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Sums of nonnegative multiplicative functions over integers without large prime factors I

by

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1. Introduction. Integers whose prime factors are relatively small arise naturally in various areas of number theory. Specifically, if $P(n)$ denotes the largest prime divisor of an integer $n > 1$, with $P(1) = 1$, then positive integers n with $P(n) \le y$ for a number y are of special interest. The well known counting function of integers not exceeding x with prime factors at most y is

$$\Psi(x,y) := \sum_{\substack{n \le x \\ P(n) \le y}} 1,$$

the summatory function of the multiplicative function defined by the relation

$$h(n) = \begin{cases} 0, & P(n) > y, \\ 1, & P(n) \le y. \end{cases}$$

The behavior of $\Psi(x,y)$ has been the subject of numerous studies. It has been documented in the survey papers of Norton [N], Moree [M], and most recently Hildebrand and Tenenbaum [HT]. There is also a comprehensive introduction and overview of main results on the asymptotic behavior of $\Psi(x,y)$ in Tenenbaum [Tn] (Chapter III.5). All this work goes back to the following result due to Dickman [Di]. He showed, with $u = \log x/\log y$, that

$$\Psi(x,y) \sim x\varrho(u), \qquad x \to \infty,$$

where $\varrho(u)$ satisfies

$$u\varrho'(u) = -\varrho(u-1), \qquad u > 1,$$

subject to the initial conditions

$$\begin{aligned} \varrho(u) &= 0, \qquad u < 0, \\ \varrho(u) &= 1, \qquad 0 \le u \le 1. \end{aligned}$$

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The function ϱ is now known as Dickman's function. Later, de Bruijn [dB] obtained results implying that for any fixed $\varepsilon > 0$,

$$\Psi(x,y) = x\varrho(u)\left\{1 + O\left(\frac{\log(u+1)}{\log y}\right)\right\} \tag{1.1}$$

uniformly within the range

$$1 \le u \le (\log x)^{3/8-\varepsilon}$$

as $x \to \infty$. De Bruijn used the so-called Buchstab identity,

$$\Psi(x,y) = \Psi(x,z) + \sum_{y<p\le z} \Psi\left(\frac{x}{p}, p\right), \qquad 1 \le z \le y \le x. \tag{1.2}$$

However, an inductive argument of Hildebrand [Hi] has turned out to be more effective. He showed that (1.1) holds uniformly for

$$x \ge 3, \qquad 1 \le u \le \log x/(\log\log x)^{5/3+\varepsilon} \tag{1.3}$$

using a Chebyshev-type functional equation for $\Psi(x,y)$, namely,

$$\Psi(x,y)\log x = \int_1^x \Psi(t,y)\frac{dt}{t} + \sum_{\substack{p^k \le x\\ p\le y}} \Psi\left(\frac{x}{p^k}, y\right)\log p, \qquad 1 \le y < x; \tag{1.4}$$

the advantage of (1.4) over the Buchstab identity is that the second argument is the same for all the Ψ functions. Our goal in the present paper is to prove an asymptotic estimate for an incomplete sum of a general class of nonnegative multiplicative functions over integers without large prime factors by use of an analog of this identity.

Let $\mathcal{M}$ denote the class of nonnegative multiplicative functions h satisfying the following conditions:

There exist constants δ, $0 < \delta < 1$, *and* $\kappa > 0$ *such that*

$$\sum_{p\le z} \frac{h(p)}{p}\log p = \kappa \log z + O((\log z)^{1-\delta}), \qquad z \ge 2, \tag{Ω_1}$$

and

there exists a constant $b > 0$ *such that*

$$\sum_{p,\, k\ge 2} \frac{h(p^k)}{p^k}\log p^k \le b. \tag{Ω_2}$$

Define

$$m_0(x,y) := \sum_{\substack{n\le x\\ P(n)\le y}} \frac{h(n)}{n}, \qquad V(y) := \prod_{p\le y}\left(1 + \sum_{k=1}^{\infty}\frac{h(p^k)}{p^k}\right).$$

Clearly, we have

$$m_0(x,y) \le \sum_{P(n)\le y} \frac{h(n)}{n} = V(y).$$

Also, for the real parameter κ occurring in (Ω_1), define j_κ to be the continuous solution of the differential difference equation (DDE)

$$uj'_\kappa(u) = \kappa j_\kappa(u) - \kappa j_\kappa(u-1), \quad u > 1, \tag{1.5}$$

and

$$j_\kappa(u) = \begin{cases} 0, & u \le 0, \\ B_\kappa u^\kappa, & 0 < u \le 1, \end{cases} \tag{1.6}$$

where $B_\kappa = e^{-\gamma\kappa}/\Gamma(\kappa+1)$, with γ Euler's constant, and Γ Euler's Gamma function. We note that Dickman's function satisfies

$$\varrho(u) = e^\gamma j'_1(u).$$

Main Theorem. *Suppose $h \in \mathcal{M}$. Then for all sufficiently large y,*

$$m_0(x,y) = V(y)\left\{ j_\kappa(u) + O\left(\frac{\log(u+1)}{(\log y)^\delta}\right)\right\} \tag{1.7}$$

uniformly for

$$1 \le u \le \exp\left(\frac{1}{c}(\log y)^\delta\right)$$

for a suitable constant c.

The conditions on $\mathcal{M}$ restrict the size of h on prime powers and condition (Ω_1) asserts h is, on average, about κ on prime numbers (see Section 3). The main result can be viewed as a quantitative version of a result of de Bruijn and van Lint [dBvL]; it can be easily seen that the conditions on $\mathcal{M}$ imply those of [dBvL]. Under those similar, but weaker, conditions, de Bruijn and van Lint [dBvL] proved results implying

$$\sum_{\substack{n\le x\\ P(n)\le y}} \frac{h(n)}{n} \sim (\log y)^\kappa L(\log y), \tag{1.8}$$

uniformly for $0 < \delta \le u \le U$ as $y \to \infty$, for any bounded U, where L is a slowly oscillating function. This result is reflected in (1.7) once we show that

$$V(y) \sim A(\log y)^\kappa,$$

with a suitable constant A, for the class of multiplicative functions $\mathcal{M}$.

Using the technique of [HR], Chapter 5, Halberstam has given mean value estimates of h satisfying the conditions of $\mathcal{M}$ in an unpublished manuscript. In 1990, Odoni [Od] proved similar results using a Tauberian theorem with essentially the same hypotheses. Halberstam's proofs are elementary and his results slightly stronger. He also gave results providing a starting point from which $m_0(x,y)$ can be estimated. We state and prove these results below.

THEOREM A. *Let $h \in \mathcal{M}$. Then for all sufficiently large x,*

$$m(x) := \sum_{n \le x} \frac{h(n)}{n} = C_\kappa (\log x)^\kappa + O((\log x)^{\kappa-\delta}), \tag{1.9}$$

where

$$C_\kappa = \frac{1}{\Gamma(\kappa+1)} \lim_{s\to 1^+} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k)}{p^{ks}}\right)\left(1 - \frac{1}{p^s}\right)^\kappa.$$

When $u \le 1$, we have $m_0(x,y) = m(x)$. Theorem A is a quantitative version of a result of Wirsing's [W]; the classical Wirsing condition on h,

$$h(p^k) \le \gamma_0 \gamma_1^k, \qquad k \ge 2,\ 0 < \gamma_1 < 2,$$

implies (Ω_2), and the proof is based on ideas in [W].

A result on $m_0(x,y)$ using the technique in [HR] is the following:

THEOREM B. *Suppose $h \in \mathcal{M}$. Then for all sufficiently large y,*

$$m_0(x,y) = V(y)\left\{ j_\kappa(u) + O\left(\frac{u^{2\kappa}\log(u+1)}{(\log y)^\delta}\right)\right\} \tag{1.10}$$

for $u > 1$.

Theorem B will be used to prove the Main Theorem for $u \le \kappa + 2$. The proof of the Main Theorem for $u > \kappa + 2$ is completed in Section 5 using the approach of [Hi].

2. Some results about $j_\kappa(\cdot)$. We state the following lemma that is needed in the proof of the Main Theorem.

LEMMA 1. *For $\kappa \ge 0$,*

(a) $j_\kappa(u)$ *is strictly increasing and positive for all $u > 0$, and converges to* 1 *as $u \to \infty$,*

(b) $j_\kappa(\kappa) \ge 1/2$.

Following pioneering work of Ankeny and Onishi [AO], the properties of $j_\kappa(u)$ and $j_\kappa'(u)$ have been studied by many others. The proof of part (a) of the lemma can be found, for example, in [dBvL]. Proof of part (b) is given in Grupp and Richert [GR], Theorem 4, and the Ph.D. thesis of F. Wheeler [Wh].

Also, using the definition of j_κ and the relation

$$(u j_\kappa(u))' = \kappa(j_\kappa(u) - j_\kappa(u-1)) + j_\kappa(u) \tag{2.1}$$

we obtain

$$(2.2)\qquad j_\kappa(u) = \frac{\kappa}{u}\int_{u-1}^{u} j_\kappa(t)\,dt + \frac{1}{u}\int_{0}^{u} j_\kappa(t)\,dt = \frac{\kappa}{u}\int_{u-1}^{u} j_\kappa(t)\,dt + \frac{1}{u}\int_{1}^{u} j_\kappa(t)\,dt + \frac{B_\kappa}{(\kappa+1)u}, \qquad u \ge 1.$$

As simple as its derivation is, this identity plays an important role in the proof of the main result: it serves as a continuous analog of the functional equation of the sum $m_0(x,y)$. It is the use of this identity in the proof of the Main Theorem that improves on the approach in the proof of Theorem B.

3. Preliminary lemma. Before embarking on the proofs of Theorems A, B and the Main Theorem, we shall establish a useful relation involving $m_0(x,y)$. First, we remark that the condition (Ω_1) implies not only that $h(p)$ is, on average, equal to κ but also that it cannot be too large for a given prime. In particular, we obtain

$$(3.1)\qquad h(p) \ll p(\log p)^{-\delta}$$

for $h \in \mathcal{M}$, from summing the right side of (Ω_1) to p and $p-1$ respectively and taking the difference. By Abel summation based on (Ω_1) we deduce that

$$(3.2)\qquad \sum_{p\le z} \frac{h(p)}{p}(\log p)^{1-\delta} \ll (\log z)^{1-\delta}$$

and

$$(3.3)\qquad \sum_{p>z} \frac{h(p)}{p(\log p)^{\delta}} \ll (\log z)^{-\delta}.$$

We see from (3.3) that

$$(3.4)\qquad \sum_{p} \frac{h(p)}{p(\log p)^{\delta}} < \infty.$$

Next, starting from an identity for $m_0(x,y)$ analogous to (1.4), namely

$$(3.5)\qquad m_0(x,y)\log x = T_0(x,y) + \sum_{\substack{mp^k\le x\\ P(mp)\le y\\ p\nmid m}} \frac{h(m)}{m}\cdot\frac{h(p^k)}{p^k}\log p^k,$$

where

$$T_0(x,y) := \int_1^x \frac{m_0(t,y)}{t}\,dt,$$

we shall derive Lemma 2 below and use it as a springboard for the proofs of first Theorem B and then the Main Theorem.

Lemma 2. *Suppose $h \in \mathcal{M}$. Then*

$$m_0(x,y)\log x = T_0(x,y) + \sum_{\substack{m\le x\\ P(m)\le y}} \frac{h(m)}{m} \sum_{p\le \min(x/m,y)} \frac{h(p)}{p}\log p + O(m_0(x,y)(\log y)^{1-\delta}), \qquad 1<y\le x. \tag{3.6}$$

Proof. We deduce from (3.5) the inequality

$$\Bigl| m_0(x,y)\log x - T_0(x,y) - \sum_{\substack{m\le x\\ P(m)\le y}} \frac{h(m)}{m} \sum_{p\le \min(x/m,y)} \frac{h(p)}{p}\log p \Bigr| \le \sum_{\substack{p,\,k\ge 1\\ p\le y}} m_0\Bigl(\frac{x}{p^{k+1}},y\Bigr)\frac{h(p^k)}{p^k}\cdot\frac{h(p)}{p}\log p + bm_0(x,y), \tag{3.7}$$

where b is the constant in (Ω_2). Indeed, if we take the terms corresponding to $k=1$ from the sum on the right of (3.5), they contribute

$$\begin{aligned} &\sum_{\substack{mp\le x\\ P(mp)\le y\\ p\nmid m}} \frac{h(m)}{m}\cdot\frac{h(p)}{p}\log p \\ &= \sum_{\substack{mp\le x\\ P(mp)\le y}} \frac{h(m)}{m}\cdot\frac{h(p)}{p}\log p - \sum_{\substack{mp\le x\\ p\mid m\\ P(mp)\le y}} \frac{h(m)}{m}\cdot\frac{h(p)}{p}\log p \\ &= \sum_{\substack{m\le x\\ P(m)\le y}} \frac{h(m)}{m} \sum_{p\le \min(x/m,y)} \frac{h(p)}{p}\log p - \sum_{\substack{lp^{k+1}\le x\\ p\nmid l,\,k\ge 1\\ P(lp)\le y}} \frac{h(l)}{l}\cdot\frac{h(p^k)}{p^k}\cdot\frac{h(p)}{p}\log p, \end{aligned} \tag{3.8}$$

whereas the remaining terms, those corresponding to $k\ge 2$, of the same sum on the right of (3.5) contribute at most

$$\sum_{p,\,k\ge 2} m_0\Bigl(\frac{x}{p^k},y\Bigr)\frac{h(p^k)}{p^k}\log p^k \le bm_0(x,y) \tag{3.9}$$

by (Ω_2). Thus, with (3.8) and (3.9) applied in (3.5), we have

$$\Bigl| m_0(x,y)\log x - T_0(x,y) - \sum_{\substack{m\le x\\ P(m)\le y}} \frac{h(m)}{m} \sum_{p\le \min(x/m,y)} \frac{h(p)}{p}\log p \Bigr|$$

$$\le \sum_{\substack{lp^{k+1}\le x\\ p\nmid l,\, k\ge 1\\ P(lp)\le y}} \frac{h(l)}{l}\cdot\frac{h(p^k)}{p^k}\cdot\frac{h(p)}{p}\log p + bm_0(x,y)$$

$$\le \sum_{\substack{p^{k+1}\le x\\ p\le y,\, k\ge 1}} m_0\left(\frac{x}{p^{k+1}}, y\right)\frac{h(p^k)}{p^k}\cdot\frac{h(p)}{p}\log p + bm_0(x,y).$$

This proves (3.7).

The double sum on the right of (3.7) is at most

$$m_0(x,y)\sum_{\substack{p^{k+1}\le x\\ p\le y}} \frac{h(p^k)}{p^k}\cdot\frac{h(p)}{p}\log p$$
$$\ll m_0(x,y)\Big\{\sum_{p\le \min(\sqrt{x},y)}\frac{h(p)}{p}(\log p)^{1-\delta} + \sum_{\substack{p,\,k\ge 2\\ p\le y}}\frac{h(p^k)}{p^k}(\log p)^{1-\delta}\Big\},$$

by (3.1), after separating terms corresponding to $k=1$ and $k\ge 2$. The expression on the right is of order

$$m_0(x,y)(\log y)^{1-\delta},\qquad 1<y\le x,$$

by (3.2) and (Ω_2). ∎

We also remark that if $1\le x\le y$, we take $y=x$ in the above lemma to obtain

$$m(x)\log x - \int_1^x m(t)\frac{dt}{t} - \sum_{m\le x}\frac{h(m)}{m}\sum_{p\le x/m}\frac{h(p)}{p}\log p \ll m(x)(\log x)^{1-\delta}, \tag{3.10}$$

a relation that is needed in the proof of Theorem A.

4. Proofs of Theorems A and B. In this section we employ a technique similar to that used in [HR], Lemmas 5.4 and 6.1, to prove Theorems A and B. In order to prove Theorem A, we consider the approximate functional equation (3.10) for $m(x)$: we may rewrite it as

$$m(x)\log x = T(x) + \sum_{m\le x}\frac{h(m)}{m}\sum_{p\le x/m}\frac{h(p)}{p}\log p + O(m(x)(\log x)^{1-\delta}), \tag{4.1}$$

with the notation

$$T(x) := \sum_{m\le x}\frac{h(m)}{m}\log\frac{x}{m} = \int_1^x \frac{m(t)}{t}\,dt. \tag{4.2}$$

The sum on the right side of (4.1) is equal to

$$\sum_{m\le x}\frac{h(m)}{m}\left\{\kappa\log\frac{x}{m}+O\Big(\Big(\log\frac{x}{m}\Big)^{1-\delta}\Big)\right\}=\kappa T(x)+O(m(x)(\log x)^{1-\delta})$$

by (Ω_1), whence

$$m(x)=\frac{\kappa+1}{\log x}\,T(x)+m(x)\varepsilon(x),$$

where

$$\varepsilon(x)\ll(\log x)^{-\delta}. \tag{4.3}$$

If x is large enough, say $x\ge x_0$, then

$$|\varepsilon(x)|\le 1/2, \tag{4.4}$$

and we have the useful expression

$$m(x)=\frac{1}{1-\varepsilon(x)}\cdot\frac{\kappa+1}{\log x}\,T(x),\qquad x\ge x_0. \tag{4.5}$$

Let

$$E(z):=\log\left(\frac{\kappa+1}{(\log z)^{\kappa+1}}\,T(z)\right),$$

and note that, by (4.2) and (4.5),

$$E'(z)=\frac{\kappa+1}{z\log z}\cdot\frac{\varepsilon(z)}{1-\varepsilon(z)}\ll\frac{1}{z(\log z)^{1+\delta}},\qquad z\ge x_0, \tag{4.6}$$

after appeal to (4.3) and (4.4). Hence

$$E_0:=\int_1^\infty E'(z)\,dz$$

converges absolutely, and therefore

$$\int_x^\infty E'(z)\,dz=E_0-E(x),\qquad x\ge x_0.$$

On writing $C=\exp(E_0)$, we obtain

$$\begin{aligned}\frac{\kappa+1}{(\log x)^{\kappa+1}}\,T(x)&=\exp(E(x))=C\exp\Big(-\int_x^\infty E'(z)\,dz\Big)\\&=C(1+O((\log x)^{-\delta})).\end{aligned}$$

When we substitute this in (4.5) and use (4.3) we obtain the result in Theorem A, except that we need to find C.

To determine C we argue as follows: if $s > 1$ we have

$$\prod_p \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k)}{p^{ks}}\right) = \sum_{m=1}^{\infty} \frac{h(n)}{n^s} = (s-1)\int_1^{\infty} \frac{m(x)}{x^s}\,dx$$
$$= (s-1)\int_1^{\infty} C\frac{(\log x)^{\kappa} + O((\log x)^{\kappa-\delta})}{x^s}\,dx$$
$$= C\frac{\Gamma(\kappa+1)}{(s-1)^{\kappa}} + O\left(\frac{1}{(s-1)^{\kappa-\delta}}\right),$$

whence

$$C = \frac{1}{\Gamma(\kappa+1)} \lim_{s\to 1+0} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k)}{p^{ks}}\right)(s-1)^{\kappa}.$$

On the other hand,

$$\lim_{s\to 1+0} (s-1)\zeta(s) = 1$$

with

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad s > 1.$$

Hence

$$C_{\kappa} = C = \frac{1}{\Gamma(\kappa+1)} \lim_{s\to 1+0} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k)}{p^{ks}}\right)\left(1 - \frac{1}{p^s}\right)^{\kappa}.$$

This proves Theorem A.

Now we go on to prove Theorem B. First, we derive an asymptotic formula for the product $V(y)$ defined in the introduction:

$$V(y) := \prod_{p\le y} \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k)}{p^k}\right).$$

Let

$$H_p(s) := \sum_{k=2}^{\infty} \frac{h(p^k)}{p^{ks}}, \qquad H_p = H_p(1),$$

so that

$$V(y) = \prod_{p\le y} \left(1 + \frac{h(p)}{p} + H_p\right). \tag{4.7}$$

Lemma 3. *Suppose $h \in \mathcal{M}$. Then*

$$\sum_{y<p\le z} \frac{h(p)}{p^s} - \frac{\kappa}{p^s} = O((\log y)^{-\delta}), \qquad z > y, \tag{4.8}$$

uniformly in $s \ge 1$.

Proof. Let

$$L(t) := \sum_{y<p\le t} \frac{h(p)-\kappa}{p}\log p.$$

Then $L(t)=0$ when $t\le y$, and

$$L(t) = O((\log t)^{1-\delta}), \qquad t>y,$$

by (Ω_1) and Mertens' prime number estimate. Hence the expression in the statement of the lemma is equal to

$$\begin{aligned}
\int_y^z \frac{dL(t)}{t^{s-1}\log t} &= \frac{L(z)}{z^{s-1}\log z} + \int_y^z \frac{L(t)}{t^s\log t}\left(s-1+\frac{1}{\log t}\right)dt \\
&\ll \frac{1}{z^{s-1}}(\log z)^{-\delta} + \int_y^z (s-1)(\log t)^{-\delta}\frac{dt}{t^s} + \int_y^z \frac{dt}{t^s(\log t)^{1+\delta}} \\
&\le \frac{1}{z^{s-1}(\log z)^{\delta}} + \frac{s-1}{(\log y)^{\delta}}\int_{\log y}^{\infty} e^{-(s-1)v}\,dv + \int_y^{\infty}\frac{dt}{t(\log t)^{1+\delta}} \\
&= \frac{1}{z^{s-1}(\log z)^{\delta}} + \frac{1}{y^{s-1}(\log y)^{\delta}} + \frac{1}{\delta(\log y)^{\delta}} \ll (\log y)^{-\delta}
\end{aligned}$$

uniformly in $s\ge 1$. ∎

Consider the product

$$\begin{aligned}
(4.9)\qquad &\prod_{y<p\le z}\left(1+\frac{h(p)}{p^s}+H_p(s)\right)\left(1-\frac{1}{p^s}\right)^{\kappa} \\
&= \exp\left\{\sum_{y<p\le z}\log\left(1+\frac{h(p)}{p^s}+H_p(s)\right)+\kappa\log\left(1-\frac{1}{p^s}\right)\right\}, \qquad s\ge 1.
\end{aligned}$$

Since

$$x-\tfrac{1}{2}x^2 \le \log(1+x)\le x,$$

we have

$$\begin{aligned}
\log\left(1+\frac{h(p)}{p^s}+H_p(s)\right) &= \frac{h(p)}{p^s}+H_p(s)+O\left(\frac{h^2(p)}{p^{2s}}+H_p^2(s)\right) \\
&= \frac{h(p)}{p^s}+O\left(H_p+\frac{h(p)}{p(\log p)^{\delta}}\right), \qquad s\ge 1,
\end{aligned}$$

by (3.1) and (Ω_2). Also, for $s\ge 1$,

$$0\ge \log\left(1-\frac{1}{p^s}\right)+\frac{1}{p^s} = -\sum_{k=2}^{\infty}\frac{1}{kp^{ks}} \ge -\frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{p^k} = -\frac{1}{2p(p-1)} \ge -\frac{1}{p^2}$$

so that the product (4.9) equals

$$\exp\Big\{\sum_{y<p\le z}\frac{h(p)-\kappa}{p^s}+O\Big(\sum_{p>y}\Big(H_p+\frac{h(p)}{p(\log p)^\delta}+\frac{1}{p^2}\Big)\Big)\Big\}$$
$$=\exp\{O((\log y)^{-\delta})\}=1+O((\log y)^{-\delta})$$

uniformly in $s\ge 1$, $z\ge y$, by Lemma 3, (3.3), and (Ω_2). By the definition of $V(y)$ and C_κ in Theorem 1, we get

$$\begin{aligned}\Gamma(\kappa+1)C_\kappa&=\prod_{p\le y}\Big(1+\frac{h(p)}{p}+H_p\Big)\Big(1-\frac{1}{p}\Big)^\kappa\{1+O((\log y)^{-\delta})\}\\&=V(y)\prod_{p\le y}\Big(1-\frac{1}{p}\Big)^\kappa\{1+O((\log y)^{-\delta})\}\\&=V(y)\frac{e^{-\gamma\kappa}}{(\log y)^\kappa}\{1+O((\log y)^{-1})\}\{1+O((\log y)^{-\delta})\}\end{aligned}$$

by the well known Mertens formula. Referring to the definition of the constant B_κ in Section 1, we conclude that

$$V(y)=\frac{C_\kappa}{B_\kappa}(\log y)^\kappa\{1+O((\log y)^{-\delta})\},\tag{4.10}$$

for all sufficiently large y.

Now we are ready to prove Theorem B. On writing

$$u=\frac{\log x}{\log y},$$

we first note that from Theorem A it follows that if $u\le 1$ then

$$\begin{aligned}m_0(x,y)=m(x)&=C_\kappa(\log x)^\kappa\{1+O((\log x)^{-\delta})\}\\&=C_\kappa u^\kappa(\log y)^\kappa\{1+O(u^{-\delta}(\log y)^{-\delta})\}\\&=B_\kappa u^\kappa V(y)\{1+O((\log y)^{-\delta})\}\{1+O(u^{-\delta}(\log y)^{-\delta})\}\end{aligned}$$

by (4.10). Hence,

$$m_0(x,y)=V(y)\Big\{j_\kappa(u)+O\Big(\frac{u^{\kappa-\delta}}{(\log y)^\delta}\Big)\Big\},\qquad 0<u\le 1.\tag{4.11}$$

From now on, assume $u>1$. The double sum on the right of (3.6) is equal to

$$\sum_{\substack{m\le x/y\\P(m)\le y}}\frac{h(m)}{m}\sum_{p\le y}\frac{h(p)}{p}\log p+\sum_{\substack{x/y<m\le x\\P(m)\le y}}\frac{h(m)}{m}\sum_{p\le x/m}\frac{h(p)}{p}\log p$$
$$=\kappa T_0(x,y)-\kappa T_0(x/y,y)+O(m_0(x,y)(\log y)^{1-\delta})$$

by (Ω_1) and partial summation; hence we obtain

$$(4.12)\qquad m_0(x,y)\log x = (\kappa+1)T_0(x,y) - \kappa T_0(x/y,y) + O(V(y)(\log y)^{1-\delta}), \qquad u>1.$$

We have

$$\frac{\partial}{\partial x}\left(\frac{T_0(x,y)}{(\log x)^{\kappa+1}}\right) = \frac{m_0(x,y)\log x - (\kappa+1)T_0(x,y)}{x(\log x)^{\kappa+2}} = -\frac{\kappa T_0(x/y,y)}{x(\log x)^{\kappa+2}} + O\left(\frac{V(y)(\log y)^{1-\delta}}{x(\log x)^{\kappa+2}}\right)$$

by (4.12) and using the definition of $T_0(x,y)$, and when we integrate this relation with respect to x from ω to ξ we obtain the approximate integral equation

$$(4.13)\qquad \frac{T_0(\xi,y)}{(\log\xi)^{\kappa+1}} = \frac{T_0(\omega,y)}{(\log\omega)^{\kappa+1}} - \kappa\int_\omega^\xi \frac{T_0(x/y,y)}{x(\log x)^{\kappa+2}}\,dx + O\left(\frac{V(y)(\log y)^{1-\delta}}{(\log\omega)^{\kappa+1}}\right), \qquad y\le\omega\le\xi.$$

With an eye on (4.13), and also by (1.5) it is plausible that $T_0(\xi,y)$ is approximable by

$$\int_1^\xi V(y)j_\kappa\left(\frac{\log t}{\log y}\right)\frac{dt}{t} = V(y)(\log y)\int_0^{\log\xi/\log y} j_\kappa(v)\,dv.$$

Define

$$J_\kappa(u) = \int_0^u j_\kappa(v)\,dv,$$

so that by (1.6) and (2.2),

$$(4.14)\qquad \begin{cases} J_\kappa(u) = \dfrac{1}{\kappa+1}B_\kappa u^{\kappa+1}, & 0<u\le 1,\\ uj_\kappa(u) = uJ'_\kappa(u) = (\kappa+1)J_\kappa(u) - \kappa J_\kappa(u-1), & u>1, \end{cases}$$

and, in particular,

$$(4.15)\qquad \frac{J_\kappa(\tau)}{\tau^{\kappa+1}} = \frac{J_k(\omega)}{\omega^{\kappa+1}} - \kappa\int_\omega^\tau \frac{J_\kappa(v-1)}{v^{\kappa+2}}\,dv.$$

We find that when we put

$$(4.16)\qquad T_0(\xi,y) = J_\kappa(\tau)V(y)\log y + R(\xi,y), \qquad \tau = \log\xi/\log y,$$

and substitute in (4.13), we arrive at

$$\frac{R(\xi,y)}{(\log\xi)^{\kappa+1}} = \frac{R(\omega,y)}{(\log\omega)^{\kappa+1}} - \kappa\int_{\omega}^{\xi}\frac{R(x/y,y)}{x(\log x)^{\kappa+2}}\,dx + O\left(\frac{V(y)(\log y)^{1-\delta}}{(\log\omega)^{\kappa+1}}\right), \quad y\le\omega\le\xi, \tag{4.17}$$

by (4.14) and (4.15). This is a relation linking just the remainders $R(\cdot,y)$. Since the second argument of the remainders is the same, it is possible to estimate the size of R by means of an inductive argument involving $\tau = \log\xi/\log y$. Suppose first that $\xi\le y$, or $\tau\le 1$. Then by (4.11),

$$T_0(\xi,y) = V(y)(\log y)\left\{J_\kappa(\tau) + O\left(\frac{\tau^{\kappa+1-\delta}}{(\log y)^\delta}\right)\right\}, \quad \tau\le 1,$$

which agrees with (4.16) when $\tau\le 1$ and

$$|R(\xi,y)| \le D_1\frac{V(y)\log y}{(\log y)^\delta}, \quad \tau\le 1, \tag{4.18}$$

with the constant D_1 at least as large as the constant in (4.11). Suppose next that $y<\xi\le y^2$, i.e. $1<\tau\le 2$. Let D_0 denote the O-constant in (4.17) and apply (4.17) with $\omega=y$. On the right of (4.17),

$$R(\omega,y) = R(y,y)$$

and in the integral we have $x/y\le\xi/y\le y$; hence (4.18) applies to each remainder $R(\cdot,y)$ on the right of (4.17), and therefore

$$\begin{aligned}\frac{|R(\xi,y)|}{(\log\xi)^{\kappa+1}} &\le D_1\frac{V(y)\log y}{(\log y)^\delta} \\ &\qquad\times\left\{\frac{1}{(\log y)^{\kappa+1}} + \kappa\int_y^{y^2}\frac{dx}{x(\log x)^{\kappa+2}} + \frac{D_0}{D_1}\cdot\frac{1}{(\log y)^{\kappa+1}}\right\} \\ &\le D_1\frac{V(y)}{(\log y)^{\kappa+\delta}}\left(1+\frac{\kappa}{\kappa+2}+\frac{D_0}{D_1}\right) \\ &\le 2D_1\frac{V(y)}{(\log y)^{\kappa+\delta}}, \quad y<\xi\le y^2,\end{aligned}$$

provided that $D_1\ge(\kappa+1)D_0$. We proceed inductively. Suppose that for $\nu=1,2,\ldots$ there exists a constant D^* independent of ν such that

$$\left|\frac{R(\xi,y)}{(\log\xi)^{\kappa+1}}\right| \le D^*\eta_0(\nu)\frac{V(y)}{(\log y)^{\kappa+\delta}}, \quad y^\nu<\xi\le y^{\nu+1}, \tag{4.19}$$

where

$$\eta_0(\nu) = \nu^\kappa\log(\nu+1).$$

We have proven (4.19) to be true for $\nu = 1$ provided

$$D^* \geq 2D_1/\log 2,$$

so we may assume $\nu \geq 2$. Set $\omega = y^\nu$ in (4.17). Applying the induction hypothesis (4.19) to each of the remainders in (4.17) we get

$$\frac{|R(\xi,y)|}{(\log\xi)^{\kappa+1}} \leq D^* \frac{V(y)}{(\log y)^{\kappa+\delta}} \times\left\{\eta_0(\nu-1) + \kappa\eta_0(\nu-1)\int_{y^\nu}^{y^{\nu+1}} \frac{(\log x-\log y)^{\kappa+1}}{x(\log x)^{\kappa+2}}\,dx + \frac{D_0}{D^*\nu^{\kappa+1}}\right\}. \tag{4.20}$$

The integral on the right is equal to

$$\int_\nu^{\nu+1} \frac{(t-1)^{\kappa+1}}{t^{\kappa+2}}\,dt \leq \int_\nu^{\nu+1}\frac{dt}{t} = \log\left(1+\frac{1}{\nu}\right) \leq \frac{1}{\nu},$$

whence

$$\begin{aligned}\frac{|R(\xi,y)|}{(\log\xi)^{\kappa+1}} &\leq D^*\frac{V(y)}{(\log y)^{\kappa+\delta}}\left\{\left(1+\frac{\kappa}{\nu}\right)\eta_0(\nu-1)+\frac{D_0}{D^*\nu}\right\}\\ &\leq D^*\eta_0(\nu)\frac{V(y)}{(\log y)^{\kappa+\delta}}\end{aligned}$$

if

$$\left(1+\frac{\kappa}{\nu}\right)\eta_0(\nu-1)+\frac{D_0}{D^*\nu} \leq \eta_0(\nu).$$

But

$$\begin{aligned}&\eta_0(\nu) - \left(1+\frac{\kappa}{\nu}\right)\eta_0(\nu-1)\\ &\qquad= \nu^\kappa\log\left(1+\frac{1}{\nu}\right) + (\nu-1)^\kappa\left\{\left(1+\frac{1}{\nu-1}\right)^\kappa - 1 - \frac{\kappa}{\nu}\right\}\log\nu\\ &\qquad> \nu^\kappa\log\left(1+\frac{1}{\nu}\right) > \nu^{\kappa-1}\left(1-\frac{1}{2\nu}\right) \geq \frac{3}{4}\nu^{\kappa-1} \geq \frac{D_0}{D^*}\nu^{-1}\end{aligned}$$

if $D^* \geq 4D_0/3$. This proves (4.19) for all $\nu \geq 1$, and we conclude that

$$|R(\xi,y)| \leq D^*\tau^{2\kappa+1}\log(\tau+1)\frac{V(y)\log y}{(\log y)^\delta}, \qquad \tau > 1.$$

By (4.16), we then obtain

$$T_0(\xi,y) = \left\{J_\kappa(\tau) + O\left(\frac{\tau^{2\kappa+1}\log(\tau+1)}{(\log y)^\delta}\right)\right\}V(y)\log y, \qquad \tau > 1,$$

which, when substituted in (4.12), once with $\xi = x$ and once with $\xi = x/y$, gives

$$m_0(x,y)\log x = V(y)(\log y)\biggl\{((\kappa+1)J_\kappa(u)-\kappa J_\kappa(u-1)) + O\biggl(\frac{u^{2\kappa+1}\log(u+1)+1}{(\log y)^\delta}\biggr)\biggr\},\qquad u>1.$$

Dividing by $\log x = u\log y$, we get, by (4.14),

$$m_0(x,y) = V(y)\biggl\{j_\kappa(u) + O\biggl(\frac{u^{2\kappa}\log(u+1)}{(\log y)^\delta}\biggr)\biggr\},\qquad u>1,$$

which is the statement of Theorem B.

5. Proof of Main Theorem. By switching the order of summation we may restate (3.6) as

$$m_0(x,y)\log x = \int_1^x m_0(t,y)\frac{dt}{t} + \sum_{p\le y} m_0\Bigl(\frac{x}{p},y\Bigr)\frac{h(p)}{p}\log p + O(m_0(x,y)(\log y)^{1-\delta}). \tag{5.1}$$

We need two additional lemmas to elucidate the right hand side of (5.1). First, we restate Theorem A: for $h\in\mathcal{M}$, there exist constants K and x_0 such that

$$|m(z) - C_\kappa(\log z)^\kappa| \le K(\log z)^{\kappa-\delta} \tag{5.2}$$

for all $z\ge x_0$.

LEMMA 4. *Suppose $h\in\mathcal{M}$. Then*

$$\int_1^y m(t,y)\frac{dt}{t} = \frac{B_\kappa}{\kappa+1}V(y)(\log y)\{1+O((\log y)^{-\delta})\} \tag{5.3}$$

for $y\ge y_1$, with y_1 the least value of y satisfying

$$(\log y)^{\kappa+1-\delta} \ge (\log x_0)^{\kappa+1}, \tag{5.4}$$

where x_0 satisfies (5.2).

Proof. Since $m(t,y)=m(t)\le V(x_0)$ when $1\le t\le x_0\le y$, we have for $y\ge y_1$,

$$\int_1^y m(t)\frac{dt}{t} = \int_1^{x_0} m(t)\frac{dt}{t} + \int_{x_0}^y \{m(t) - C_\kappa(\log t)^\kappa\}\frac{dt}{t} + \frac{C_\kappa}{\kappa+1}\{(\log y)^{\kappa+1} - (\log x_0)^{\kappa+1}\}$$

whence

$$\left|\int_1^y m(t)\frac{dt}{t} - \frac{C_\kappa}{\kappa+1}(\log y)^{\kappa+1}\right| \le V(x_0)\log x_0 + K\int_{x_0}^y (\log t)^{\kappa-\delta}\,\frac{dt}{t} + \frac{C_\kappa}{k+1}(\log x_0)^{\kappa+1}. \tag{5.5}$$

We have, by (4.10),

$$\frac{V(x_0)\log x_0}{V(y)\log y} \ll \left(\frac{\log x_0}{\log y}\right)^{\kappa+1} \le (\log y)^{-\delta}$$

by (5.4). The integral on the right of (5.5) is obviously $O((\log y)^{\kappa+1-\delta})$, and the third expression on the right is of the same order of magnitude, also by (5.4). Finally, by (4.10),

$$\frac{C_\kappa}{\kappa+1}(\log y)^{\kappa+1} = \frac{B_\kappa}{\kappa+1}V(y)(\log y)\{1+O((\log y)^{-\delta})\}.$$

This proves the lemma. ∎

LEMMA 5. *Suppose $h \in \mathcal{M}$. Let θ be a fixed number such that $1/2 \le \theta \le 1$. Then*

$$\sum_{p\le y^\theta} j_\kappa\left(u-\frac{\log p}{\log y}\right)\frac{h(p)}{p}\log p = \kappa(\log y)\int_0^\theta j_\kappa(u-v)\,dv + O((\log y)^{1-\delta}) \tag{5.6}$$

for $y \ge 2$, $u \ge 2$.

Proof. Let $s(z)$ denote the sum $\sum_{p\le z}\frac{h(p)}{p}\log p$. By (Ω_1) we have

$$s(z) - \kappa\log z = r(z),$$

where

$$|r(z)| \le A(\log z)^{1-\delta}, \qquad z \ge 2,$$

for a suitable constant A. The sum on the left hand side of (5.6) is equal to

$$\begin{aligned} \sum_{p\le y^\theta} j_\kappa\left(u-\frac{\log p}{\log y}\right)\frac{h(p)}{p}\log p &= \int_1^{y^\theta} j_\kappa\left(u-\frac{\log t}{\log y}\right)d\{s(t)-\kappa\log t\} + \kappa\int_1^{y^\theta} j_\kappa\left(u-\frac{\log t}{\log y}\right)\frac{dt}{t} \\ &= \kappa\int_1^{y^\theta} j_\kappa\left(u-\frac{\log t}{\log y}\right)\frac{dt}{t} + \int_1^{y^\theta} j_\kappa\left(u-\frac{\log t}{\log y}\right)dr(t) \end{aligned} \tag{5.7}$$

where the second integral is at most of order

$$j_\kappa(u-\theta)|r(y^\theta)| - \int_1^{y^\theta} r(t)\frac{d}{dt}j_\kappa\left(u-\frac{\log t}{\log y}\right)dt$$
$$= j_\kappa(u-\theta)|r(y^\theta)| + O\left((\log y)^{1-\delta}\int_1^{y^\theta} j'_\kappa\left(u-\frac{\log t}{\log y}\right)dt\right)$$
$$= j_\kappa(u-\theta)|r(y^\theta)| + O(j_\kappa(u-\theta)(\log y)^{1-\delta})$$
$$\ll (\log y)^{1-\delta}.$$

With the change of variable $v = \log t/\log y$, from (5.7) we obtain

$$\sum_{p\le y^\theta} j_\kappa\left(u-\frac{\log p}{\log y}\right)\frac{h(p)}{p}\log p = \kappa(\log y)\int_0^\theta j_\kappa(u-v)\,dv + O((\log y)^{1-\delta}). \quad \blacksquare \tag{5.8}$$

Now we are in a position to prove the Main Theorem. First, we note that the theorem is already proved for u in some bounded range by virtue of Theorem B, let us say for $y \ge y_0$ and $1 \le u \le \kappa+2$. Thus,

$$m_0(x,y) = V(y)\{j_\kappa(u) + R(y,u)\}, \qquad u > 1, \tag{5.9}$$

where

$$|R(y,u)| \ll (\log y)^{-\delta}, \tag{5.10}$$

if $1 < u \le \kappa+2$. If $u > \kappa$, then we have

$$m_0(x,y) = V(y)j_\kappa(u)\left\{1+\frac{R(y,u)}{j_\kappa(u)}\right\} = V(y)j_\kappa(u)\{1+\Delta(y,u)\}, \tag{5.11}$$

where $\Delta(y,u) \le 2R(y,u)$, by Lemma 1(b). Let

$$\Delta^*(y,u) = \sup_{\kappa\le u'\le u} |\Delta(y,u')|.$$

Then we have

$$\Delta^*(y,u) \ll (\log y)^{-\delta}, \qquad \kappa < u \le \kappa+2, \tag{5.12}$$

by (5.10), and now it suffices to show that

$$\Delta^*(y,u) \ll \frac{\log u}{(\log y)^\delta} \tag{5.13}$$

uniformly for $u \ge \kappa+2$, and for all sufficiently large y in order to prove the theorem.

We first consider the integral in (5.1), namely

$$\int_1^x m_0(t,y)\,\frac{dt}{t} = \int_1^y m(t)\,\frac{dt}{t} + \int_y^x m_0(t,y)\,\frac{dt}{t}.$$

The first integral on the right is

$$\frac{B_\kappa}{\kappa+1}V(y)(\log y)\{1+O((\log y)^{-\delta})\}$$

by Lemma 4. The portion of the second integral that corresponds to $y \le t \le y^{\kappa+2}$ is, by (5.9) and (5.10),

$$V(y)\int_y^{y^{\kappa+2}} j_\kappa\biggl(\frac{\log t}{\log y}\biggr)\frac{dt}{t} + O_\kappa(V(y)(\log y)^{1-\delta})$$
$$= V(y)(\log y)\int_1^{\kappa+2} j_\kappa(v)\,dv + O_\kappa(V(y)(\log y)^{1-\delta}),$$

and the remaining portion contributes, with the notation of (5.11),

$$V(y)\int_{y^{\kappa+2}}^x j_\kappa\biggl(\frac{\log t}{\log y}\biggr)\biggl\{1+\varDelta\biggl(y,\frac{\log t}{\log y}\biggr)\biggr\}\frac{dt}{t}$$
$$= V(y)(\log y)\int_{\kappa+2}^u j_\kappa(v)\{1+\varDelta(y,v)\}\,dv.$$

Hence

$$\begin{aligned}(5.14)\qquad \int_1^x m_0(t,y)\frac{dt}{t} &= V(y)(\log y)\frac{B_\kappa}{\kappa+1}\\ &\quad + V(y)(\log y)\biggl\{\int_1^u j_\kappa(v)\,dv + \int_{\kappa+2}^u j_\kappa(v)\varDelta(y,v)\,dv\biggr\}\\ &\quad + O(V(y)(\log y)^{1-\delta}),\qquad y\ge y_0,\ u\ge\kappa+2.\end{aligned}$$

Next we turn to the second expression on the right of (5.1). By (5.11), for $u > \kappa+2$, we have

$$\begin{aligned}(5.15)\qquad &\sum_{p\le y} m_0\biggl(\frac{x}{p},y\biggr)\frac{h(p)}{p}\log p\\ &= V(y)\sum_{p\le y} j_\kappa\biggl(u-\frac{\log p}{\log y}\biggr)\frac{h(p)}{p}(\log p)\biggl\{1+\varDelta\biggl(y,u-\frac{\log p}{\log y}\biggr)\biggr\}.\end{aligned}$$

By Lemma 5, this is equal to

$$V(y)\kappa(\log y)\int_0^1 j_\kappa(u-t)\,dt + O(V(y)(\log y)^{1-\delta})$$
$$+V(y)\sum_{p\le y} j_\kappa\Big(u-\frac{\log p}{\log y}\Big)\frac{h(p)}{p}(\log p)\Delta\Big(y,u-\frac{\log p}{\log y}\Big).$$

Dividing both sides of the equation (5.1) by $V(y)j_\kappa(u)\log x$, we get, by (5.14) and (5.15),

$$\begin{aligned}1+\Delta(y,u) &= \frac{B_\kappa}{(\kappa+1)uj_\kappa(u)}+\frac{1}{uj_\kappa(u)}\int_1^u j_\kappa(v)\,dv+\frac{\kappa}{uj_\kappa(u)}\int_0^1 j_\kappa(u-v)\,dv\\ &\quad+\frac{1}{uj_\kappa(u)\log y}\sum_{p\le y} j_\kappa\Big(u-\frac{\log p}{\log y}\Big)\frac{h(p)}{p}(\log p)\Delta\Big(y,u-\frac{\log p}{\log y}\Big)\\ &\quad+\frac{1}{uj_\kappa(u)}\int_{\kappa+2}^u j_\kappa(v)\Delta(y,v)\,dv\\ &\quad+O\Big(\frac{1}{uj_\kappa(u)(\log y)^\delta}\Big),\qquad u>\kappa+2.\end{aligned}$$

Hence, by (2.2),

$$\begin{aligned}(5.16)\qquad \Delta(y,u)&\\ &=\frac{1}{uj_\kappa(u)\log y}\sum_{p\le y} j_\kappa\Big(u-\frac{\log p}{\log y}\Big)\frac{h(p)}{p}(\log p)\Delta\Big(y,u-\frac{\log p}{\log y}\Big)\\ &\quad+\frac{1}{uj_\kappa(u)}\int_{\kappa+2}^u j_\kappa(v)\Delta(y,v)\,dv\\ &\quad+O_\kappa\Big(\frac{1}{uj_\kappa(u)(\log y)^\delta}\Big).\end{aligned}$$

The terms in the sum on the right of (5.16) that correspond to $p\le y^{1/2}$ contribute at most

$$\Delta^*(y,u)\sum_{p\le y^{1/2}} j_\kappa\Big(u-\frac{\log p}{\log y}\Big)\frac{h(p)}{p}\log p$$
$$=\Delta^*(y,u)\Big(\kappa(\log y)\int_0^{1/2} j_\kappa(u-t)\,dt+O((\log y)^{-\delta})\Big)$$

and the remaining terms contribute at most

$$\Delta^*(y,u-1/2)\sum_{y^{1/2}<p\le y} j_\kappa\Big(u-\frac{\log p}{\log y}\Big)\frac{h(p)}{p}\log p$$
$$=\Delta^*(u,y-1/2)\Big(\kappa(\log y)\int_{1/2}^{1} j_\kappa(u-t)\,dt+O((\log y)^{-\delta})\Big)$$

by Lemma 5. The integral on the right of (5.16) is at most

$$\Delta^*(y,u-1)\int_1^{u-1} j_\kappa(v)\,dv+\Delta^*(y,u)\int_{u-1}^{u} j_\kappa(v)\,dv,$$

and thus

$$(5.17)\qquad |\Delta(y,u)|\le \Delta^*(y,u)\Big\{\alpha(u)+\frac{1}{uj_\kappa(u)}\int_{u-1}^{u} j_\kappa(v)\,dv\Big\}$$
$$+\Delta^*(y,u-1/2)\Big\{\beta(u)+\frac{1}{uj_\kappa(u)}\int_1^{u-1} j_\kappa(v)\,dv\Big\}$$
$$+O\Big(\frac{1+\Delta^*(y,u)}{u(\log y)^\delta}\Big),$$

where

$$\alpha(u)=\frac{\kappa}{uj_\kappa(u)}\int_0^{1/2} j_\kappa(u-v)\,dv,\qquad \beta(u)=\frac{\kappa}{uj_\kappa(u)}\int_{1/2}^{1} j_\kappa(u-v)\,dv.$$

We note that

$$\alpha(u),\beta(u)\le \kappa/(2u),\qquad u>1.$$

Introduce

$$\alpha_1(u)=\alpha(u)+\frac{1}{uj_\kappa(u)}\int_{u-1}^{u} j_\kappa(v)\,dv$$

and note, by (2.2), that

$$\beta(u)+\frac{1}{uj_\kappa(u)}\int_1^{u-1} j_\kappa(v)\,dv=1-\alpha_1(u)-\frac{B_\kappa}{(\kappa+1)uj_\kappa(u)}\le 1-\alpha_1(u);$$

hence (5.17) simplifies to

$$(5.18)\qquad |\Delta(y,u)|\le \Delta^*(y,u)\alpha_1(u)+\Delta^*(y,u-1/2)(1-\alpha_1(u))$$
$$+O\Big(\frac{1+\Delta^*(y,u)}{u(\log y)^\delta}\Big),\qquad u\ge\kappa+2.$$

We next claim that, by (5.18),

$$|\varDelta(y,u)| \le \frac{1}{2}(\varDelta^*(y,u)+\varDelta^*(y,u-1/2)) + O\left(\frac{1+\varDelta^*(y,u)}{u(\log y)^\delta}\right) \tag{5.19}$$

uniformly for $u \ge \kappa+2$ and if y is sufficiently large. Indeed,

$$\begin{aligned}\frac{1}{2}(\varDelta^*(y,u)+\varDelta^*(y,u-1/2)) - (\varDelta^*(y,u)\alpha_1(u)+\varDelta^*(y,u-1/2)(1-\alpha_1(u)))\\ = \left(\frac{1}{2}-\alpha_1(u)\right)(\varDelta^*(y,u)-\varDelta^*(y,u-1/2)),\end{aligned}$$

and this quantity is nonnegative for we have

$$(\varDelta^*(y,u)-\varDelta^*(y,u-1/2)) \ge 0$$

by the monotonicity of $\varDelta^*$, and

$$\begin{aligned}\alpha_1(u) &= \alpha(u) + \frac{1}{uj_\kappa(u)}\int_{u-1}^{u} j_\kappa(v)\,dv\\ &\le \frac{\kappa}{2u}+\frac{1}{u} = \frac{\kappa+2}{2u} \le \frac{1}{2}, \qquad u \ge \kappa+2.\end{aligned}$$

This proves (5.19).

In order to show that (5.13) holds for $u \ge \kappa+2$, first suppose that $u-1/2 \le u' \le u$. Let A denote the O-constant in (5.19). By the monotonicity of $\varDelta^*$,

$$\begin{aligned}|\varDelta(y,u')| &\le \frac{1}{2}(\varDelta^*(y,u')+\varDelta^*(y,u'-1/2)) + A\left(\frac{1+\varDelta^*(y,u')}{u'(\log y)^\delta}\right)\\ &\le \frac{1}{2}(\varDelta^*(y,u)+\varDelta^*(y,u-1/2)) + \frac{2A}{3}\left(\frac{1+\varDelta^*(y,u)}{u(\log y)^\delta}\right).\end{aligned}$$

Now, if $\kappa+2 \le u' \le u-1/2$, then

$$|\varDelta(y,u')| \le \varDelta^*(y,u-1/2) \le \frac{1}{2}(\varDelta^*(y,u)-\varDelta^*(y,u-1/2)),$$

and it follows, by taking the supremum on the right of (5.19), that, uniformly for $u \ge \kappa+2$,

$$\varDelta^*(y,u) \le \frac{1}{2}(\varDelta^*(y,u)+\varDelta^*(y,u-1/2)) + \frac{2A}{3}\left(\frac{1+\varDelta^*(y,u)}{u(\log y)^\delta}\right).$$

After rearranging terms we arrive at the inequality

$$\varDelta^*(y,u) \le \varDelta^*(y,u-1/2) + \frac{4A}{3}\left(\frac{1+\varDelta^*(y,u)}{u(\log y)^\delta}\right), \tag{5.20}$$

which we iterate to get

$$\varDelta^*(y,u) \le \varDelta^*(y,u_0) + \frac{4A}{3}\left(\frac{1+\varDelta^*(y,u)}{(\log y)^\delta}\log u\right),$$

where

$$\kappa + 3/2 \le u_0 \le \kappa + 2.$$

By (5.12) we have

$$\Delta^*(y, u_0) \le A_1 \frac{1}{(\log y)^\delta},$$

where A_1 is an appropriate constant, and thus

$$\Delta^*(y, u) \le A^* \left((1 + \Delta^*(y, u)) \frac{\log u}{(\log y)^\delta} \right), \qquad u \ge \kappa + 2, \tag{5.21}$$

where $A^* = \max(4A/3, A_1)$. This proves the theorem.

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