in \mathcal{S}$ the group $E_{(a,b,c)}(\mathbb{Q})$ has rank at least two. ■

REMARK 6.4. Observe that the point

$$(c^2, abc) = -2\left(\frac{1}{2}(a + b - c)^2, \frac{1}{2}(a + b)(a + b - c)^2\right)$$

is on the curve

$$y^2 = x(x - a^2)(x - b^2).$$

The point $(\frac{1}{2}(a + b - c)^2, \frac{1}{2}(a + b)(a + b - c)^2)$ corresponds to the point

$$(2(t - 1)^2, 2(t - 1)^2(-1 + 2t + t^2))$$

via the inverse of the map $(x, y) \mapsto (x(a - c)^2/4, y(c - a)^3/8)$. The point

$$(2(t - 1)^2, 2(t - 1)^2(-1 + 2t + t^2))$$

is a generator of the free part of the Mordell–Weil group over $\mathbb{Q}(t)$ on the curve

$$y^2 = x(x - (t^2 - 1)^2)(x - 4t^2).$$

We prove this fact in the next lemma.

LEMMA 6.5. *The group $E_2(\overline{\mathbb{Q}}(t))$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The free part is generated by*

$$P_1 = (2(1 + \sqrt{2})(-1 + t)^2 t, 2\sqrt{-1}(1 + \sqrt{2})(-1 + (\sqrt{2} - t)^2)(-1 + t)^2 t),$$

$$P_2 = (2(t - 1)^2, 2(-1 + t)^2(-1 + 2t + t^2)).$$

The torsion part is generated by

$$T_1 = (-4t^2, 0),$$

$$T_2 = (2(-t + t^3), 2\sqrt{-1}(t^2 - 1)t(-1 - 2t + t^2)).$$

The group $E_2(\mathbb{Q}(t))$ is generated by P_2 , T_1 and $2T_2 = (0, 0)$.

Proof. The torsion subgroup is computed as in the proof of Lemma 6.1. We put $K = \overline{\mathbb{Q}}(t)$. Let $(E_2(K)/E_2(K)_{\text{tors}}, \langle \cdot, \cdot \rangle_{E_2})$ be the Mordell–Weil lattice with height pairing $\langle \cdot, \cdot \rangle_{E_2}$. We easily compute $\langle P_1, P_1 \rangle_{E_2} = 1/2$, $\langle P_2, P_2 \rangle_{E_2} = 1$ and $\langle P_1, P_2 \rangle_{E_2} = 0$.

In general, for each $P, Q \in E_2(K)/E_2(K)_{\text{tors}}$ the value of $\langle P, Q \rangle_{E_2}$ lies in $\frac{1}{4}\mathbb{Z}$, which follows from the type of singular fibers (see Table 2).

Consider the lattice $\Lambda = (E_2(K)/E_2(K)_{\text{tors}})$ with the pairing $\langle \cdot, \cdot \rangle = 4\langle \cdot, \cdot \rangle_{E_2}$. Let Λ' be generated by P_1 and P_2 . It is a sublattice of Λ of a finite index $n = [\Lambda : \Lambda']$. For the lattice Λ we have $\langle P_1, P_1 \rangle = 2$, $\langle P_2, P_2 \rangle = 4$ and $\langle P_1, P_2 \rangle = 0$. Hence, the following equality holds for the discriminants of the lattices Λ and Λ' with respect to the pairing $\langle \cdot, \cdot \rangle$:

$$8 = \Delta(\Lambda') = n^2 \Delta(\Lambda).$$

Hence, n divides 2. We want to show that $n = 1$. Suppose to the contrary that $n = 2$. There exists a point $R \in E_2(K)$ of infinite order such that

$$2R = aP_1 + bP_2 + T$$

for some $a, b \in \{0, 1\}$ and $T \in E_2(K)_{\text{tors}}$. So

$$4\langle R, R \rangle = \langle 2R, 2R \rangle = 2a^2 + 4b^2 = 2(a^2 + 2b^2).$$

This implies $2 \mid (a^2 + 2b^2)$. For $a, b \in \{0, 1\}$ there are pairs $(a, b) = (0, 0)$ and $(a, b) = (0, 1)$. For $(a, b) = (0, 0)$ we obtain the equation

$$2R = T$$

for a K -rational torsion point T . This implies that R is of finite order, hence a contradiction. For the pair $(a, b) = (0, 1)$ we obtain the equation

$$2R = P_2 + T$$

with a K -rational torsion point T . We consider only the cases $T = O$, $T = T_1$, $T = T_2$ and $T = T_1 + T_2$, since one can add a point from $2E_2(K)_{\text{tors}}$ to both sides.

Consider the 2-descent homomorphism

$$\psi : E_2(K)/2E_2(K) \hookrightarrow K^*/(K^*)^2 \times K^*/(K^*)^2.$$

For nontorsion points (x, y) in $E_2(K)$ it is defined by $\psi(x, y) = (x - e_1, x - e_2)$, where $e_1 = 0$ and $e_2 = 4t^2$. We check using MAGMA that $\psi(P_2 + T) \neq (1, 1)$ for $T \in \{O, T_1, T_2, T_1 + T_2\}$. This proves that the assumption $n = 2$ leads to a contradiction. Hence $\Lambda = \Lambda'$, proving that the rank of $E_2(K)$ is two.

Now we prove that the group $E_2(\mathbb{Q}(t))$ is generated by P_2 , T_1 and $2T_2$. For the torsion part, observe that $E_2(\mathbb{Q}(t))_{\text{tors}} \subset E_2(\overline{\mathbb{Q}}(t))_{\text{tors}}$. The group $E_2(\overline{\mathbb{Q}}(t))_{\text{tors}}$ is generated by T_1 and T_2 . Since T_2 is not $\mathbb{Q}(t)$ -rational, it follows that the group $E_2(\mathbb{Q}(t))_{\text{tors}}$ is generated by T_1 and $2T_2 = (0, 0)$. We know that $\text{rank } E_2(\overline{\mathbb{Q}}(t)) = 2$. Hence, the rank of $E_2(\mathbb{Q}(t))$ is at most 2.

Assume it equals 2. Then there exists a point R defined over $\mathbb{Q}(t)$ such that $R = aP_1 + bP_2 + T$ for some integers $a \neq 0$ and b and a torsion point T . Since $4T = O$, we have

$$(6.2) \quad 4R = 4aP_1 + 4bP_2.$$

Recall that

$$P_1 = (2(1 + \sqrt{2})(-1 + t)^2t, 2\sqrt{-1}(1 + \sqrt{2})(-1 + (\sqrt{2} - t)^2)(-1 + t)^2t).$$

We choose an automorphism $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which acts on the coefficients of rational functions in the coordinates of P_1 by the formula $\sigma(\sqrt{-1}) = -\sqrt{-1}$, $\sigma(\sqrt{2}) = \sqrt{2}$. The action of σ commutes with the addition morphism on the curve E_2 which is defined over $\mathbb{Q}(t)$. Applying σ to both sides of (6.2) we get $8aP_1 = O$, because $\sigma(P_1) = -P_1$ and $\sigma(P_2) = P_2$. This gives a contradiction since P_1 is a nontorsion point.

If the Mordell–Weil lattice $E_2(\mathbb{Q}(t))/E_2(\mathbb{Q}(t))_{\text{tors}}$ were not generated by P_2 , then the lattice generated by this point would be of finite index greater than 1 in the full Mordell–Weil lattice. Then the lattice generated by P_1 and P_2 would be of index greater than 1 in the full Mordell–Weil lattice $E_2(\mathbb{Q}(t))/E_2(\mathbb{Q}(t))_{\text{tors}}$, which contradicts what has been proven already. ■

Proof of Theorem 1.3. This follows from Lemma 6.5 and the fact that the curves

$$\begin{aligned} y^2 &= x(x-1)\left(x - \left(\frac{2t}{t^2-1}\right)^2\right), \\ y^2 &= x(x - (t^2 - 1)^2)(x - 4t^2) \end{aligned}$$

are isomorphic over $\mathbb{Q}(t)$. ■

REMARK 6.6. It is natural to ask what is the rank of the Mordell–Weil group of the curve

$$y^2 = x(x - \alpha a^2)(x - \beta b^2),$$

where $\alpha a^2 + \beta b^2 + \gamma c^2 = 0$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$. In particular, one would like to know what is the upper bound of the rank in such a big family. We hope to return to this question in the future.

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