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Iwasawa λ_3-invariants of certain cubic fields

by

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1. Introduction. Let l be a prime number and k a finite extension of $\mathbb{Q}$. We denote by $\lambda_l(k)$ (resp. $\mu_l(k)$) the Iwasawa λ (resp. μ)-invariant of the cyclotomic $\mathbb{Z}_l$-extension k_∞/k. If $k/\mathbb{Q}$ is an abelian extension, then it was shown by Ferrero and Washington [FW] that $\mu_l(k) = 0$ for any prime l. In [G1], Greenberg conjectured that $\lambda_l(k) = \mu_l(k) = 0$ for any totally real number field k. For a cyclic l-extension of $\mathbb{Q}$, one can deduce the following result from [I1] and [I3]:

THEOREM A. *Let l be an odd prime number and p a prime number which is congruent to* 1 *modulo l. Denote by $\mathbb{Q}^{(l)}(p)$ the unique subfield of $\mathbb{Q}(\zeta_p)$ with $[\mathbb{Q}^{(l)}(p) : \mathbb{Q}] = l$, where ζ_p is a primitive pth root of unity. If either*

$$l^{(p-1)/l} \not\equiv 1 \pmod{p} \quad \textit{or} \quad p \not\equiv 1 \pmod{l^2},$$

then $\lambda_l(\mathbb{Q}^{(l)}(p)) = \mu_l(\mathbb{Q}^{(l)}(p)) = 0$.

The authors of [FKOT] considered the case where $l = 3$ and both of the conditions of Theorem A are not satisfied. Put $k = \mathbb{Q}^{(3)}(p)$. They have shown that $\lambda_3(k) = \mu_3(k) = 0$ if $(E_{\mathbb{Q}_1} : N_{k_1/\mathbb{Q}_1} E_{k_1}) = 9$, where k_1 (resp. $\mathbb{Q}_1$) is the first layer of the cyclotomic $\mathbb{Z}_3$-extension of k (resp. $\mathbb{Q}$), E_{k_1} (resp. $E_{\mathbb{Q}_1}$) is the unit group of k_1 (resp. $\mathbb{Q}_1$) and $N_{k_1/\mathbb{Q}_1}$ is the norm map from k_1 to $\mathbb{Q}_1$.

Recently, Komatsu investigated the field $k = \mathbb{Q}^{(3)}(73)$ and proved that $\lambda_3(k) = \mu_3(k) = 0$ (see [K]). (Note that $3^{(73-1)/3} \equiv 1 \pmod{73}$, $73 \equiv 1 \pmod{3^2}$ and $(E_{\mathbb{Q}_1} : N_{k_1/\mathbb{Q}_1} E_{k_1}) = 3$.)

In the present paper, we give simple sufficient conditions on p for $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ and verify $\lambda_3 = \mu_3 = 0$ for many $\mathbb{Q}^{(3)}(p)$'s including the case where $(E_{\mathbb{Q}_1} : N_{k_1/\mathbb{Q}_1} E_{k_1}) < 9$. Specifically, we show that $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ for all $p < 10000$ with $p \equiv 1 \pmod{3}$ except for $p = 3907, 7219, 8011, 8677$.

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2. Results. Our main criterion for $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ is the following:

THEOREM 1. *Let $k = \mathbb{Q}^{(3)}(p)$ be a cubic field with conductor p, where p is a prime number such that* (a) $3^{(p-1)/3} \equiv 1 \pmod{p}$ *and* (b) $p \equiv 1 \pmod{9}$. *Put $z = g^{(p-1)/9}$ for a primitive root g modulo p. If*

$$(*) \qquad \left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \not\equiv 1 \pmod{p},$$

then $\lambda_3(k) = \mu_3(k) = 0$.

Proof. First, we note that condition (a) holds if and only if 3 is decomposed in k and that condition (b) holds if and only if p is decomposed in $\mathbb{Q}_1$. Let $\mathbb{Q}_1(p)$ be the mod p ray class field of $\mathbb{Q}_1$ and $\mathbb{Q}_1(p)^{(3)}$ be the maximal subextension of $\mathbb{Q}_1(p)/\mathbb{Q}_1$ whose Galois group over $\mathbb{Q}_1$ is an elementary abelian 3-group. Then $k_1 \subseteq \mathbb{Q}_1(p)^{(3)}$ and $\mathbb{Q}_1(p)^{(3)}/k_1$ is unramified, because a prime of $\mathbb{Q}_1$ lying over p ramifies in k_1 and its ramification index in $\mathbb{Q}_1(p)^{(3)}/\mathbb{Q}_1$ is 3. Denote by $L^{\mathrm{ab}}(k_1)$ the maximal unramified abelian 3-extension field of k_1 which is abelian over $\mathbb{Q}_1$. Then $\mathbb{Q}_1(p)^{(3)} \subseteq L^{\mathrm{ab}}(k_1)$. Since $\mathbb{Q}_1$ has class number prime to 3 and the ramification index of every ramified prime in $L^{\mathrm{ab}}(k_1)/\mathbb{Q}_1$ is 3, $L^{\mathrm{ab}}(k_1)/\mathbb{Q}_1$ has no cyclic subextension of degree 9. Hence $L^{\mathrm{ab}}(k_1)/\mathbb{Q}_1$ is an elementary abelian 3-extension of conductor p. Therefore $\mathbb{Q}_1(p)^{(3)} = L^{\mathrm{ab}}(k_1)$. We put $F := \mathbb{Q}_1(p)^{(3)} = L^{\mathrm{ab}}(k_1)$ for simplicity. For a generator σ of $\mathrm{Gal}(k_1/\mathbb{Q}_1)$, $\mathrm{Gal}(F/k_1) \simeq A(k_1)/A(k_1)^{\sigma-1}$ by class field theory, where $A(k_1)$ stands for the 3-Sylow subgroup of the ideal class group of k_1. Also by class field theory,

$$(1) \qquad \mathrm{Gal}(F/\mathbb{Q}_1) \simeq (I_p/S_p)/(I_p/S_p)^3,$$

where I_p is the group of the fractional ideals of $\mathbb{Q}_1$ which are prime to p, and $S_p = \{\alpha\mathcal{O}_{\mathbb{Q}_1} \mid \alpha \in \mathbb{Q}_1^{\times},\ \alpha \equiv 1 \pmod{p}\} \subseteq I_p$, $\mathcal{O}_{\mathbb{Q}_1}$ being the integer ring of $\mathbb{Q}_1$. Since the class number of $\mathbb{Q}_1$ is prime to 3, we get the exact sequence

$$(2) \qquad E_{\mathbb{Q}_1} \to (\mathcal{O}_{\mathbb{Q}_1}/p)^{\times}/((\mathcal{O}_{\mathbb{Q}_1}/p)^{\times})^3 \to (I_p/S_p)/(I_p/S_p)^3 \to 0,$$

where $E_{\mathbb{Q}_1}$ stands for the unit group of $\mathbb{Q}_1$. Because

$$(\mathcal{O}_{\mathbb{Q}_1}/p)^{\times} \simeq \bigoplus_{\mathfrak{p}|p} (\mathcal{O}_{\mathbb{Q}_1}/\mathfrak{p})^{\times} \simeq (\mathbb{Z}/(p-1)\mathbb{Z})^{\oplus 3},$$

we get the isomorphism

$$(3) \qquad (\mathcal{O}_{\mathbb{Q}_1}/p)^{\times}/((\mathcal{O}_{\mathbb{Q}_1}/p)^{\times})^3 \overset{(p-1)/3}{\underset{\sim}{\longrightarrow}} (\mathcal{O}_{\mathbb{Q}_1}/p)^{\times(p-1)/3}.$$

Therefore it follows from (1)–(3) that

$$(4) \qquad \mathrm{Gal}(F/\mathbb{Q}_1) \simeq \mathrm{Coker}(E_{\mathbb{Q}_1} \overset{(p-1)/3}{\longrightarrow} (\mathcal{O}_{\mathbb{Q}_1}/p)^{\times(p-1)/3}).$$

Since $(\mathcal{O}_{\mathbb{Q}_1}/p)^{\times(p-1)/3} \simeq \bigoplus_{\mathfrak{p}|p}(\mathcal{O}_{\mathbb{Q}_1}/\mathfrak{p})^{\times(p-1)/3} \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus 3}$ we obtain 3-rank$(A(k_1)/A(k_1)^{\sigma-1}) =$ 3-rank$(\mathrm{Gal}(F/k_1)) \le 2$ by (4).

Let $\zeta = \zeta_9$ be a primitive 9th root of unity. We put $\pi = (\zeta-1)(\zeta^{-1}-1)$. Then $\pi \in \mathbb{Q}_1$ and $3\mathcal{O}_{\mathbb{Q}_1} = \pi^3\mathcal{O}_{\mathbb{Q}_1}$ in $\mathbb{Q}_1$. Now we choose $\gamma \in \mathrm{Gal}(\mathbb{Q}_1/\mathbb{Q})$ such that $\pi^\gamma = (\zeta^2-1)(\zeta^{-2}-1)$. If we put

$$C = \langle \pi^{\gamma-1}, \pi^{\gamma(\gamma-1)}, \pi^{\gamma^2(\gamma-1)}\rangle \subseteq E_{\mathbb{Q}_1},$$

then C is a subgroup of the cyclotomic units of $\mathbb{Q}_1 = \mathbb{Q}(\zeta_9)^+$ whose index is prime to 3. Since the class number of $\mathbb{Q}_1$ is prime to 3, we have $3 \nmid [E_{\mathbb{Q}_1} : C]$. Therefore

$$\begin{aligned}\text{(5)}\qquad \mathrm{Gal}(F/\mathbb{Q}_1) &\simeq \mathrm{Coker}(C \xrightarrow{(p-1)/3} (\mathcal{O}_{\mathbb{Q}_1}/p)^{\times(p-1)/3}) \\ &= \mathrm{Coker}(\langle \eta, \eta^\gamma, \eta^{\gamma^2}\rangle \xrightarrow{\varphi} (\mathcal{O}_{\mathbb{Q}_1}/p)^{\times(p-1)/3}),\end{aligned}$$

by (4), where $\eta = (\pi^{\gamma-1})^{(p-1)/3}$ and φ is the natural projection map.

From the above isomorphism, we find that 3-rank$(A(k_1)/A(k_1)^{\sigma-1}) \le 1$ if and only if $\mathrm{Im}(\varphi) \ne 0$. Also, since $\pi^{(p-1)/3} \cdot \pi^{(p-1)\gamma/3} \cdot \pi^{(p-1)\gamma^2/3} = 3^{(p-1)/3} \equiv 1 \pmod p$ from assumption (a), we obtain $\eta^\gamma = (\pi^{\gamma^2-\gamma})^{(p-1)/3} \equiv (\pi^{(p-1)/3}\cdot\pi^{(p-1)\gamma/3})^{-1}\cdot(\pi^{(p-1)\gamma/3})^{-1} \equiv (\pi^{(p-1)/3})^{-2\gamma-1} \equiv (\pi^{(p-1)/3})^{\gamma-1} = \eta \pmod p$. Hence

$$\text{(6)}\qquad \varphi(\eta) = \varphi(\eta^\gamma) = \varphi(\eta^{\gamma^2}).$$

We deduce from (5) and (6) that $\mathrm{Im}(\varphi) \ne 0$ is equivalent to $\eta \not\equiv 1 \pmod p$, and that $A(k_1)/A(k_1)^{\sigma-1} \ne 0$. For $z \in \mathbb{Z}$ in the statement of the theorem, there exists a prime ideal $\mathfrak{P}$ of $\mathbb{Q}(\zeta_9)$ lying above p such that $\zeta_9 \equiv z \pmod{\mathfrak{P}}$. We denote by $\mathfrak{p}$ the prime ideal of $\mathbb{Q}_1$ below $\mathfrak{P}$. Then $\eta \not\equiv 1 \pmod{\mathfrak{p}}$ if and only if condition $(*)$ holds. If $\eta \equiv 1 \pmod{\mathfrak{p}}$, then $\eta \equiv \eta^{\gamma^i} \equiv 1 \pmod{\mathfrak{p}^{\gamma^i}}$ for $i = 1, 2$ by (6), hence $\eta \equiv 1 \pmod p$. Therefore $\mathrm{Im}(\varphi) \ne 0$ is equivalent to $\eta \not\equiv 1 \pmod{\mathfrak{p}}$, which in turn is equivalent to condition $(*)$. Consequently, the three statements: 3-rank$(A(k_1)/A(k_1)^{\sigma-1}) = 1$, $\mathrm{Im}(\varphi) = \langle \eta \bmod p\rangle \ne 0$, and condition $(*)$, are equivalent.

Next we show that if $(*)$ is satisfied, then the natural map $D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}$ is a non-zero map, where $D(k_1)$ is the subgroup of $A(k_1)$ consisting of the ideal classes which contain a product of prime ideals lying over 3. One can see that $D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}$ is a non-zero map if and only if $\pi\mathcal{O}_{\mathbb{Q}_1}$ is not totally decomposed in $F/\mathbb{Q}_1$ by the canonical isomorphism $\mathrm{Gal}(F/k_1) \simeq A(k_1)/A(k_1)^{\sigma-1}$ and the fact that the prime $\pi\mathcal{O}_{\mathbb{Q}_1}$ splits in k_1. This is equivalent to

$$\pi^{(p-1)/3} \bmod p \notin \mathrm{Im}(\varphi),$$

which in turn is equivalent to

$$\pi^{(p-1)/3} \not\equiv \eta^a \pmod p \quad \text{for any } a \in \mathbb{Z},$$

because $\mathrm{Im}(\varphi) = \langle \eta \bmod p \rangle$ by (6). Now we assume that $\pi^{(p-1)/3} \equiv \eta^a \pmod p$ for some $a \in \mathbb{Z}$. Then $\pi^{(p-1)\gamma/3} \equiv \eta^{a\gamma} \equiv \eta^a \equiv \pi^{(p-1)/3} \pmod p$ by (6), hence

$$\eta = \pi^{(p-1)(\gamma-1)/3} \equiv 1 \pmod p.$$

But this contradicts the fact that $\mathrm{Im}(\varphi) \neq 0$, which is equivalent to assumption $(*)$. Hence assumption $(*)$ implies that the map $D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}$ is non-zero. Also if condition $(*)$ holds, then $A(k_1)/A(k_1)^{\sigma-1} \simeq \mathbb{Z}/3\mathbb{Z}$. Hence $D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}$ is surjective. Since $D(k_1)$ is a $\mathrm{Gal}(k_1/\mathbb{Q}_1)$-submodule of $A(k_1)$, the above surjection shows that $A(k_1) = D(k_1)$ by Nakayama's lemma. $D(k_1)$ capitulates in k_n for sufficiently large n, since Leopoldt's conjecture is valid for an abelian number field k (see [G1] and [B]). Therefore, $\lambda_3(k) = \mu_3(k) = 0$ by [O, Theorem], since $A(k_1)$ capitulates in k_∞. ■

Remark. From the above proof, one can find that condition $(*)$ holds if and only if $A(k_1) = D(k_1)$ under assumptions (a) and (b).

In the case where condition $(*)$ does not hold, we give the following sufficient condition on p for $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$:

Theorem 2. *Let $k = \mathbb{Q}^{(3)}(p)$ be a cyclic cubic field with conductor p, where p is a prime number such that $p \equiv 1 \pmod 9$ and $3^{(p-1)/3} \equiv 1 \pmod p$. Denote by χ a 3-adic Dirichlet character associated with k. Assume that*

$$\left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \equiv 1 \pmod p,$$

$$((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod p,$$

and that $f(T,\chi)$ is irreducible in $\mathbb{Z}_3[\chi(\mathrm{Gal}(k/\mathbb{Q}))][[T]]$, where z is as in the statement of Theorem 1 and $f(T,\chi)$ is the Iwasawa power series associated with the 3-adic L-function $L_3(s,\chi)$, namely, $f(4^s-1,\chi) = L_3(s,\chi)$ for $s \in \mathbb{Z}_3$. Then $\lambda_3(k) = \mu_3(k) = 0$.

Proof. Let k_∞/k be the cyclotomic $\mathbb{Z}_3$-extension, and k_n the nth layer of k_∞/k. Denote by $A(k_n)$ the 3-Sylow subgroup of the ideal class group of k_n. Put $G = \mathrm{Gal}(k/\mathbb{Q})$, $\Lambda = \mathbb{Z}_3[[\mathrm{Gal}(k_\infty/k)]]$, $X = \mathrm{Gal}(L(k_\infty)/k_\infty)$ and $\mathfrak{X} = \mathrm{Gal}(M(k_\infty)/k_\infty)$, where $L(k_\infty)/k_\infty$ and $M(k_\infty)/k_\infty$ are the maximal unramified pro-3 abelian extension and the maximal 3-ramified pro-3 abelian extension, respectively. Then X and $\mathfrak{X}$ are finitely generated torsion $\Lambda[G]$-modules (see [I2]). Let $\widetilde{\gamma} \in \mathrm{Gal}(k_\infty(\zeta_3)/k(\zeta_3))$ be such that $\zeta^{\widetilde{\gamma}} = \zeta^4$ for any 3-power-th root of unity ζ, where ζ_3 is a primitive 3rd root of unity, and put $\gamma = \widetilde{\gamma}|_{k_\infty} \in \mathrm{Gal}(k_\infty/k)$. In what follows we identify Λ with $\mathbb{Z}_3[[T]]$ via the correspondence $\gamma \leftrightarrow 1+T$. For any $\mathbb{Z}_3[G]$-module M, we put

$$M_\chi = M \bigotimes_{\mathbb{Z}_3[G]} \mathbb{Z}_3[\chi(G)],$$

where G acts on $\mathbb{Z}_3[\chi(G)]$ via χ.

We now show that $X_\chi \simeq X$ and $\mathfrak{X}_\chi \simeq \mathfrak{X}$. Since the kernel of the map

$$\mathbb{Z}_3[G] \to \mathbb{Z}_3[\chi(G)], \quad \sum a_g g \mapsto \sum a_g \chi(g),$$

is $N_G \mathbb{Z}_3[G]$ ($N_G := \sum_{g\in G} g$), we have $M_\chi = M/N_G M$ for any $\mathbb{Z}_3[G]$-module M. Because the class number of the nth layer of the cyclotomic $\mathbb{Z}_3$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ is prime to 3, $N_G A(k_n) = 0$. Hence it follows from $X \simeq \varprojlim A(k_n)$ that $X_\chi \simeq X$ (as $\Lambda[G]$-modules), where the projective limit is taken with respect to the norm map. Next we show $\mathfrak{X}_\chi \simeq \mathfrak{X}$. It is enough to prove $\mathfrak{X}^G = 0$ since $N_G\mathfrak{X} \subseteq \mathfrak{X}^G$. Let σ be a generator of G. Then $\mathfrak{X}/(\sigma-1)\mathfrak{X} \simeq \mathrm{Gal}(M(k_\infty)^{\mathrm{ab}}/k_\infty)$, where $M(k_\infty)^{\mathrm{ab}}$ is the maximal intermediate field of $M(k_\infty)/\mathbb{Q}_\infty$ which is abelian over $\mathbb{Q}_\infty$. Let $\mathfrak{P}$ be a prime of $\mathbb{Q}_\infty$ lying above p and $I_\mathfrak{P}$ the inertia subgroup of $\mathrm{Gal}(M(k_\infty)^{\mathrm{ab}}/\mathbb{Q}_\infty)$ for $\mathfrak{P}$. Then $I_\mathfrak{P} \simeq \mathbb{Z}/3\mathbb{Z}$ and $\sum_{\mathfrak{P}|p} I_\mathfrak{P} = \mathrm{Gal}(M(k_\infty)^{\mathrm{ab}}/\mathbb{Q}_\infty)$ because $\mathbb{Q}_\infty$ has no proper 3-ramified pro-3 abelian extension. Since the number of primes of $\mathbb{Q}_\infty$ lying above p is finite, $\mathfrak{X}/\mathfrak{X}^{\sigma-1} \simeq \mathrm{Gal}(M(k_\infty)^{\mathrm{ab}}/k_\infty)$ is finite. From the exact sequence of Λ-modules

$$0 \to \mathfrak{X}^G \to \mathfrak{X} \xrightarrow{\sigma-1} \mathfrak{X} \to \mathfrak{X}/\mathfrak{X}^{\sigma-1} \to 0,$$

it follows that $\mathrm{char}_\Lambda(\mathfrak{X}^G) = \mathrm{char}_\Lambda(\mathfrak{X}/\mathfrak{X}^{\sigma-1}) = \Lambda$, where $\mathrm{char}_\Lambda(M)$ denotes the characteristic ideal of M for any finitely generated torsion Λ-module M. Hence $\mathfrak{X}^G$ is finite. Because $\mathfrak{X}$ does not have non-trivial finite Λ-submodules ([G2]), we obtain $\mathfrak{X}^G = 0$. Thus $\mathfrak{X}_\chi \simeq \mathfrak{X}$.

From the Mazur–Wiles theorem ([MW, p. 214, Theorem])

$$\mathrm{char}_{\Lambda[G]_\chi}(\mathfrak{X}_\chi) = f(4(1+T)^{-1}-1,\chi)\Lambda[G]_\chi$$

and the surjection $\mathfrak{X}_\chi \to X_\chi$, it follows that

$$\mathrm{char}_{\Lambda[G]_\chi}(X_\chi) \supseteq f(4(1+T)^{-1}-1,\chi)\Lambda[G]_\chi.$$

Now assume that X_χ is infinite. Since $f(4(1+T)^{-1}-1,\chi)$ is irreducible in $\Lambda[G]_\chi = \mathbb{Z}_3[\chi(G)][[T]]$ by assumption, we see that

$$\mathrm{char}_{\Lambda[G]_\chi}(\mathfrak{X}_\chi) = \mathrm{char}_{\Lambda[G]_\chi}(X_\chi),$$

and $\mathrm{Ker}(\mathfrak{X}_\chi \to X_\chi)$ is finite. Since $\mathfrak{X}_\chi \simeq \mathfrak{X}$ and $\mathfrak{X}$ does not have non-trivial finite Λ-submodules, $\mathrm{Ker}(\mathfrak{X}_\chi \to X_\chi) = 0$, which implies $\mathfrak{X} \simeq \mathfrak{X}_\chi \simeq X_\chi \simeq X$. From the assumptions $\big(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\big)^{(p-1)/3} \equiv 1 \pmod p$ and $((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod p$, we can see that $D(k_1) \neq 0$ as in the proof of Theorem 1, since $\mathrm{Im}(\varphi) = 0$ from the first assumption, and $\pi^{(p-1)/3} \not\equiv 1 \pmod p$, i.e., $\pi^{(p-1)/3} \bmod p \notin \mathrm{Im}(\varphi) = 0$ from the second (notations as in the proof of Theorem 1). We write $D(k_n)$ for the subgroup of $A(k_n)$ consisting of the ideal classes which contain a product of prime ideals of k_n lying above 3. Because Leopoldt's conjecture is valid for k (see [B]), $\#D(k_n)$ is bounded (see [G1, Proposition 1]). Hence $\varprojlim D(k_n)$ is a non-trivial finite Λ-submodule of $\varprojlim A(k_n) \simeq X \simeq \mathfrak{X}$, because the norm map $D(k_m) \to D(k_n)$ is surjective for $m \geq n \geq 0$ and $D(k_1) \neq 0$. This contradicts the fact that $\mathfrak{X}$ does not have non-trivial finite Λ-submodules. Thus we have shown that $X \simeq X_\chi$ is finite, which is equivalent to $\lambda_3(k) = \mu_3(k) = 0$. ∎

We obtain the following corollary to Theorem 2:

Corollary 3. *Let p be a prime number such that $p \equiv 1 \pmod 9$ and $3^{(p-1)/3} \equiv 1 \pmod p$. Assume that*

$$\left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \equiv 1 \pmod p,$$

$$((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod p,$$

where z is as in the statement of Theorem 1. Denote by χ and ω a 3-adic Dirichlet character corresponding to $\mathbb{Q}^{(3)}(p)$ and the Teichmüller character for the prime 3, respectively. If

$$B_{1,\chi\omega^{-1}} \not\equiv 0 \pmod 3,$$

then $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$, where $B_{1,\chi\omega^{-1}}$ is the generalized Bernoulli number.

Proof. It is sufficient to show that $f(T,\chi) \in \mathbb{Z}_3[\zeta_3][[T]]$ is irreducible in $\mathbb{Z}_3[\zeta_3][[T]]$ by Theorem 2. Note that $-B_{1,\chi\omega^{-1}}$ is the constant term of $f(T,\chi)$, and that $g(T) \in \mathbb{Z}_3[\zeta_3][[T]]^\times$ if and only if $g(0) \in \mathbb{Z}_3[\zeta_3]$ is a unit. Hence we see immediately that $f(T,\chi)$ is irreducible in $\mathbb{Z}_3[\zeta_3][[T]]$. ∎

One can easily check whether the conditions of Theorem 1 and Corollary 3 hold or not by computer. We give some examples below.

We consider the prime numbers $p \leq 10000$ congruent to 1 modulo 3. There exist 611 such p's. By Theorem A, one can verify $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ for 547 among them. The remaining 64 prime numbers are as follows:

73, 271, 307, 523, 577, 613, 757, 919, 991, 1009, 1117, 1531, 1549, 1621, 1783, 2179, 2251, 2269, 2287, 2341, 2971, 3079, 3187, 3529, 3853, 3889, 3907, 4177, 4339, 4483, 4933, 4951, 4969, 5059, 5077, 5113, 5527, 5851, 6067, 6211, 6247, 6301, 6481, 6553, 6967, 7219, 7507, 7561, 7669, 7687, 8011, 8191, 8461, 8677, 8803, 8893, 8929, 9001, 9109, 9181, 9343, 9613, 9901, 9973.

For 42 prime numbers p among these, we can show that $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ by using Theorem 1. The remaining 22 are:

991, 1117, 1549, 2251, 2269, 2341, 3907, 4483, 4933, 5527, 6247, 6481, 6967, 7219, 7669, 7687, 8011, 8677, 8803, 9001, 9181, 9901.

For 10 prime numbers p among these, we can show that $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ by using Corollary 3. The remaining 12 are:

2269, 3907, 4933, 5527, 6247, 6481, 7219, 7687, 8011, 8677, 9001, 9901.

For 6 prime numbers p among these, we can verify that $f(T,\chi)$ is irreducible and $((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod{p}$ by computer, hence $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ by Theorem 2. The remaining 6 are:

2269, 3907, 6481, 7219, 8011, 8677.

For these, one can verify that $f(T,\chi)$ is reducible in the case $p = 7219$ and 8677. Also, one can verify that $((z-1)(z^{-1}-1))^{(p-1)/3} \equiv 1 \pmod{p}$ in the case $p = 3907$ and 8011. For $p = 2269$ and 6481, we do not know whether $f(T,\chi)$ is irreducible or not. In what follows we give another method to show $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ for $p = 2269$ and 6481.

We would like to express our thanks to Prof. Masato Kurihara who communicated to us the following theorem. It gives an upper bound of the λ_3-invariant of $\mathbb{Q}^{(3)}(p)$:

THEOREM 4 (M. Kurihara). *Let $k = \mathbb{Q}^{(3)}(p)$ be a cyclic cubic field with conductor p, where p is a prime number such that $p \equiv 1 \pmod{9}$ and $3^{(p-1)/3} \equiv 1 \pmod{p}$. If*

$$((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod{p},$$

then $\lambda_3(k) \le 2$. ($\mu_3(k) = 0$ by the Ferrero–Washington theorem.)

Proof. If $\left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \not\equiv 1 \pmod{p}$, then $\lambda_3(k) = 0$ by Theorem 1. Hence we may assume that $\left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \equiv 1 \pmod{p}$.

Let σ be a generator of $G = \mathrm{Gal}(k_\infty/\mathbb{Q}_\infty)$. As in the proof of Theorem 1, we can see that

$$A(k_1)/A(k_1)^{\sigma-1} \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}, \tag{7}$$

and

$$\mathrm{Im}(D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}) \simeq \mathbb{Z}/3\mathbb{Z}, \tag{8}$$

since $\left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \equiv 1 \pmod{p}$ and $((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod{p}$. Let $A'(k_n)$ be the 3-Sylow subgroup of the 3-ideal class group of k_n, namely, $A'(k_n) = A(k_n)/D(k_n)$. Then

$$A'(k_1)/A'(k_1)^{\sigma-1} \simeq \mathrm{Coker}(D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}) \simeq \mathbb{Z}/3\mathbb{Z} \tag{9}$$

from (7) and (8).

We now also show that

$$A'(k_2)/A'(k_2)^{\sigma-1} \simeq \mathbb{Z}/3\mathbb{Z}. \tag{10}$$

Denote by $\mathbb{Q}_2(p)^{(3)}$ the maximal abelian extension over $\mathbb{Q}_2$ (the second layer of the cyclotomic $\mathbb{Z}_3$-extension over $\mathbb{Q}$) of conductor p whose Galois group over $\mathbb{Q}_2$ is an elementary abelian 3-group. Then, as in the proof of Theorem 1, we find that $\mathbb{Q}_2(p)^{(3)}$ is the maximal unramified abelian 3-extension over k_2 which is abelian over $\mathbb{Q}_2$ and that $\mathrm{Gal}(\mathbb{Q}_2(p)^{(3)}/k_2) \simeq A(k_2)/A(k_2)^{\sigma-1}$, because the class number of $\mathbb{Q}_2$ is prime to 3.

Firstly, we consider the case $p \not\equiv 1 \pmod{27}$. In this case, as in the proof of Theorem 1, we have $\mathrm{Gal}(\mathbb{Q}_2(p)^{(3)}/\mathbb{Q}_2) \simeq \mathrm{Coker}(E_{\mathbb{Q}_2}^{(p^3-1)/3} \to ((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p^3-1)/3})$ and $3\text{-rank}(((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p^3-1)/3}) = 3$ because the prime p decomposes into three primes of degree three in $\mathbb{Q}_2$ by the assumption $p \not\equiv 1 \pmod{27}$. Hence $3\text{-rank}(A(k_2)/A(k_2)^{\sigma-1}) = 3\text{-rank}(\mathrm{Gal}(\mathbb{Q}_2(p)^{(3)}/k_2)) \le 2$. Since the norm maps $A(k_2) \to A(k_1)$ and $D(k_2) \to D(k_1)$ are surjective, it follows from (7) and (8) that $3\text{-rank}(A(k_2)/A(k_2)^{\sigma-1}) = 2$ and $\mathrm{Im}(D(k_2) \to A(k_2)/A(k_2)^{\sigma-1}) \ne 0$. Hence $3\text{-rank}(A'(k_2)/A'(k_2)^{\sigma-1}) \le 1$. Since the norm map $A'(k_2)/A'(k_2)^{\sigma-1} \to A'(k_1)/A'(k_1)^{\sigma-1}$ is surjective, the group $A'(k_2)/A'(k_2)^{\sigma-1}$ is non-trivial, so we have proved (10).

Next, we consider the case $p \equiv 1 \pmod{27}$. As in the case $p \not\equiv 1 \pmod{27}$, we deduce $\mathrm{Im}(D(k_2) \to A(k_2)/A(k_2)^{\sigma-1}) \ne 0$ from (8). Since this image is generated by an ideal class containing a prime of k_2 lying above 3, we have

$$\mathrm{Im}(D(k_2) \to A(k_2)/A(k_2)^{\sigma-1}) \simeq \mathbb{Z}/3\mathbb{Z}. \tag{11}$$

Let $\mathfrak{p}_2$ be a prime ideal of k_2 lying above p and $\mathfrak{p}_1$ the prime ideal of k_1 below $\mathfrak{p}_2$. We choose a primitive 9th root of unity ζ_9 such that $(z-1)(z^{-1}-1) \equiv (\zeta_9 - 1)(\zeta_9^{-1} - 1) \pmod{\mathfrak{p}_1}$. Further, we choose a primitive 27th root of unity ζ_{27} such that $N_{\mathbb{Q}_2/\mathbb{Q}_1}((\zeta_{27}-1)(\zeta_{27}^{-1}-1)) = (\zeta_9 - 1)(\zeta_9^{-1} - 1)$. Let γ be a topological generator of $\Gamma = \mathrm{Gal}(k_\infty/k)$ such that $((\zeta_9 - 1)(\zeta_9^{-1} - 1))^\gamma = (\zeta_9^2 - 1)(\zeta_9^{-2} - 1)$, and γ_n the restriction of γ to k_n. Then γ_n is a generator of $\Gamma_n = \mathrm{Gal}(k_n/k)$. For simplicity, we put $\pi_i = (\zeta_{3^{i+1}} - 1)(\zeta_{3^{i+1}}^{-1} - 1)$ and $\eta_i = (\pi_i^{\gamma_i - 1})^{(p-1)/3}$ for $i = 1, 2$. Then it follows from the assumption $((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod{p}$ that

$$\pi_1^{(p-1)/3} \not\equiv 1 \pmod{p}. \tag{12}$$

Also it follows from the assumption $\left(\frac{(z^2-1)(z^{-2}-1)}{(z-1)(z^{-1}-1)}\right)^{(p-1)/3} \equiv 1 \pmod{p}$ that

$$\eta_1 \equiv 1 \pmod{p} \tag{13}$$

as in the proof of Theorem 1.

In this case, we have

$$\mathrm{Gal}(\mathbb{Q}^{(3)}(p)/\mathbb{Q}_2) \simeq \mathrm{Coker}(\eta_2^{\mathbb{Z}[\Gamma_2]} \to ((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p-1)/3})$$

as in the proof of Theorem 1 since $[E_{\mathbb{Q}_2} : \eta_2^{\mathbb{Z}[\Gamma_2]}]$ is prime to 3. Note that we have $((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p-1)/3} \simeq \bigoplus_{i=0}^{8}((\mathcal{O}_{\mathbb{Q}_2}/\mathfrak{p}_2^{\gamma^i})^\times)^{(p-1)/3} \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus 9}$. Let $I = \mathrm{Ann}_{\mathbb{F}_3[\Gamma_2]}(\eta_2 \bmod p) \subseteq \mathbb{F}_3[\Gamma_2]$ be the annihilator ideal of $\eta_2 \bmod p \in ((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p-1)/3}$.

Then $\mathrm{Im}(\eta_2^{\mathbb{Z}[\Gamma_2]} \to ((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p-1)/3}) \simeq \mathbb{F}_3[\Gamma_2]/I$. We claim that $I = (\gamma_2-1)^6\mathbb{F}_3[\Gamma_2] = (1+\gamma_2^3+\gamma_2^6)\mathbb{F}_3[\Gamma_2]$. Since $\eta_2^{1+\gamma_2^3+\gamma_2^6} = \eta_1 \equiv 1 \pmod p$ by (13), we have $(1+\gamma_2^3+\gamma_2^6)\mathbb{F}_3[\Gamma_2] = (\gamma_2-1)^6\mathbb{F}_3[\Gamma_2] \subseteq I$. Now, assume that $(\gamma_2-1)^6\mathbb{F}_3[\Gamma_2] \subsetneq I$. Since $\mathbb{F}_3[\Gamma_2] \simeq \mathbb{F}_3[T]/(T^9-1) = \mathbb{F}_3[T]/(T-1)^9$ and $\mathbb{F}_3[T]$ is a principal ideal domain, we must have $(\gamma_2-1)^5 \in I$. Then

$$\begin{aligned}1 &\equiv \eta_2^{(\gamma_2-1)^5} = ((\pi_2^{\gamma_2-1})^{(p-1)/3})^{(\gamma_2-1)^5} = (\pi_2^{(\gamma_2-1)^6})^{(p-1)/3}\\ &\equiv (\pi_2^{1+\gamma_2^3+\gamma_2^6})^{(p-1)/3} = \pi_1^{(p-1)/3} \pmod p,\end{aligned}$$

which contradicts (12). Thus we obtain $I = (\gamma_2-1)^6\mathbb{F}_3[\Gamma_2]$, hence

$$\begin{aligned}\mathrm{Gal}(\mathbb{Q}^{(3)}(p)/\mathbb{Q}_2) &\simeq \mathrm{Coker}(\eta_2^{\mathbb{Z}[\Gamma_2]} \to ((\mathcal{O}_{\mathbb{Q}_2}/p)^\times)^{(p-1)/3})\\ &\simeq \mathbb{F}_3[\Gamma_2]/(\gamma_2-1)^6 \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus 3}.\end{aligned}$$

Therefore we have $A(k_2)/A(k_2)^{\sigma-1} \simeq \mathrm{Gal}(\mathbb{Q}^{(3)}(p)/k_2) \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ and $A'(k_2)/A'(k_2)^{\sigma-1} \simeq \mathbb{Z}/3\mathbb{Z}$ by (11). Thus we have proved (10).

Let $L'(k_\infty)$ be the maximal unramified pro-3 abelian extension of k_∞ in which every prime of k_∞ lying above 3 splits completely, and $X' = \mathrm{Gal}(L'(k_\infty)/k_\infty)$. Then $X' \simeq \varprojlim A'(k_n)$, and it follows from $A'(k) = 0$ that $X'/\nu_n X' \simeq A'(k_n)$, where $\nu_n = 1+\gamma+\gamma^2+\ldots+\gamma^{3^n-1}$ (see [I2]). Hence

$$(X'/X'^{\sigma-1})/\nu_n(X'/X'^{\sigma-1}) \simeq A'(k_n)/A'(k_n)^{\sigma-1} \tag{14}$$

for all $n \geq 1$.

We need the following lemma:

LEMMA 5. *Let $\Lambda = \mathbb{Z}_l[[\Gamma]]$, where l is any prime number and Γ is a pro-l group isomorphic to $\mathbb{Z}_l$. For a topological generator γ of Γ, put $\nu_n = 1+\gamma+\gamma^2+\ldots+\gamma^{l^n-1} \in \Lambda$. For a finitely generated Λ-module M and some $n \geq 0$, if the identity map $M \simeq M$ induces the isomorphism*

$$M/\nu_{n+1}M \simeq M/\nu_n M,$$

then $\nu_n M = 0$.

Proof. By assumption,

$$\mathrm{Ker}(M/\nu_{n+1}M \to M/\nu_n M) \simeq \nu_n M/\nu_{n+1}M = \nu_n M/(\nu_{n+1}/\nu_n)\nu_n M = 0.$$

Since ν_{n+1}/ν_n is contained in the maximal ideal of Λ, we have $\nu_n M = 0$ by Nakayama's lemma. ■

We can apply Lemma 5 to $M = X'/X'^{\sigma-1}$ and $n = 1$ by (9), (10) and (14). Then we get $\nu_1(X'/X'^{\sigma-1}) = 0$. Hence

$$X'/X'^{\sigma-1} \simeq X'/X'^{\sigma-1}/\nu_1(X'/X'^{\sigma-1}) \simeq A'(k_1)/A'(k_1)^{\sigma-1} \simeq \mathbb{Z}/3\mathbb{Z}$$

by (9) and (14). Therefore, there exists an $x' \in X'$ such that $X' = \mathbb{Z}_3[G]x'$ by Nakayama's lemma. Since $(1 + \sigma + \sigma^2)x' = 0$, there is a surjection $\mathbb{Z}_3^{\oplus 2} \simeq \mathbb{Z}_3[G]/N_G Z_3[G] \to X'$. Hence $\operatorname{rank}_{\mathbb{Z}_3} X' \le 2$. Because $\varprojlim D(k_n)$ is finite, we have $\lambda_3(k) = \operatorname{rank}_{\mathbb{Z}_3}(\varprojlim A(k_n)) = \operatorname{rank}_{\mathbb{Z}_3}(\varprojlim A(k_n)/\varprojlim D(k_n)) = \operatorname{rank}_{\mathbb{Z}_3} X' \le 2$. Thus we have proved Theorem 2. ∎

COROLLARY 6. *Let k satisfy the assumptions of Theorem 4. If $\lambda_3(k) \neq 0$, then*

$$X' \simeq \mathbb{Z}_3^{\oplus 2} \quad \text{and} \quad \operatorname{Tor}_{\mathbb{Z}_3}(X) = D,$$

where $X = \varprojlim A(k_n)$ and $D = \varprojlim D(k_n)$.

Proof. We consider a surjective homomorphism

$$f : \mathbb{Z}_3[\zeta_3] \simeq \mathbb{Z}_3[G]/N_G Z_3[G] \to X', \qquad \zeta_3 \mapsto \sigma \mapsto \sigma(x'),$$

where x' is as in the proof of Theorem 3 ($X' = \mathbb{Z}_3[G]x'$). If $\operatorname{Ker}(f) \neq 0$, then X' is finite because any non-zero ideal of $\mathbb{Z}_3[\zeta_3]$ has finite index in $\mathbb{Z}_3[\zeta_3]$. Hence $\lambda_3(k) = \operatorname{rank}_{\mathbb{Z}_3}(X') = 0$. Therefore, under the assumption $\lambda_3(k) \neq 0$, f must be injective, hence an isomorphism. This shows that $X' \simeq \mathbb{Z}_3^{\oplus 2}$. Further, from the exact sequence $0 \to D \to X \to X' \to 0$ and the fact that the order of D is finite, we see immediately that $\operatorname{Tor}_{\mathbb{Z}_3}(X) = D$. ∎

Now, we investigate the case $p = 2269$. The number field computations in what follows where done by using KASH, version 2.2. The prime number $p = 2269$ satisfies the assumption of Theorem 4. Let M be the decomposition field of the polynomial $x^3 + 124794x^2 + 5186218509x + 71770829079384$ over $\mathbb{Q}$. Then M is a cubic subfield of k_1 different from k and $\mathbb{Q}_1$, where $k = \mathbb{Q}^{(3)}(2269)$. We find that $A(M) \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$, where $A(F)$ denotes the 3-Sylow subgroup of the ideal class group of F for any number field F. Let $\mathfrak{p}$ and $\mathfrak{l}$ be the prime ideals of M lying above 2269 and 3, respectively. For a number field F and a fractional ideal $\mathfrak{a}$ of F, we denote by $\pi_F(\mathfrak{a})$ the projection of $\mathfrak{a}$ to $A(F)$. Then $\pi_M(\mathfrak{p}) \neq 0$ and $\pi_M(\mathfrak{l}) = 0$. Hence

$$\pi_{k_1}(\mathfrak{p}_1) \notin D(k_1) \tag{15}$$

for any prime ideal $\mathfrak{p}_1$ of k_1 lying above $\mathfrak{p}$, because

$$N_{k_1/M}(\pi_{k_1}(\mathfrak{p}_1)) = \pi_M(\mathfrak{p}) \notin N_{k_1/M}(D(k_1)) = \langle \pi_M(\mathfrak{l}) \rangle = 0.$$

We shall apply the following lemma:

LEMMA 7. *Let $l \ge 2$ be a prime number and $k/\mathbb{Q}$ a cyclic extension with $[k : \mathbb{Q}] = l$. Denote by F_∞/F and F_n the cyclotomic $\mathbb{Z}_l$-extension and its nth layer, respectively, for any number field F. Let $\mathfrak{p}$ be a prime ideal of k_n which* *ramifies in $k_n/\mathbb{Q}_n$. If $\mathfrak{p}$ splits completely in k_m and l-rank $A'(k_m) < l^{m-n}$ for some $m \geq n$, then $\pi_{k_n}(\mathfrak{p}) \in A(k_n)$ capitulates in k_∞, where $A'(k_n)$ is the l-Sylow subgroup of the l-ideal class group of k_n, and π_{k_n} denotes the natural projection map from the ideal group of k_n to the l-Sylow subgroup $A(k_n)$ of the ideal class group of k_n.*

Proof. We write I_K, $A'(K)$ and $A(K)$ for the ideal group of K, the l-Sylow subgroup of the l-ideal class group of K, and the l-Sylow subgroup of the ideal class group of K, respectively, for any subfield K of $\overline{\mathbb{Q}}$. Also we denote by π'_K and π_K the natural projection maps from I_K to $A'(K)$ and $A(K)$, respectively.

We write $\mathfrak{P}$ for a prime ideal of k_m lying above $\mathfrak{p}$. Since $\mathfrak{P}^l \in I_{\mathbb{Q}_m}$ and since $A'(\mathbb{Q}_m) = 0$, we see that $\pi'_{k_m}(\mathfrak{P})^l = 0$. We consider the map $\psi : \mathbb{Z}/l\mathbb{Z}[\mathrm{Gal}(k_m/k_n)] \to A'(k_m)[l]$, $f \mapsto f\pi'_{k_m}(\mathfrak{P})$, where $A'(k_m)[l]$ is the subgroup of $A'(k_m)$ consisting of the elements whose order divides l. It follows from the assumption that

$$\#(A'(k_m)[l]) < l^{l^{m-n}} = \#\mathbb{Z}/l\mathbb{Z}[\mathrm{Gal}(k_m/k_n)].$$

Hence $\mathrm{Ker}(\psi) \neq 0$, which implies $\mathrm{Ker}(\psi)^{\mathrm{Gal}(k_m/k_n)} \neq 0$. Because

$$\mathbb{Z}/l\mathbb{Z}[\mathrm{Gal}(k_m/k_n)]^{\mathrm{Gal}(k_m/k_n)} = \mathbb{Z}/l\mathbb{Z} \sum_{\gamma \in \mathrm{Gal}(k_m/k_n)} \gamma,$$

we have $\sum_{\gamma \in \mathrm{Gal}(k_m/k_n)} \gamma \in \mathrm{Ker}(\psi)$. So $\pi'_{k_m}(\mathfrak{p}) = \sum_{\gamma \in \mathrm{Gal}(k_m/k_n)} \gamma\pi'_{k_m}(\mathfrak{P}) = 0$, namely, $\pi_{k_m}(\mathfrak{p}) \in D(k_m)$ since $A'(k_m) = A(k_m)/D(k_m)$, where $D(k_m)$ is the subgroup of $A(k_m)$ consisting of the ideal classes which contain a product of prime ideals of k_m lying above l. Because $k/\mathbb{Q}$ is abelian, $D(k_m)$ capitulates in k_∞. Therefore $\pi_{k_n}(\mathfrak{p})$ capitulates in k_∞. ■

Assume that $\lambda_3(k) \neq 0$. Since $2269 \equiv 1 \pmod{27}$, a prime ideal $\mathfrak{p}_1$ of k_1 lying above 2269 decomposes into three prime ideals in k_2. Then $\pi_{k_1}(\mathfrak{p}_1)$ capitulates in k_∞ by Lemma 7, because 3-rank$(A'(k_2)) \leq$ 3-rank$(X') = 2$ by Corollary 6. Thus $\pi_{k_1}(\mathfrak{p}_1) \in \mathrm{Ker}(A(k_1) \to A(k_\infty)) = \mathrm{Im}(\mathrm{Tor}_{\mathbb{Z}_3} X \to A(k_1))$ by [O, Proposition]. But this contradicts (15) since $\mathrm{Tor}_{\mathbb{Z}_3} X = D$ and $\mathrm{Im}(D \to A(k_1)) = D(k_1)$ by Corollary 6. Therefore, we have shown that $\lambda_3(k) = \mu_3(k) = 0$.

In the case $p = 6481$, one can verify $\lambda_3(\mathbb{Q}^{(3)}(6481)) = \mu_3(\mathbb{Q}^{(3)}(6481)) = 0$ in the same manner.

Consequently, we have the following result:

THEOREM 8. $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ *for all prime numbers* $p <$ 10000 *with* $p \equiv 1 \pmod 3$ *but* $p = 3907, 7219, 8011, 8677$.

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