ACTA ARITHMETICA
160.2 (2013)

A zero density result for the Riemann zeta function

by

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1. Introduction. In recent years, it has become apparent that explicit results concerning prime numbers are required to solve important problems in number theory. In particular, the impressive works of Ramaré [18], Tao [29], and Helfgott [13] related to Goldbach's conjecture highlight the need of better explicit bounds for finite sums over primes. For instance, they make use of [4], [21], [22], [24], [25], [27]. Moreover articles of Rosser and Schoenfeld ([24], [25]–[27]), Dusart ([5]–[8]), and Ramaré and Rumely ([23]) are extensively used in a wide range of fields including Diophantine approximation, cryptography, and computer science. These results on primes rely heavily on explicit estimates of sums over the non-trivial zeros of the Riemann zeta function. More precisely, they rely on three key ingredients: a numerical verification of the Riemann Hypothesis (RH), an explicit zero-free region, and explicit bounds for the number of zeros in the critical strip up to a fixed height T.

In 1986, van de Lune et al. [33] established that RH had been verified for all zeros ϱ verifying $|\mathfrak{Im}\,\varrho| \le H_0$ with $H_0 = 545\,439\,823$. In 2011, Platt ([15], [16]) proved that $H_0 = 30\,610\,046\,000$ is admissible. Previously, Wedeniwski [34] in 2003 and Gourdon [11] in 2004 had announced higher values for H_0. As Platt's computations are more rigourous (he employs interval arithmetic), we decide to use his value throughout this article:

$$H_0 = 3.061 \cdot 10^{10}.$$

For the latest explicit results about zero-free regions for the Riemann zeta function, we refer the reader to [14] and [10].

Let $\sigma \ge 0.55$. We consider $N(\sigma, T)$, the number of zeros of the Riemann zeta function in the region $\sigma \le \mathfrak{Re}\, s \le 1$ and $0 \le \mathfrak{Im}\, s \le T$. Trivially we have $N(\sigma, T) = 0$ for all $T \le H_0$. We prove here an explicit bound for $N(\sigma, T)$ valid in the range $T \ge H_0$.

2010 *Mathematics Subject Classification*: Primary 11M06, 11M26; Secondary 11Y35.
Key words and phrases: Riemann zeta function, zero density, explicit results.

DOI: 10.4064/aa160-2-6

Theorem 1.1. *Let* $\sigma \geq 0.55$ *and* $T \geq H_0$. *Let* σ_0 *and* H *be such that* $0.5208 < \sigma_0 < 0.9723$, $\sigma_0 < \sigma$, *and* $10^3 \leq H \leq H_0$. *Then there exist* b_1, b_2, b_3, *positive constants depending on* σ, σ_0, H, *such that*

$$N(\sigma, T) \leq b_1(T - H) + b_2 \log(TH) + b_3.$$

The b_i*'s are defined in* (6.3).

We rewrite this as $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$ for $T \geq H_0$. Numerical values of the b_i's and c_i's are recorded at the end of this article in Table 1. For example, for $\sigma \geq 17/20$ and $T \geq H_0$, we have

$$N(\sigma, T) \leq 0.5561T + 0.7586 \log T - 268\,658.$$

Let $N(T)$ be the number of non-trivial zeros ϱ with $0 \leq \mathfrak{Im}\,\varrho \leq T$. We recall that Rosser [24] proved

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq a \log T + b \log\log T + c, \tag{1.1}$$

with $a = 0.137$, $b = 0.443$, $c = 1.588$. Rosser's result got recently improved by Trudgian [32, Corollary 1] with $a = 0.111$, $b = 0.275$, $c = 2.450$. A trivial bound for $N(\sigma, T)$ follows from the inequalities $N(\sigma, T) \leq \frac{1}{2} N(T)$ and (1.1):

$$N(\sigma, T) \leq \frac{T}{4\pi} \log\left(\frac{T}{2\pi e}\right)(1 + o(1)).$$

Note that when T is asymptotically large, then a factor of $\log T$ is saved. Moreover, we have

$$c_1 \sim \frac{\log \zeta(2\sigma_0)}{4\pi(\sigma - \sigma_0)}$$

where σ_0 is a parameter whose value can be chosen to make c_1 as small as possible. Another feature of Theorem 1.1 is the factor $T - H$: when T is near H_0, we choose H to be close to H_0 so as to make $N(\sigma, T)$ of size $\log H_0$. This saves a factor of size H_0. As an example, for $\sigma \geq 17/20$ and $T = H_0 + 1$, we choose $H = H_0 - 1$ and σ_0 as in Table 1 and obtain $N(\sigma, H_0 + 1) \leq 156$, while (1.1) gives $5.2 \cdot 10^{10}$ (with either Rosser's or Trudgian's values).

The key motivation for establishing Theorem 1.1 is to use it in place of (1.1) and thus to provide improved explicit bounds for Chebyshev's prime counting functions. We prove in [9] that, for all $x \geq e^b$,

$$|\psi(x) - x| \leq \epsilon_b x, \tag{1.2}$$

where b is a fixed positive constant, and ϵ_b is an effective positive constant. For example, for $x \geq e^{50}$ we obtain $\epsilon_{50} = 9.461 \cdot 10^{-10}$, while Dusart [7, Theorem 2] obtained $0.905 \cdot 10^{-7}$.

Despite a very rich history of asymptotic results, there were almost no explicit bounds for $N(\sigma, T)$. Ramaré proved in an unpublished manuscript [19] that, for $T \geq 2000$, $Q \geq 10$, and $T \geq Q$,

$$\sum_{q \leq Q} \sum_{\chi \bmod^* q} N(\sigma, T, \chi) \leq 157(Q^5 T^3)^{1-\sigma} \log^{4-\sigma}(Q^2 T) + 6Q^2 \log^2(Q^2 T),$$

where $\sum_{\chi \bmod^* q}$ denotes the sum over primitive Dirichlet characters χ to modulus q, and $N(\sigma, T, \chi)$ counts the number of zeros ϱ of the Dirichlet L-function $L(s, \chi)$ satisfying $\sigma < \mathfrak{Re}\,\varrho < 1$ and $0 < \mathfrak{Im}\,\varrho < T$. Taking $Q = 10$ and restricting the left sum to $q = 1$, it follows that

$$N(\sigma, T) \leq 157(100\,000 T^3)^{1-\sigma} \log^{4-\sigma}(100T) + 600 \log^2(100T). \tag{1.3}$$

Our main theorem improves Ramaré's result for certain values of σ and T: he obtains $N(17/20, 10 \cdot H_0) \leq 2.675 \cdot 10^{12}$, while we have $N(17/20, 10 \cdot H_0) \leq 3.404 \cdot 10^{10}$. In 2010, Cheng [3] obtained the weaker result

$$N(\sigma, T) \leq 453\,472.54\, T^{8/3(1-\sigma)} (\log T)^5 \tag{1.4}$$

for all $\sigma \geq 5/8$ and $T \geq \exp(\exp(18)) \simeq 10^{28\,515\,762}$. His method is based on Ford's [10] effective version of Korobov–Vinogradov's bound for the Riemann zeta function. He applied (1.4) to deduce explicit results on primes between consecutive cubes. Note that Cheng's result is not valid in the region $T \leq \exp(\exp(18))$, while most applications require bounds for T as small as H_0.

In order to prove Theorem 1.1 we establish two intermediate theorems about $\zeta(s)$ in the critical strip: an effective version of a Dirichlet polynomial approximation, and an explicit estimate for the second moment.

THEOREM 1.2. *Let $t_0 > 0$, $s = \sigma + it$ with $\sigma \geq 1/2$, $t \geq t_0$ and $c > 1/2\pi$. Then*

$$\zeta(s) = \sum_{1 \leq n < ct} \frac{1}{n^s} + R(s)$$

with $|R(s)| \leq C(\sigma, c) t^{-\sigma}$, and

$$C(\sigma, c) = \left(c + \frac{1}{2} + \frac{3\sqrt{1 + 1/t_0^2}}{2\pi} \left(\frac{\zeta(2)}{2\pi c} + 1 + \frac{1}{2\pi c - 1} \right) \right) c^{-\sigma}. \tag{1.5}$$

We apply the theorem for $c = 1$ and for t_0 the height of the first zero of zeta.

COROLLARY 1.3. *Let $\sigma \geq 1/2$ and $t \geq 14.1347$. Then*

$$\left| \zeta(s) - \sum_{1 \leq n < t} \frac{1}{n^s} \right| \leq c_0 t^{-\sigma}, \quad \textit{where} \quad c_0 = 2.1946. \tag{1.6}$$

This is to compare with Proposition 1 of Cheng [2] who obtained 5.505 instead of $2.1946 t^{-\sigma}$. When $\sigma \geq 1/2$ and $0 \leq t \leq 15$, a Mathematica computation gives $|\zeta(s) - \sum_{1 \leq n < t} n^{-s}| \leq 43 t^{-\sigma}$.

THEOREM 1.4. *Let* $0.5208 < \sigma_0 < 0.9723$ *and* $10^3 \le H \le H_0$. *We define*

$$\epsilon_1(\sigma_0, H) = \frac{4H_0}{H_0 - H}\bigg(\frac{(\log H_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{(2\sigma_0 - 1)\log H_0}{2(1-\sigma_0)H_0} + \frac{\max\big(0, \frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{\zeta(2\sigma_0)}{2}\big)}{H_0} + \frac{(2-\sigma_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)^2} - \frac{\sigma_0 H_0^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{H_0^{-2\sigma_0}}{2(2\sigma_0 - 1)} + \frac{H_0^{-2\sigma_0-1}}{2}\bigg), \tag{1.7}$$

$$\epsilon_2(\sigma_0, H) = \frac{c_0^2}{2\sigma_0 - 1}\,\frac{H^{-(2\sigma_0-1)} - H_0^{-(2\sigma_0-1)}}{H_0 - H}, \tag{1.8}$$

$$\epsilon_3(\sigma_0, H) = 2\sqrt{\epsilon_2(\sigma_0, H)(\zeta(2\sigma_0) + \epsilon_1(\sigma_0, H))}, \tag{1.9}$$

$$\mathcal{E}_1 = \epsilon_1 + \epsilon_2 + \epsilon_3. \tag{1.10}$$

Then, for all $T \ge H_0$, *we have*

$$\frac{1}{T-H}\int_H^T |\zeta(\sigma_0 + it)|^2\,dt \le \zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H)$$

and

$$\int_H^T \log|\zeta(\sigma_0 + it)|dt \le \frac{T-H}{2}\log(\zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H)).$$

For the rest of this article H_0, H, T, σ_0, σ_1 and σ satisfy

$$\begin{aligned} &H_0 = 3.061\cdot 10^{10}, 10^3 \le H \le H_0 \le T,\\ &0.5208 < \sigma_0 < 0.9723, \sigma_1 = 1.5002, \sigma_0 < \sigma < \sigma_1. \end{aligned} \tag{1.11}$$

2. Approximate formula for $\zeta(\sigma + it)$; proof of Theorem 1.2. Let $s = \sigma + it$ with $1/2 < \sigma < 1$ and $t \ge 2$. Let $x = ct$ with $c > 1/2\pi$, and let N be a positive integer. Theorem 1.2 gives an explicit version of an approximation formula for zeta, as proven by Hardy and Littlewood in [12].

Proof of Theorem 1.2. We start with the classical identity [30, (3.5.3)]

$$\zeta(s) - \sum_{1\le n<x}\frac{1}{n^s} = \sum_{x\le n\le N}\frac{1}{n^s} + s\int_N^\infty \frac{((u))}{u^{s+1}}\,du - \frac{N^{1-s}}{1-s} - \frac{1}{2}N^{-s}, \tag{2.1}$$

where $((u)) = [u] - u + 1/2$. The summation formula [30, (2.1.2)] gives

$$\begin{aligned}\sum_{x\le n<N}\frac{1}{n^s} &= \int_x^N \frac{du}{u^s} - \frac{((x))}{x^s} + s\int_x^N \frac{((u))}{u^{1+s}}\,du\\ &= \frac{N^{1-s} - x^{1-s}}{1-s} - \frac{((x))}{x^s} + s\int_x^N \frac{((u))}{u^{1+s}}\,du.\end{aligned}$$

We have the bounds

$$\left|\frac{x^{1-s}}{1-s}\right| \le \frac{x^{1-\sigma}}{t}, \qquad \left|\frac{((x))}{x^s}\right| \le \frac{x^{-\sigma}}{2},$$

$$\left|s\int_N^\infty \frac{((u))}{u^{s+1}}\,du\right| \le \frac{|s|}{2}\int_N^\infty \frac{1}{u^{\sigma+1}}\,du = \frac{|s|}{2\sigma N^\sigma}.$$

Thus

$$\left|\zeta(s) - \sum_{1\le n<x}\frac{1}{n^s}\right| \le x^{1-\sigma}t^{-1} + \frac{x^{-\sigma}}{2} + \left|s\int_x^N \frac{((u))}{u^{1+s}}\,du\right| + \frac{|s|}{2\sigma}N^{-\sigma} + \frac{1}{2}N^{-\sigma}. \tag{2.2}$$

The choice $x = ct$ is made to balance the error term $x^{1-\sigma}t^{-1} + x^{-\sigma}/2$. We appeal to the Fourier series of $((x))$ to obtain a smaller bound for the integral expression. For $u \notin \mathbb{N}$, we have [30, p. 74]

$$((u)) = [u] - u + 1/2 = \frac{1}{\pi}\sum_{\nu=1}^\infty \frac{\sin(2\pi\nu u)}{\nu}.$$

Lebesgue's bounded convergence theorem applies, and we can exchange the order of the integral and the summation. We obtain

$$\int_x^N \frac{((u))}{u^{1+s}}\,du = \frac{1}{\pi}\sum_{\nu=1}^\infty \frac{1}{\nu}\int_x^N \frac{\sin(2\pi\nu u)}{u^{1+s}}\,du = \sum_{\nu=1}^\infty \frac{I(\nu)-I(-\nu)}{\nu}, \tag{2.3}$$

where the integral I is given by

$$I(h) = \frac{1}{2\pi i}\int_x^N \frac{e^{2i\pi(hu - \frac{t\log u}{2\pi})}}{u^{\sigma+1}}\,du = \frac{1}{2\pi}\int_x^N F(h,u)\,d(e^{2\pi i(f(u)+hu)}) \tag{2.4}$$

with $F(h,u) = \frac{u^{-\sigma}}{t-2\pi uh}$ and $f(u) = -\frac{t\log u}{2\pi}$. Since

$$\frac{\partial}{\partial u}F(h,u) = u^{-\sigma}\frac{-\sigma t u^{-1} + 2\pi h(\sigma+1)}{(t-2\pi uh)^2},$$

it is easy to check that $F(-\nu,u)$ is positive and decreases with u, and that $F(\nu,u)$ is negative and increases with u.

We now apply the second mean value theorem from [31, Section 12.3]:

LEMMA 2.1. *If $j(x)$ is an integrable function over (a,b), and $\phi(x)$ is positive, bounded, and non-increasing, then there exists $\xi \in (a,b)$ such that*

$$\int_a^b \phi(x)j(x)\,dx = \phi(a+0)\int_a^\xi j(x)\,dx.$$

First, we consider $I(-\nu)$. We separate the real and imaginary part in $d(e^{2\pi i(f(u)+hu)})$ in (2.4) and we apply the lemma for $\phi(u) = F(-\nu,u)$. We consider $j(u)du = d(\cos(2\pi(f(u)-\nu u)))$ and $j(u)du = d(\sin(2\pi(f(u)-\nu u)))$

respectively. We deduce that there exist $\xi_1, \xi_2 \in (x, N)$ such that

$$\begin{aligned} 2\pi I(-\nu) &= F(-\nu, x)\cos(2\pi(f(\xi_1) - \nu\xi_1)) - F(-\nu, x)e^{2\pi i(f(x)-\nu x)} \\ &\quad + iF(-\nu, x)\sin(2\pi(f(\xi_2) - \nu\xi_2)). \end{aligned}$$

It follows that

$$|I(-\nu)| \le \frac{3}{2\pi}F(-\nu, x) = \frac{3}{2\pi}\frac{(ct)^{-\sigma}}{t + 2\pi ct\nu} \le \frac{3}{(2\pi)^2}\frac{c^{-\sigma-1}t^{-\sigma-1}}{\nu}. \tag{2.5}$$

A similar argument applies to $I(\nu)$. We obtain

$$|I(\nu)| \le -\frac{3}{2\pi}F(\nu, ct) = \frac{3}{2\pi}\frac{(ct)^{-\sigma}}{2\pi ct\nu - t} \le \begin{cases} \dfrac{3}{2\pi}\dfrac{c^{-\sigma}t^{-\sigma-1}}{\nu - 1} & \text{if } \nu \ge 2, \\ \dfrac{3}{2\pi}\dfrac{c^{-\sigma}t^{-\sigma-1}}{2\pi c - 1} & \text{if } \nu = 1. \end{cases} \tag{2.6}$$

Using the simplification $\sum_{\nu=2}^{\infty}\frac{1}{\nu(\nu-1)} = 1$, $\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \zeta(2)$, and $|s|/t \le \sqrt{1 + 1/t^2}$, we put together (2.3), (2.5), and (2.6), and obtain the bound

$$\begin{aligned} \left| s\int_x^N \frac{((u))}{u^{1+s}}\,du\right| &\le |s|\sum_{\nu=1}^{\infty}\frac{|I(\nu)| + |I(-\nu)|}{\nu} \\ &\le \frac{3\sqrt{1 + 1/t^2}}{2\pi}\left(1 + \frac{1}{2\pi c - 1} + \frac{\zeta(2)}{2\pi c}\right)c^{-\sigma}t^{-\sigma}. \end{aligned}$$

On letting $N \to \infty$, inequality (2.2) becomes

$$\left|\zeta(s) - \sum_{1\le n<ct}\frac{1}{n^s}\right| \le \left(c + \frac{1}{2} + \frac{3\sqrt{1 + 1/t^2}}{2\pi}\left(1 + \frac{1}{2\pi c - 1} + \frac{\zeta(2)}{2\pi c}\right)\right)(ct)^{-\sigma}. \ \blacksquare$$

Remark 2.2. A careful reading of Cheng's proof shows that his error term has size $\mathcal{O}(t^{1-2\sigma})$, instead of our $\mathcal{O}(t^{-\sigma})$. This comes from the fact that the bounds directly the terms $N^{1-s}/(1 - s)$, instead of eliminating them as we did.

3. Explicit upper bound for the second moment of zeta; proof of Theorem 1.4. We recall that σ_0, T, H are as in (1.11). By Theorem 1.2, we have the identity

$$\begin{aligned} &\frac{1}{T - H}\int_H^T |\zeta(\sigma_0 + it)|^2\,dt \\ &\quad = D(\sigma_0, T, H) + E_1(\sigma_0, T, H) + E_2(\sigma_0, T, H) + E_3(\sigma_0, T, H), \end{aligned} \tag{3.1}$$

where

$$D(\sigma_0, T, H) = \frac{1}{T-H} \int_H^T \sum_{1 \le n < t} \frac{1}{n^{2\sigma_0}}\, dt,$$

$$E_1(\sigma_0, T, H) = \frac{2}{T-H} \int_H^T \sum_{1 \le n < m < t} \frac{\cos(t \log(m/n))}{(nm)^{\sigma_0}}\, dt,$$

$$E_2(\sigma_0, T, H) = \frac{1}{T-H} \int_H^T |R(\sigma_0 + it)|^2\, dt,$$

$$E_3(\sigma_0, T, H) = \frac{2}{T-H} \mathfrak{Re} \int_H^T \sum_{1 \le n < t} \frac{R(\sigma_0 + it)}{n^{\sigma_0 + it}}\, dt.$$

We record here some basic inequalities that we use throughout the following argument. Let $A, B \in \mathbb{N}$. If f is decreasing and positive, then

$$\sum_{A \le j \le B} f(j) \le f(A) + \int_A^B f(u)\, du. \tag{3.2}$$

For $\sigma_0 > 1/2$, we bound trivially the diagonal term:

$$D(\sigma_0, T, H) \le \zeta(2\sigma_0). \tag{3.3}$$

We interchange summation order in the off-diagonal terms $E_1(\sigma_0, T, H)$ and use the fact that $\int_u^v \cos(at)\, dt \le 2/a$ when $a \ne 0$:

$$E_1(\sigma_0, T, H) \le \frac{4}{T-H} \sum_{1 \le n < m < T} \frac{(nm)^{-\sigma_0}}{\log(m/n)}.$$

We further use the fact that, for $\lambda > 1$ and $\sigma < 1$, $\frac{1}{\log \lambda} \le 1 + \frac{\lambda^{1-\sigma}}{\lambda - 1}$. Taking $\lambda = m/n$, we obtain

$$E_1(\sigma_0, T, H) \le \frac{4}{T-H} \sum_{1 \le n < m < T} (nm)^{-\sigma_0} + \frac{4}{T-H} \sum_{1 \le n < m < T} \frac{m^{1-2\sigma_0}}{m-n}. \tag{3.4}$$

For the first sum, we complete the square:

$$\begin{aligned} \sum_{1 \le n < m < T} (nm)^{-\sigma_0} &= \frac{1}{2}\Big(\sum_{k<T} k^{-\sigma_0}\Big)^2 - \frac{1}{2} \sum_{k<T} k^{-2\sigma_0} \\ &= \frac{1}{2}\Big(\sum_{k<T} k^{-\sigma_0}\Big)^2 - \frac{1}{2}\Big(\zeta(2\sigma_0) - \sum_{k \ge T} k^{-2\sigma_0}\Big), \end{aligned}$$

and use (3.2) with $f(t) = t^{-\sigma_0}$ and $f(t) = t^{-2\sigma_0}$ to bound the resulting sums. We obtain

$$(3.5)\quad \sum_{1\le n<m<T} (nm)^{-\sigma_0} \le \frac{T^{2(1-\sigma_0)}}{2(1-\sigma_0)^2} - \frac{\sigma_0 T^{1-\sigma_0}}{(1-\sigma_0)^2} + \frac{\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0) - \frac{T^{1-2\sigma_0}}{2(1-2\sigma_0)} + \frac{T^{-2\sigma_0}}{2}.$$

We consider $k = m - n$ and separate variables in the second sum of (3.4) and use (3.2), with $f(t) = t^{1-2\sigma_0}$ and $f(t) = t^{-1}$, to bound the resulting sums:

$$(3.6)\quad \sum_{1\le n<m<T} \frac{m^{1-2\sigma_0}}{m-n} \le \Big(\sum_{1\le m<T} m^{1-2\sigma_0}\Big)\Big(\sum_{1\le k<T} k^{-1}\Big)$$
$$\le \frac{(\log T)T^{2(1-\sigma_0)}}{2(1-\sigma_0)} + \frac{T^{2(1-\sigma_0)}}{2(1-\sigma_0)} + \log T + 1 - \frac{1}{2(1-\sigma_0)} - \frac{\log T}{2(1-\sigma_0)}.$$

Combining (3.4)–(3.6), we obtain

$$E_1(\sigma_0, T, H) \le \frac{4T}{T-H}\bigg(\frac{(\log T)T^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0-1}{2(1-\sigma_0)}\frac{\log T}{T} + \bigg(\frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0)\bigg)\frac{1}{T} + \frac{2-\sigma_0}{2(1-\sigma_0)^2}T^{1-2\sigma_0} - \frac{\sigma_0 T^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{T^{-2\sigma_0}}{2(2\sigma_0-1)} + \frac{1}{2}T^{-2\sigma_0-1}\bigg).$$

We denote

$$E_{11}(\sigma_0, T) = \frac{(\log T)T^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0-1}{2(1-\sigma_0)}\frac{\log T}{T},$$
$$E_{12}(\sigma_0, T) = \bigg(\frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0)\bigg)\frac{1}{T},$$
$$E_{13}(\sigma_0, T) = \frac{2-\sigma_0}{2(1-\sigma_0)^2}T^{1-2\sigma_0} - \frac{\sigma_0 T^{-\sigma_0}}{(1-\sigma_0)^2},$$
$$E_{14}(\sigma_0, T) = \frac{T^{-2\sigma_0}}{2(2\sigma_0-1)} + \frac{1}{2}T^{-2\sigma_0-1},$$

and we now study their behavior with respect to $T \ge H_0$. It is immediate that E_{14} decreases with T. Considering the fact that $\frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{1}{2}\zeta(2\sigma_0)$ changes sign at $\sigma_0 = 0.679785\ldots$, we obtain

$$E_{12}(\sigma_0, T) \le \max(0, E_{12}(\sigma_0, H_0)).$$

For $0.5208 < \sigma_0 < 1$, we find

$$\frac{\partial E_{11}(\sigma_0,T)}{\partial T} = \frac{-(T^{2(1-\sigma_0)}-1)((2\sigma_0-1)(\log T)-1)+2(1-\sigma_0)}{2(1-\sigma_0)T^2} \le 0,$$

and, when $\sigma_0 \le 0.9723$,

$$\frac{\partial E_{13}(\sigma_0,T)}{\partial T} = \left(-\frac{(2-\sigma_0)(2\sigma_0-1)}{2}T^{1-\sigma_0}+\sigma_0^2\right)\frac{T^{-1-\sigma_0}}{(1-\sigma_0)^2} \le 0.$$

Thus $E_{11}(\sigma_0,T)$ and $E_{13}(\sigma_0,T)$ decrease with $T \ge H_0$. We conclude that, for $T \ge H_0$ and $0.5208 \le \sigma_0 \le 0.9723$,

$$(3.7)\qquad \begin{aligned} &E_1(\sigma_0,T,H)\\ &\le \frac{4H_0}{H_0-H}\Bigg(\frac{(\log H_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)} - \frac{2\sigma_0-1}{2(1-\sigma_0)}\frac{\log H_0}{H_0}\\ &\quad + \frac{\max\left(0, \frac{1-3\sigma_0+3\sigma_0^2}{2(1-\sigma_0)^2} - \frac{\zeta(2\sigma_0)}{2}\right)}{H_0}\\ &\quad + \frac{(2-\sigma_0)H_0^{1-2\sigma_0}}{2(1-\sigma_0)^2} - \frac{\sigma_0 H_0^{-\sigma_0}}{(1-\sigma_0)^2} + \frac{H_0^{-2\sigma_0}}{2(2\sigma_0-1)} + \frac{H_0^{-2\sigma_0-1}}{2}\Bigg).\end{aligned}$$

Theorem 1.2 gives

$$(3.8)\qquad \begin{aligned} E_2(\sigma_0,T,H) &\le c_0^2\frac{1}{T-H}\int_H^T t^{-2\sigma_0}\,dt\\ &\le \frac{c_0^2}{2\sigma_0-1}\,\frac{H^{-(2\sigma_0-1)}-H_0^{-(2\sigma_0-1)}}{H_0-H}.\end{aligned}$$

We use the Cauchy–Schwarz inequality to bound E_3:

$$(3.9)\qquad \begin{aligned} &E_3(\sigma_0,T,H)\\ &\le 2\left(\frac{1}{T-H}\int_H^T |\,\mathfrak{Re}\,R(s)|^2\,dt\right)^{1/2}\left(\frac{1}{T-H}\int_H^T\Big|\sum_{1\le n<t}\frac{1}{n^{\sigma_0+it}}\Big|^2\,dt\right)^{1/2}\\ &\le 2\sqrt{E_2(\sigma_0,T,H)(D(\sigma_0,T,H)+E_1(\sigma_0,T,H))}\\ &\le 2\sqrt{\epsilon_2(\sigma_0,H)(\zeta(2\sigma_0)+\epsilon_1(\sigma_0,H))}.\end{aligned}$$

The definitions of $\epsilon_1,\epsilon_2,\epsilon_3$ follow from (3.7)–(3.9). The proof is completed by putting together (3.1), (3.3), (3.7), (3.8), (3.9), and by applying the following bound for concave functions:

$$\int_H^T \log|\zeta(\sigma_0+it)|\,dt \le \frac{T-H}{2}\log\left(\frac{1}{T-H}\int_H^T |\zeta(\sigma_0+it)|^2\,dt\right). \blacksquare$$

4. A lower bound for $\log|\zeta(s)|$ when $\sigma > 1$

LEMMA 4.1. *Let* $2 \le H \le T$ *and* $\sigma_1 = 1.5002$. *Then*

$$\int_H^T \log|\zeta(\sigma_1+it)|\,dt \ge -\mathcal{E}_2 \qquad \textit{with } \mathcal{E}_2 = 1.7655. \tag{4.1}$$

Proof. Let $s = \sigma_1 + it$. It follows from the Euler product that

$$\log|\zeta(s)| = \mathfrak{Re} \sum_{n\ge 2} \frac{\Lambda(n)}{(\log n)n^s}.$$

Thus

$$\begin{aligned}\int_H^T \log|\zeta(\sigma_1+it)|\,dt &= \sum_{n\ge 2} \frac{\Lambda(n)\big(\sin(T\log n) - \sin(H\log n)\big)}{(\log n)^2 n^{\sigma_1}}\\ &\ge -2\sum_{n\ge 2}\frac{\Lambda(n)}{(\log n)^2 n^{\sigma_1}}.\end{aligned}$$

We truncate the sum at $N_0 = 10^3$ and bound the tail:

$$\sum_{n>N_0} \frac{\Lambda(n)}{(\log n)^2 n^{\sigma_1}} \le \frac{1}{(\log N_0)^2}\bigg(-\frac{\zeta'}{\zeta}(\sigma_1) - \sum_{n\le N_0}\frac{\Lambda(n)}{n^{\sigma_1}}\bigg).$$

We obtain

$$\int_H^T \log|\zeta(\sigma_1+it)|dt \ge -2\bigg(\frac{-\frac{\zeta'}{\zeta}(\sigma_1)}{(\log N_0)^2} + \sum_{n\le N_0}\frac{\Lambda(n)}{n^{\sigma_1}}\bigg(\frac{1}{(\log n)^2} - \frac{1}{(\log N_0)^2}\bigg)\bigg),$$

and a numerical calculation with Maple gives the value for the above left term. ∎

5. Explicit bounds for $\int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau+iT)\,d\tau$

LEMMA 5.1. *Let* $\eta = 0.0001$, $\sigma_1 = 3/2 + 2\eta = 1.5002$. *Let* σ_0, T, H *satisfy* $\sigma_0 < \sigma_1$, $2 \le H \le T$. *Then*

$$\int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau+iT)\,d\tau - \int_{\sigma_0}^{\sigma_1}\arg\zeta(\tau+iH)\,d\tau \le \mathcal{E}_3(\sigma_0)\log(HT) + \mathcal{E}_4(\sigma_0,H)$$

with

$$\mathcal{E}_3(\sigma_0) = \frac{\pi(1+2\eta)(\sigma_1-\sigma_0)}{4\log 2} \tag{5.1}$$

and

$$(5.2)\qquad \mathcal{E}_4(\sigma_0, H) = \frac{\pi(\sigma_1-\sigma_0)}{\log 2} \times \log\left(3\frac{H+3(1+\eta)}{H-(1+2\eta)}\left(\frac{3(1+\eta)/H+1}{2\pi}\right)^{(1+2\eta)/2}\frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2}\right).$$

It suffices to bound an integral of the form

$$\int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau+it)\,d\tau$$

with $t \geq H$. We only make use of the convexity bound for $\zeta(s)$.

Proof of Lemma 5.1. Let $\omega \in \mathbb{C}$ and $N \in \mathbb{N}$. Following Rosser's modification of Backlund's trick ([1, equation (32)] and [24, p. 223]), we introduce $f_t(\omega) = \frac{1}{2}(\zeta(\omega+it)^N + \zeta(\omega-it)^N)$. We denote by n the number of real zeros of $f_t(\tau) = \mathfrak{Re}\,\zeta(\tau+it)^N$ in the interval $\sigma_0 < \tau < \sigma_1$. The interval is split into $n+1$ subintervals and on each of them $\arg\zeta(\tau+it)^N$ changes by at most π. Thus

$$(5.3)\qquad \left|\int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau+it)\,d\tau\right| = \frac{1}{N}\left|\int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau+it)^N\,d\tau\right| \leq \frac{(\sigma_1-\sigma_0)(n+1)\pi}{N}.$$

We write $n(r)$ for the number of zeros of f_t in the circle centered at $1+\eta+it$ and with radius r. For $r \geq 1/2+\eta$, the segment $[\sigma_0, \sigma_1]$ is contained in $[1+\eta-r, 1+\eta+r]$, thus $n \leq n(r)$. The following version of Jensen's formula [28, p. 137, equation (2)]:

$$\log|f_t(1+\eta)| + \int_0^{1+2\eta} \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} \log|f_t(1+\eta+(1+2\eta)e^{i\theta})|\,d\theta,$$

allows us to deduce an upper bound for n:

$$(5.4)\qquad n \leq \frac{1}{\log 2}\int_0^{1+2\eta}\frac{n(r)}{r}\,dr \leq \frac{1}{2\pi\log 2}\int_{-\pi/2}^{3\pi/2} \log|f_t(1+\eta+(1+2\eta)e^{i\theta})|\,d\theta - \frac{\log|f_t(1+\eta)|}{\log 2}.$$

We write $\zeta(1+\eta+it) = Re^{i\phi}$. Thus $f_t(1+\eta) = \mathfrak{Re}(\zeta(1+\eta+it)^N) = R^N\cos(N\phi)$. We choose a sequence of N's with $\lim_{N\to\infty} N\phi = 0 \pmod{2\pi}$.

Thus

$$\begin{aligned}
\log|f_t(1+\eta)| &= N\log((1+o(1))R) \\
&= N\log((1+o(1))|\zeta(1+\eta+it)|) \\
&\geq N\log\left(\frac{\zeta(2(1+\eta))}{\zeta(1+\eta)}\right) + o_N(1),
\end{aligned} \tag{5.5}$$

where $o_N(1) \to 0$ when $N \to \infty$.

We now split the integral in the left term of inequality (5.4) depending on the sign of $\cos\theta$. For $\theta \in (-\pi/2, \pi/2)$, $\mathfrak{Re}(1+\eta+(1+2\eta)e^{i\theta} \pm it) > 1+\eta > 1$, and we use the trivial bound

$$|\zeta(1+\eta+(1+2\eta)e^{i\theta} \pm it)| \leq \zeta(1+\eta),$$

giving

$$\int_{-\pi/2}^{\pi/2} \log|f_t(1+\eta+(1+2\eta)e^{i\theta})|\, d\theta \leq N\pi\log(\zeta(1+\eta)). \tag{5.6}$$

For $\theta \in (\pi/2, 3\pi/2)$, we use Rademacher's bound [17, equation (7.4)]

$$|\zeta(s)| \leq 3\frac{|1+s|}{|1-s|}\left(\frac{|1+s|}{2\pi}\right)^{(1+\eta-\mathfrak{Re}\, s/2)}\zeta(1+\eta)$$

with $s = 1+\eta+(1+2\eta)e^{i\theta} \pm it$. Since

$$|1+s| \leq t+3(1+\eta), \quad |1-s| \geq |\mathfrak{Im}\, s| \geq t-(1+2\eta), \quad 0 \leq 1+\eta-\mathfrak{Re}\, s \leq 1+2\eta,$$

we have

$$\begin{aligned}
&\int_{\pi/2}^{3\pi/2} \log|f_t(1+\eta+(1+2\eta)e^{i\theta})|\, d\theta \\
&\qquad \leq N\pi\log\left(3\frac{t+3(1+\eta)}{t-(1+2\eta)}\left(\frac{t+3(1+\eta)}{2\pi}\right)^{(1+2\eta)/2}\zeta(1+\eta)\right).
\end{aligned} \tag{5.7}$$

From (5.4)–(5.7), we deduce

(5.8)

$$\begin{aligned}
n &\leq \frac{N}{2\log 2}\log\left(3\frac{t+3(1+\eta)}{t-(1+2\eta)}\left(\frac{3(1+\eta)+t}{2\pi}\right)^{(1+2\eta)/2}\frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2}\right) + o_N(1) \\
&\leq \frac{N(1+2\eta)}{4\log 2}\log t \\
&\quad + \frac{N}{2\log 2}\log\left(3\frac{t+3(1+\eta)}{t-(1+2\eta)}\left(\frac{3(1+\eta)/t+1}{2\pi}\right)^{(1+2\eta)/2}\frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2}\right) + o_N(1).
\end{aligned}$$

Combining this with (5.3) and letting $N \to \infty$, we obtain

$$\Big|\int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + it)\, d\tau\Big| \le \frac{\pi(1+2\eta)(\sigma_1 - \sigma_0)}{4\log 2} \log t$$
$$+ \frac{\pi(\sigma_1 - \sigma_0)}{2\log 2} \log\bigg(3\frac{t+3(1+\eta)}{t-(1+2\eta)}\bigg(\frac{3(1+\eta)/t+1}{2\pi}\bigg)^{(1+2\eta)/2} \frac{\zeta(1+\eta)^4}{\zeta(2(1+\eta))^2}\bigg).$$

Observing that the second term decreases with $t \ge H$ completes the proof. ■

6. Explicit upper bounds for $N(\sigma, T)$; proof of Theorem 1.1. We recall that σ, σ_0, σ_1, H, and T satisfy (1.11). We consider the number $N(\sigma,T)$ of zeros $\varrho = \beta + i\gamma$ of zeta in the rectangle $\sigma < \beta < 1$ and $H < \gamma < T$. Since $N(\sigma, H) = 0$, we have

$$N(\sigma,T) \le \frac{1}{\sigma - \sigma_0} \int_{\sigma_0}^{\sigma_1} (N(\tau,T) - N(\tau,H))\, d\tau. \tag{6.1}$$

It follows from a lemma of Littlewood (see [30, (9.9.1)]) that

$$\int_{\sigma_0}^{\sigma_1} (N(\tau,T) - N(\tau,H))\, d\tau = -\frac{1}{2\pi i} \int_{\mathcal{R}} \log \zeta(s)\, ds,$$

where $\mathcal{R}$ is the rectangle with vertices $\sigma_0 + iH$, $\sigma_1 + iH$, $\sigma_1 + iT$, and $\sigma_0 + iT$. Thus

$$N(\sigma,T) \le \frac{1}{2\pi(\sigma - \sigma_0)} \bigg(\int_H^T \log|\zeta(\sigma_0 + it)|\, dt + \int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau + iT)\, d\tau - \int_{\sigma_0}^{\sigma_1} \arg \zeta(\tau + iH)\, d\tau - \int_H^T \log|\zeta(\sigma_1 + it)|\, dt\bigg). \tag{6.2}$$

We use Theorem 1.4, Lemma 4.1, and Lemma 5.1 respectively to bound these integrals:

$$\int_H^T \log|\zeta(\sigma_0 + it)|\, dt \le \frac{T-H}{2} \log(\zeta(2\sigma_0) + \mathcal{E}_1(\sigma_0, H)),$$
$$-\int_H^T \log|\zeta(\sigma_1 + it)|\, dt \le \mathcal{E}_2,$$
$$\int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau + iT)\, d\tau - \int_{\sigma_0}^{\sigma_1} \arg\zeta(\tau + iH)\, d\tau \le \mathcal{E}_3(\sigma_0) \log(HT) + \mathcal{E}_4(\sigma_0, H),$$

where the $\mathcal{E}_i$'s are defined respectively in (1.10), (4.1), (5.1), and (5.2). We obtain

$$N(\sigma,T) \le b_1(\sigma_0,H)(T-H) + b_2(\sigma_0,H)\log(TH) + b_3(\sigma_0,H)$$

with

$$\begin{aligned} b_1(\sigma_0,H) &= \frac{\log(\zeta(2\sigma_0)+\mathcal{E}_1(\sigma_0,H))}{4\pi(\sigma-\sigma_0)}, \\ b_2(\sigma_0,H) &= \frac{\mathcal{E}_3(\sigma_0)}{2\pi(\sigma-\sigma_0)}, \\ b_3(\sigma_0,H) &= \frac{\mathcal{E}_2+\mathcal{E}_4(\sigma_0,H)}{2\pi(\sigma-\sigma_0)}. \end{aligned} \tag{6.3}$$

Table 1. $N(\sigma,T) \le b_1(T-H) + b_2\log(TH) + b_3$ and $N(\sigma,T) \le c_1T + c_2\log T + c_3$ for $T \ge H_0$

σ	σ_0	H	$b_1 = c_1$	$b_2 = c_2$	b_3	c_3
0.60	0.5229	19 399	4.2288	2.2841	333	−81 673
0.65	0.5552	40 105	2.4361	1.7965	262	−97 414
0.70	0.5873	91 470	1.4934	1.4609	213	−136 370
0.75	0.6096	169 119	1.0031	1.1442	167	−169 449
0.76	0.6136	188 973	0.9355	1.0921	160	−176 604
0.77	0.6175	210 645	0.8750	1.0437	153	−184 134
0.78	0.6213	234 346	0.8205	0.9986	146	−192 120
0.79	0.6250	260 321	0.7714	0.9566	140	−200 644
0.80	0.6287	288 853	0.7269	0.9176	134	−209 795
0.81	0.6324	320 270	0.6864	0.8812	129	−219 667
0.82	0.6361	354 951	0.6495	0.8473	124	−230 367
0.83	0.6398	393 341	0.6156	0.8157	119	−242 009
0.84	0.6435	435 955	0.5846	0.7862	115	−254 724
0.85	0.6472	483 393	0.5561	0.7586	111	−268 658
0.86	0.6510	536 357	0.5297	0.7327	107	−283 978
0.87	0.6548	595 670	0.5053	0.7085	104	−300 872
0.88	0.6587	662 291	0.4827	0.6857	101	−319 555
0.89	0.6626	737 343	0.4617	0.6644	97	−340 272
0.90	0.6667	822 142	0.4421	0.6443	95	−363 301
0.91	0.6708	918 225	0.4238	0.6253	92	−388 959
0.92	0.6750	1 027 390	0.4066	0.6075	89	−417 606
0.93	0.6793	1 151 729	0.3905	0.5906	87	−449 647
0.94	0.6838	1 293 683	0.3754	0.5747	84	−485 543
0.95	0.6883	1 456 079	0.3612	0.5596	82	−525 807
0.96	0.6930	1 642 194	0.3478	0.5452	80	−571 018
0.97	0.6977	1 855 803	0.3352	0.5316	78	−621 815
0.98	0.7026	2 101 249	0.3232	0.5187	76	−678 911
0.99	0.7077	2 383 498	0.3118	0.5063	74	−743 087

It follows that

$$N(\sigma, T) \leq c_1 T + c_2 \log T + c_3,$$
$$\text{with } c_1 = b_1,\ c_2 = b_2,\ c_3 = -b_1 H + b_2 \log H + b_3. \ \blacksquare$$

Table 1 records the values of the b_i's and c_i's computed for $H_0 = 3.061 \cdot 10^{10}$. Specific choices of parameters σ_0 and H are made in order to obtain good bounds for $N(\sigma, T)$ when T is asymptotically large. The values of σ_0, b_1, c_1, b_2, and c_2 displayed in the table are rounded up to four decimal places. We take the ceiling of the values of H, b_3, and c_3.

Acknowledgements. I would like to thank Olivier Ramaré for his comments on this article.

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Received on 25.1.2013
and in revised form on 19.5.2013 (7329)