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The Diophantine equation $x(x+1)\ldots(x+(m-1))+r=y^n$

by

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Dedicated to Robert Tijdeman on the occasion of his 60th birthday

1. Introduction. Erdős and Selfridge [7] proved that a product of consecutive integers can never be a perfect power. That is, the equation $x(x+1)\ldots(x+(m-1))=y^n$ has no solutions in positive integers x, y, m, n with $m,n>1$. A natural problem is to study the equation

$$x(x+1)(x+2)\ldots(x+(m-1))+r=y^n \tag{1}$$

with a *non-zero* integral (or rational) parameter r. M. J. Cohen [6] proved that (1) has finitely many solutions with $m=n$, and Yuan Ping-zhi [13] used the classical theorems of Baker and Schinzel–Tijdeman to show that, with some obvious exceptions, there are at most finitely many solutions with a fixed m. (See Theorem 1.2 below.) Some special cases were completely solved by Abe [1] and Alemu [2].

In this paper we prove that (1) has finitely many solutions (x,y,m,n) when r is not a perfect power.

THEOREM 1.1. *Let r be a non-zero rational number which is not a perfect power in $\mathbb{Q}$. Then* (1) *has at most finitely many solutions* (x,y,m,n) *satisfying*

$$x,m,n\in\mathbb{Z},\qquad y\in\mathbb{Q},\qquad m,n>1. \tag{2}$$

Moreover, *all the solutions can be explicitly determined.*

We deduce Theorem 1.1 from three more particular results, one of which is the above-mentioned result of Yuan. First of all, let us display two infinite series of solutions which occur for two special values of r. For $r=1/4$ we have the solutions

$$x\in\mathbb{Z},\qquad y=\pm(x+1/2),\qquad m=n=2. \tag{3}$$

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For $r = 1$ we have infinitely many solutions

$$x \in \mathbb{Z}, \quad y = \pm(x^2 + 3x + 1), \quad m = 4, \quad n = 2. \tag{4}$$

In the following theorem m is fixed, and we solve (1) in x, y, n.

THEOREM 1.2 (Yuan). *Let r be a non-zero rational number and $m > 1$ an integer.*

1. *Assume that $(m, r) \notin \{(2, 1/4), (4, 1)\}$. Then* (1) *has at most finitely many solutions (x, y, n) satisfying*

$$x, n \in \mathbb{Z}, \quad y \in \mathbb{Q}, \quad n > 1, \tag{5}$$

and all the solutions can be explicitly determined.

2. *Assume that $(m, r) = (2, 1/4)$ or $(m, r) = (4, 1)$. Then, besides the solutions from* (3), *respectively* (4), *equation* (1) *has at most finitely many solutions (x, y, n) satisfying* (5), *and all these solutions can be explicitly determined.*

Yuan formulates his result in a slightly different (and non-equivalent) form, and his proof is about three pages long. For the convenience of the reader, we give in Section 2 a concise proof of Theorem 1.2, following Yuan's argument with some changes.

Theorem 1.2 implies that n is bounded in terms of m and r. It turns out that, when $r \neq \pm 1$, it is bounded in terms of r only.

THEOREM 1.3. *Let r be a rational number distinct from 0 and ± 1. Then there exists an effective constant $C(r)$ with the following property. If (x, y, m, n) is a solution of* (1) *satisfying* (2) *then $n \le C(r)$.*

Now change the roles: n is fixed, m is variable.

THEOREM 1.4. *Let r be a non-zero rational number and $n > 1$ an integer. Assume that r is not an nth power in $\mathbb{Q}$. Then* (1) *has at most finitely many solutions (x, y, m) satisfying*

$$x, m \in \mathbb{Z}, \quad y \in \mathbb{Q}, \quad m > 1, \tag{6}$$

and all the solutions can be explicitly determined.

In [8] this theorem is extended (non-effectively) to the equation $x(x+1)\ldots(x+(m-1)) = g(y)$, where $g(y)$ is an arbitrary irreducible polynomial.

Theorem 1.1 is an immediate consequence of Theorems 1.3 and 1.4. Indeed, assume that r is not a perfect power. Theorem 1.3 implies that n is effectively bounded in terms of r. In particular, we have finitely many possible n. Theorem 1.4 implies that for each n there are at most finitely many possibilities of (x, y, m). This proves Theorem 1.1.

REMARK 1.5. It is interesting to compare (1) with the classical equation of Catalan $x^m - y^n = 1$. This equation has been effectively solved by

Tijdeman [12], and recently Mihăilescu [9] (see also [5]) solved it completely. However, much less is known about the equation $x^m - y^n = r$ for $r \neq \pm 1$. In contrast, for equation (1) the case $r = \pm 1$ seems to be the most difficult.

2. Proof of Theorem 1.2. In this section $m > 1$ is an integer and $f_m(x) = x(x+1)\dots(x+m-1)$.

PROPOSITION 2.1. *Let* λ *be a complex number. Then the polynomial* $f_m(x) - \lambda$ *has at least two simple roots if*

$$(m,\lambda) \notin \left\{ \left(2, -\frac{1}{4}\right), \left(3, \frac{\pm 2}{3\sqrt{3}}\right), (4,-1) \right\}.$$

It has at least three simple roots if $m > 2$ *and*

$$(m,\lambda) \notin \left\{ \left(3, \frac{\pm 2}{3\sqrt{3}}\right), (4,-1), \left(4, \frac{9}{16}\right), \left(6, \frac{16(10 \pm 7\sqrt{7})}{27}\right) \right\}.$$

Proof. By the theorem of Rolle, $f_m'(x)$ has $m-1$ distinct real roots. Hence $f_m(x) - \lambda$ may have roots of order at most 2. Beukers, Shorey and Tijdeman [4, Proposition 3.4] proved that for even m at most two double roots are possible, and for odd m only one double root may occur. It follows that for $m \notin \{2,3,4,6\}$ the polynomial $f(x) - \lambda$ has at least three simple roots.

We are left with $m \in \{2,3,4,6\}$. Since the polynomial $f(x) - \lambda$ has multiple roots if and only if λ is a stationary value of $f(x)$ (that is, $\lambda = f(\alpha)$ where α is a root of $f'(x)$), it remains to determine the stationary values of each of the polynomials f_2, f_3, f_4, f_6 and count the simple roots of corresponding translates. The details are routine and we omit them. ∎

COROLLARY 2.2. *Let* r *be a non-zero rational number. The polynomial* $f_m(x) + r$ *has at least two simple roots if* $(m,r) \notin \{(2,1/4),(4,1)\}$. *It has at least three simple roots if* $m > 2$ *and* $(m,r) \notin \{(4,1),(4,-9/16)\}$. ∎

We shall use the classical results of Baker [3] and of Schinzel–Tijdeman [10] on the superelliptic equation

$$f(x) = y^n. \tag{7}$$

In Baker's theorem $n \in \mathbb{Z}$ is fixed.

THEOREM 2.3 (A. Baker). *Assume that* $f(x) \in \mathbb{Q}[x]$ *has at least three simple roots and* $n > 1$, *or* $f(x)$ *has at least two simple roots and* $n > 2$. *Then* (7) *has only finitely many solutions in* $x \in \mathbb{Z}$ *and* $y \in \mathbb{Q}$, *and the solutions can be effectively computed.* ∎

(A non-effective version of this theorem goes back to Siegel [11].)

In the theorem of Schinzel and Tijdeman n becomes a variable.

Theorem 2.4 (Schinzel and Tijdeman). *Let $f(x) \in \mathbb{Q}[x]$ be a polynomial having at least two distinct roots. Then there exists an effective constant $N(f)$ such that any solution of* (7) *in $x, n \in \mathbb{Z}$, $y \in \mathbb{Q}$ satisfies $n \le N(f)$.* ∎

Corollary 2.5. *Let $f(x) \in \mathbb{Q}[x]$ be a polynomial having at least three simple roots. Then* (7) *has at most finitely many solutions in $x, n \in \mathbb{Z}$, $y \in \mathbb{Q}$ satisfying $n > 1$. If $f(x)$ has two simple roots then* (7) *has only finitely many solutions with $n > 2$. In both cases the solutions can be explicitly determined.* ∎

Proof of Theorem 1.2. Corollaries 2.2 and 2.5 imply that the theorem is true if $m > 2$ and $(m, r) \notin \{(4, 1), (4, -9/16)\}$. It remains to consider the cases $m = 2$, $(m, r) = (4, 1)$ and $(m, r) = (4, -9/16)$.

Case 1: $m = 2$, $r \neq 1/4$. In this case $f_2(x) + r$ has two simple roots, and Corollary 2.5 implies that $f_2(x) + r = y^n$ has at most finitely many solutions with $n > 2$ (and these solutions can be explicitly determined). We are left with the equation $x(x + 1) + r = y^2$, which is equivalent to the equation $(x + 1/2 + y)(x + 1/2 - y) = 1/4 - r$, having finitely many solutions.

Case 2: $m = 2$, $r = 1/4$. In this case we have the equation $(x + 1/2)^2 = y^n$. It has infinitely many solutions given by (3) and no other solutions. Indeed, if (x, y, n) is a solution with $n > 2$ then $x + 1/2$ is a perfect power, which is impossible because its denominator is 2.

Case 3: $m = 4$, $r = 1$. In this case we have the equation $(x^2 + 3x + 1)^2 = y^n$. It has infinitely many solutions given by (4) and only finitely many other solutions, all of which can be explicitly determined.

Indeed, let (x, y, n) be a solution with $n > 2$. If n is odd, then y is a perfect square: $y = z^2$ and $x^2 + 3x + 1 = \pm z^n$. Since $x^2 + 3x + 1$ has two simple roots, the latter equation has, by Corollary 2.5, only finitely many solutions with $n \ge 3$.

If $n = 2n_1$ is even then $x^2 + 3x + 1 = \pm y^{n_1}$, which has finitely many solutions with $n_1 \ge 3$. We are left with $n = 4$, in which case $x^2 + 3x + 1 = \pm y^2$. The equation $x^2 + 3x + 1 = y^2$ is equivalent to $(2x + 3 + 2y)(2x + 3 - 2y) = 5$, which has finitely many solutions. The equation $x^2 + 3x + 1 = -y^2$ is equivalent to $(2x + 3)^2 + 4y^2 = 5$, which has finitely many solutions as well.

Case 4: $m = 4$, $r = -9/16$. In this case we have the equation $(x+3/2)^2 \times (x^2 + 3x - 1/4) = y^n$. Since its left-hand side has two simple roots, this equation has, by Corollary 2.5, only finitely many solutions with $n > 2$. We are left with the equation $(x + 3/2)^2(x^2 + 3x - 1/4) = y^2$, which is equivalent to the equation $16(x^2 + 3x + 1 - y)(x^2 + 3x + 1 + y) = 25$, having only finitely many solutions.

Theorem 1.2 is proved. ∎

3. Proof of Theorems 1.3 and 1.4. Let α be a non-zero rational number and p a prime number. Recall that $\mathrm{ord}_p(\alpha)$ is the integer t such that $p^{-t}\alpha$ is a p-adic unit. The proofs of both theorems rely on the following simple observation.

PROPOSITION 3.1. *Let p be a prime number and $t=\mathrm{ord}_p(r)$. Then for any solution (x,y,m,n) of* (1), *satisfying* (2), *one has either $m<(t+1)p$ or $n\,|\,t$.*

Proof. Assume that $m\ge (t+1)p$. Then

$$\mathrm{ord}_p(x(x+1)(x+2)\dots(x+(m-1)))\ge t+1.$$

Hence

$$\mathrm{ord}_p(x(x+1)(x+2)\dots(x+(m-1))+r)=t,$$

that is, $\mathrm{ord}_p(y^n)=t$, which implies that $n\,|\,t$. ∎

Proof of Theorem 1.3. Since $r\ne\pm1$, there exists a prime number p such that $t=\mathrm{ord}_p(r)\ne0$. Theorem 1.2 implies that for every $m>1$ there exists an effective constant $N(m)$ such that for any solution of (1) satisfying (2) we have $n\le N(m)$. Put $C'(r)=\max\{N(m):2\le m<(t+1)p\}$ if $t>0$ and $C'(r)=0$ if $t<0$. Then $n\le C'(r)$ when $m<(t+1)p$, and $n\le|t|$ by Proposition 3.1 when $m\ge(t+1)p$. Thus, in any case $n\le C(r):=\max\{C'(r),|t|\}$. ∎

Proof of Theorem 1.4. The proof splits into two cases.

CASE 1: *there is a prime p such that n does not divide $t=\mathrm{ord}_p(r)$.* In this case Proposition 3.1 implies that $m\le(t+1)p$. Also, $(n,r)\notin\{(2,1/4),(4,1)\}$, because in both these cases r is an nth power. Now Theorem 1.2 implies that we may have only finitely many solutions.

CASE 2: *n is even and $r=-r_1^n$, where $r_1\in\mathbb{Q}$.* Write $z=(y/r_1)^{n/2}$. Let p be prime number congruent to 3 mod 4 and such that $\mathrm{ord}_p(r)=0$. If $m\ge p$ then

$$\mathrm{ord}_p(1+z^2)=\mathrm{ord}_p(r^{-1}x(x+1)\dots(x+m-1))>0,$$

which implies that -1 is a quadratic residue $\bmod\,p$, a contradiction. Thus, $m<p$ and Theorem 1.2 again implies that we may have only finitely many solutions. ∎

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