ACTA ARITHMETICA
142.1 (2010)

Non-converging continued fractions related to the Stern diatomic sequence

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1. Introduction. Given an integer $a \geq 2$, it was recently observed in [7] that the regular continued fraction

$$\mathcal{C}(a) = a + \cfrac{1}{a^2 + \cfrac{1}{a^4 + \cfrac{1}{\ddots + \cfrac{1}{a^{2^n} + \cfrac{1}{\ddots}}}}}, \tag{1.1}$$

denoted subsequently as usual by $[a, a^2, a^4, \ldots, a^{2^n}, \ldots]$, is transcendental. This is a consequence of Roth's theorem and follows directly from a result of Davenport and Roth [5] concerning the growth of denominators of convergents to an algebraic number. Quite surprisingly, the author of the present note was not able to pick up the scent of this simple example in the older literature though a function field analogue previously appeared in [12]. Indeed, viewed as a Laurent series in $\mathbb{F}_2((1/x))$, the continued fraction $\mathcal{C}(x)$ has the remarkable property of being a cubic element over the field $\mathbb{F}_2(x)$. More precisely, it is the unique root in $\mathbb{F}_2((1/x))$ of the polynomial

$$t^3 + xt^2 + 1.$$

This follows from a simple computation using the fact that squaring here has a very transparent effect: if $f(x) = [a_1(x), a_2(x), \ldots] \in \mathbb{F}_2((1/x))$ then $f(x)^2 = [a_1(x^2), a_2(x^2), \ldots]$.

Curiously, the fact that $\mathcal{C}(x)$ is algebraic over $\mathbb{F}_2(x)$ with degree larger than 2 suggests that many other evaluations of $\mathcal{C}$ should be transcendental. For instance, for any field $\mathbb{K}$ of zero characteristic the continued fraction

2010 *Mathematics Subject Classification*: 11J70, 11J81, 11B39.
Key words and phrases: continued fractions, Stern diatomic sequence, Mahler's method.

DOI: 10.4064/aa142-1-6

$\mathcal{C}(x)$, viewed as an element of $\mathbb{K}((1/x))$, is transcendental over $\mathbb{K}(x)$. This is a consequence of the function field analogue in zero characteristic of Roth's theorem obtained in [11]. Notice also that the continued fraction $\mathcal{C}(q)$ converges for every complex number q with $|q| > 1$. As a direct application of a classical result of Mahler [9] (see Theorem M in Section 4 below), we extend the result of [7] mentioned above as follows.

THEOREM 1.1. *Let q be an algebraic number with $|q| > 1$. Then the real number $\mathcal{C}(q)$ is transcendental.*

As we will see, the situation is more intriguing when $\mathcal{C}$ is evaluated at complex numbers lying in the open unit disc. Indeed, when q is a non-zero complex number of modulus less than 1, the continued fraction $\mathcal{C}(q)$ is no longer convergent. This follows from the classical Stern–Stolz theorem (see for instance [8, p. 94]). However, the Stern–Stolz theorem tells us that $\mathcal{C}(q)$ is almost convergent in the sense that both

$$(1.2) \qquad \lim_{\substack{n\to\infty \\ n \text{ even}}} [q, q^2, q^4, \ldots, q^{2^n}] \quad \text{and} \quad \lim_{\substack{n\to\infty \\ n \text{ odd}}} [q, q^2, q^4, \ldots, q^{2^n}]$$

do exist for every complex number q with $0 < |q| < 1$. The authors of [7] discovered a nice relation between these limits and the Stern diatomic sequence (see Theorem DS below). We now briefly recall this connexion.

The Stern diatomic sequence, numbered as A002487 in Sloane's list, is a remarkable sequence of positive integers that has been studied by various authors (see for instance the references in [6]). It is defined by the following recurrence relation:

$$a_{2n} = a_n, \qquad a_{2n+1} = a_n + a_{n-1}, \qquad \forall n \geq 1,$$

with $a_0 = a_1 = 1$. The Stern sequence is also related to the Fibonacci sequence. Indeed the maximum of the Stern sequence between two consecutive powers of 2, say between 2^{n-2} and 2^{n-1}, is the nth Fibonacci number F_n. This maximum is attained twice at the indices

$$\alpha_n := \frac{1}{3}\,(2^n + (-1)^{n+1}) \quad \text{and} \quad \beta_n := \frac{1}{3}\,(5 \cdot 2^{n-2} + (-1)^n).$$

In [6] the authors introduced a polynomial analogue of the Stern sequence. These polynomials are defined by the recurrence relation

$$a(2n; x) = a(n; x^2), \qquad a(2n+1; x) = xa(n; x^2) + a(n+1; x),$$

with $a(0; x) = a(1; x) = 1$. The polynomial $a(n; x)$ is termed the nth *Stern polynomial*. In [7], the same authors studied the subsequence of Stern polynomials with index α_n and with index β_n. For every positive integer n, they define the two polynomials

$$f_n(x) := a(\alpha_n; x) \quad \text{and} \quad \bar{f}_n(x) := a(\beta_n; x).$$

These polynomials can be thought of as polynomial analogues or q-analogues (replacing x by q) of the Fibonacci numbers. Also, the authors proved that the sequences of polynomials $(f_{2n}(x))_{n\geq 1}$ and $(\overline{f}_{2n+1}(x))_{n\geq 2}$ converge to the same formal power series

$$F(x) = 1 + x + x^2 + x^5 + x^6 + x^8 + x^9 + x^{10} + x^{21} + x^{22} + x^{24} + \cdots ,$$

and the sequences of polynomials $(f_{2n+1}(x))_{n\geq 1}$ and $(\overline{f}_{2n}(x))_{n\geq 2}$ converge to the same formal power series

$$G(x) = 1 + x + x^3 + x^4 + x^5 + x^{11} + x^{12} + x^{13} + x^{16} + x^{17} + x^{19} + \cdots .$$

The following result obtained in [7] links the values of the functions F and G with the unusual continued fractions defined in (1.2).

Theorem DS. *For every algebraic number q with $0 < |q| < 1$, we have*

$$\mathcal{C}_{\mathrm{ev}}(q) := \lim_{\substack{n\to\infty \\ n \text{ even}}} [q, q^2, q^4, \ldots, q^{2^n}] = \frac{qF(q^3)}{G(q^6)},$$

$$\mathcal{C}_{\mathrm{od}}(q) := \lim_{\substack{n\to\infty \\ n \text{ odd}}} [q, q^2, q^4, \ldots, q^{2^n}] = \frac{G(q^3)}{q^2F(q^6)}.$$

These authors also derived many functional equations satisfied by F and/or G. We will combine such relations with more involved material about Mahler's method that is contained in the monograph of K. Nishioka [10] (Theorems N1 and N2 in Section 3 below) to prove the following result.

Theorem 1.2. *Let q be an algebraic number with $0 < |q| < 1$. Then $\mathcal{C}_{\mathrm{ev}}(q)$ and $\mathcal{C}_{\mathrm{od}}(q)$ are both transcendental.*

Note that $\mathcal{C}(1)$ is well-defined and algebraic, for we easily get $\mathcal{C}(1) = (1+\sqrt{5})/2$. It would be interesting to determine more precisely the behavior of the continued fraction $\mathcal{C}(q)$ when q runs along the unit circle.

The authors of [7] also asked about transcendence results concerning the functions F and G but they did not obtain anything conclusive. They mentioned a paper of Loxton and van der Poorten dealing with the so-called Mahler method, but observed that the main theorem in that paper cannot be applied to F and G. In a subsequent paper [4] Coons proved that both functions $F(x)$ and $G(x)$ are transcendental over $\mathbb{Q}(x)$. This follows from a simple application of a classical theorem of Fatou. In the same vein as Theorem 1.2, we will prove the following stronger result.

Theorem 1.3. *Let q be an algebraic number with $0 < |q| < 1$. Then $F(q)$ and $G(q)$ are transcendental numbers.*

Our paper is organized as follows. Before proving our main results, we observe in Section 2 that F and G turn out to be examples of so-called 2-automatic functions. This connection with the theory of finite automata leads to some useful observations. The proof of Theorem 1.3 is given in Section 3 while Section 4 is devoted to the proofs of Theorems 1.1 and 1.2.

2. Functional equations, finite automata and transcendence over function fields. Among the various functional equations derived in [7] one finds the following ones (equations (5.3) and (5.4) in that paper):

$$F(x) = G(x^2) + xF(x^4), \tag{2.1}$$

$$G(x) = xF(x^2) + G(x^4). \tag{2.2}$$

The sets of integers appearing as exponents of the power series F and G seem to enjoy some regularity inherited from equations (2.1) and (2.2). Let us denote by

$$\begin{aligned} \varPhi &:= \{0, 1, 2, 5, 6, 8, 9, 10, 21, 22, \ldots\}, \\ \varGamma &:= \{0, 1, 3, 4, 5, 11, 12, 13, 16, 17, \ldots\} \end{aligned} \tag{2.3}$$

these sets of integers. The authors of [7] claimed that these sets are examples of so-called "self-generating sequences of integers". More precisely, they stated without proof that $\varPhi$ and $\varGamma$ are the minimal sets of non-negative integers such that 0 and 1 belong to $\varPhi \cap \varGamma$ and

$$\varPhi \supseteq (4\varPhi + 1) \cup 2\varGamma, \qquad \varGamma \supseteq (2\varPhi + 1) \cup 4\varGamma.$$

We describe now a different, and perhaps more natural, way to describe the sets $\varPhi$ and $\varGamma$. This involves the theory of finite automata. Actually, $\varPhi$ and $\varGamma$ turn out to be 2-*automatic* sets of integers (also sometimes called 2-*recognizable* or 2-*regular*). This means that there exists a finite automaton that accepts exactly the finite words corresponding to the binary expansions of the integers that belong to $\varPhi$; the same holds for $\varGamma$. This notion of automatic sequence is of great importance in theoretical computer science and combinatorics on words. We refer the reader to the monograph of Allouche and Shallit [2] for precise definitions and more material on this topic.

Proposition 2.1. *Both sets $\varPhi$ and $\varGamma$ are recognizable by a finite* 2-*automaton.*

It follows from Proposition 2.1 in [6] that the coefficients of the power series F and G only take the values 0 and 1. There thus exist two binary sequences $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ such that we can rewrite F and G as $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$. Thus, for every prime number p, we can reduce these power series modulo p and define

$$F_p(x) := \sum_{n\geq 0} f_n x^n \in \mathbb{F}_p((x)) \quad \text{and} \quad G_p(x) := \sum_{n\geq 0} g_n x^n \in \mathbb{F}_p((x)).$$

With this notation, we get the following result.

THEOREM 2.2. *Both functions $F_2(x)$ and $G_2(x)$ are algebraic over $\mathbb{F}_2(x)$. If $p \geq 3$ is a prime number, then $F_p(x)$ and $G_p(x)$ are transcendental over $\mathbb{F}_p(x)$.*

Note that Theorem 2.2 strengthens Theorem 4.1 of [4], for it directly implies that F and G are transcendental over $\mathbb{Q}(x)$. Furthermore, it offers a first ready-made result concerning the transcendence of values of F and G. Indeed, since F_2 and G_2 are algebraic irrational Laurent series over $\mathbb{F}_2(x)$, Theorem 7 in [1] implies that for every integer $b \geq 2$, both real numbers $F(1/b)$ and $G(1/b)$ are transcendental.

Proposition 2.1 now follows from Theorem 2.2 and Christol's theorem:

Proof of Proposition 2.1. By Theorem 2.2, $F_2(x)$ and $G_2(x)$ are algebraic over $\mathbb{F}_2(x)$. Then it follows from a classical theorem of Christol (see [2, Theorem 12.2.5]) that the sets of integers appearing as exponents of the power series F_2 and G_2 are recognizable by finite 2-automata. Since these exponents are the same as those of F and G, this ends the proof. ∎

We end this section with the proof of Theorem 2.2.

Proof of Theorem 2.2. The main point is to consider the following functional equations obtained in Proposition 5.1 of [7]:

$$F(x) = (1 + x + x^2)F(x^4) - x^4F(x^{16}), \tag{2.4}$$

$$xG(x) = (1 + x + x^2)G(x^4) - G(x^{16}). \tag{2.5}$$

Now, we can use a classical trick when working in positive characteristic, say p. In that case, taking pth powers of elements in the field of Laurent power series $\mathbb{F}_p((x))$ leads to very simple expressions. Indeed, we recall that for any $f(x) \in \mathbb{F}_p((x))$ we have the following fundamental equality:

$$f(x)^p = f(x^p). \tag{2.6}$$

Thus, reducing (2.4) and (2.5) modulo 2, we infer from (2.6) that

$$F_2(x) = (1 + x + x^2)F_2(x)^4 + x^4F_2(x)^{16},$$
$$xG_2(x) = (1 + x + x^2)G_2(x)^4 - G_2(x)^{16}.$$

Consequently, F_2 and G_2 are algebraic functions over $\mathbb{F}_2(X)$, as claimed.

On the other hand, it is easy to infer from (2.4) and (2.5) that F and G are not rational functions: this follows from some easy considerations involving the degree of rational functions; it also follows from other simple observations as shown in [4] where the transcendence of both F and G is derived. Then, combining Christol's theorem and a classical theorem of Cobham (see [2, Theorem 11.2.1]), we deduce that F_p and G_p are transcendental over $\mathbb{F}_p(X)$ for every prime number $p \neq 2$. Note that the idea to combine

Christol's and Cobham's theorems in such a way dates back to [3]. This concludes the proof. ∎

3. Proof of Theorem 1.3. All the material we will need for proving Theorem 1.3 can be found in the monograph of K. Nishioka [10], which serves as a reference about Mahler's method.

Actually, we will derive Theorem 1.3 from the following result concerning algebraic independence of values of the function G at algebraic points.

PROPOSITION 3.1. *Let q be a complex number such that $0 < |q| < 1$. Then the complex numbers $G(q)$ and $G(q^4)$ are algebraically independent.*

Before proving Proposition 3.1, we recall some results about Mahler's method. We will need in particular the following two results from [10]. Theorem N1 below corresponds to a particular case of Theorem 4.2.1 in [10].

THEOREM N1. *Let m and d be two integers not smaller than 2 and let $f_1, \ldots, f_m$ be analytic functions that converge in the complex open unit disc. Suppose that $f_1, \ldots, f_m$ satisfy the following system of functional equations:*

$$\begin{pmatrix} f_1(z^d) \\ \vdots \\ f_m(z^d) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix},$$

where $A(z)$ is an $m \times m$ matrix with entries in $\mathbb{Q}(z)$. If q is a non-zero algebraic number with $|q| < 1$ and such that, for every positive integer k, q^{d^k} is not a pole of $A(z)$, then

$$\operatorname{trans.deg}_{\mathbb{Q}} \mathbb{Q}(f_1(q), \ldots, f_m(q)) \geq \operatorname{trans.deg}_{\mathbb{Q}(z)} \mathbb{Q}(z)(f_1(z), \ldots, f_m(z)).$$

In the case where $m = 2$ in Theorem N1, the following result appears to be very useful to prove that the functions f_1 and f_2 are algebraically independent over $\mathbb{Q}(z)$. It corresponds to Theorem 5.2 in [10].

THEOREM N2. *Use the notation of Theorem N1 with $m = 2$. For every positive integer n, let $g_{11}^{(n)}(z)$, $g_{12}^{(n)}(z)$, $g_{21}^{(n)}(z)$ and $g_{22}^{(n)}(z)$ be the polynomials defined by*

$$\begin{pmatrix} g_{11}^{(n)}(z) & g_{12}^{(n)}(z) \\ g_{21}^{(n)}(z) & g_{22}^{(n)}(z) \end{pmatrix} := A(z^{d^{n-1}}) A(z^{d^{n-2}}) \ldots A(z^d) A(z).$$

If at least one of the functions f_1 and f_2 is transcendental over $\mathbb{C}(z)$, and if f_1 and f_2 are algebraically dependent over $\mathbb{C}(z)$, then there exists a positive integer n_0 such that at least one of the following conditions holds:

(i) $g_{12}^{(n)}(z) = 0$ *for every* $n = kn_0$, $k = 1, 2, \ldots;$

(ii) $g_{21}^{(n)}(z) = 0$ *for every* $n = kn_0$, $k = 1, 2, \ldots$;

(iii) *there exist a positive integer* e *and relatively prime polynomials* $a(z)$ *and* $b(z)$ *in* $\mathbb{C}[z]$ *such that*

$$\frac{b(z)}{a(z)} = \frac{b(z^{4^n})g_{11}^{(n)}(z^e) + a(z^{4^n})g_{21}^{(n)}(z^e)}{b(z^{4^n})g_{12}^{(n)}(z^e) + a(z^{4^n})g_{22}^{(n)}(z^e)} \quad \text{for every } n = kn_0, k = 1, 2, \ldots.$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Our starting point is equation (2.5) that we recall below:

$$zG(z) = (1 + z + z^2)G(z^4) - G(z^{16}). \tag{3.1}$$

Set $f_1(z) := G(z)$ and let f_2 be the analytic function defined by $f_2(z) := G(z^4)$. Then we infer from (3.1) that f_1 and f_2 satisfy the following system of functional equations:

$$\begin{pmatrix} f_1(z^4) \\ f_2(z^4) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}, \tag{3.2}$$

where

$$A(z) := \begin{pmatrix} 0 & 1 \\ -z & 1 + z + z^2 \end{pmatrix}.$$

Now we easily infer from Theorem N1 with $m = 2$ and $d = 4$ that, if the functions f_1 and f_2 are algebraically independent over the field $\mathbb{C}(z)$, then the complex numbers $f_1(q)$ and $f_2(q)$ are algebraically independent for every complex number such that $0 < |q| < 1$, as claimed.

To end the proof of Proposition 3.1, it remains to prove that f_1 and f_2 are algebraically independent functions over $\mathbb{C}(z)$. To do so, we will use Theorem N2. For every positive integer n, we define the polynomials $g_{11}^{(n)}(z)$, $g_{12}^{(n)}(z)$, $g_{21}^{(n)}(z)$ and $g_{22}^{(n)}(z)$ by

$$A_n(z) := \begin{pmatrix} g_{11}^{(n)}(z) & g_{12}^{(n)}(z) \\ g_{21}^{(n)}(z) & g_{22}^{(n)}(z) \end{pmatrix} := A(z^{4^{n-1}})A(z^{4^{n-2}}) \ldots A(z^4)A(z).$$

From now on, we assume that f_1 and f_2 are algebraically dependent and we aim at deriving a contradiction. Since we already observed that $f_1 = G$ is a transcendental function over $\mathbb{Q}(z)$ and thus over $\mathbb{C}(z)$ (the coefficients of G are integers), we infer that at least one of the conditions (i), (ii) and (iii) of Theorem N2 holds.

We first prove that (i) and (ii) both lead to a contradiction. Indeed, we can easily deduce from their formulation the following recurrence relations linking the polynomials $g_{ij}^n(z)$:

$$g_{11}^{(n+1)}(z)=g_{21}^{(n)}(z),\qquad g_{12}^{(n+1)}(z)=g_{22}^{(n)}(z),\tag{3.3}$$

$$g_{21}^{(n+1)}(z)=-z^4g_{11}^{(n)}(z)+(1+z^{4^n}+z^{24^n})g_{21}^{(n)}(z),\tag{3.4}$$

$$g_{22}^{(n+1)}(z)=-z^4g_{12}^{(n)}(z)+(1+z^{4^n}+z^{24^n})g_{22}^{(n)}(z).\tag{3.5}$$

From these relations we can show by induction that, for every positive integer n,

$$\deg g_{21}^{(n+1)}(z)=2\cdot 4^n+\deg g_{21}^{(n)}(z),\qquad \deg g_{22}^{(n+1)}(z)=2\cdot 4^n+\deg g_{22}^{(n)}(z).$$

We then obtain the following equality:

$$\begin{pmatrix} \deg g_{11}^{(n)}(z) & \deg g_{12}^{(n)}(z) \\ \deg g_{21}^{(n)}(z) & \deg g_{22}^{(n)}(z)\end{pmatrix}=\begin{pmatrix} 2\big(\frac{4^{n-1}-1}{3}\big)-1 & 2\big(\frac{4^{n-1}-1}{3}\big) \\ 2\big(\frac{4^{n}-1}{3}\big)-1 & 2\big(\frac{4^{n}-1}{3}\big)\end{pmatrix}.$$

Since all degrees increase, we deduce that neither (i) nor (ii) can hold true.

It remains to prove that condition (iii) cannot be satisfied. Let us assume that (iii) holds. Then there exist a positive integer e and relatively prime polynomials $a(z)$ and $b(z)$ in $\mathbb{C}[z]$ such that

$$\frac{b(z)}{a(z)}=\frac{b(z^{4^n})g_{11}^{(n)}(z^e)+a(z^{4^n})g_{21}^{(n)}(z^e)}{b(z^{4^n})g_{12}^{(n)}(z^e)+a(z^{4^n})g_{22}^{(n)}(z^e)}\quad\text{for every } n=kn_0,\ k=1,2,\ldots.$$

We can rewrite this equality as follows:

$$\frac{a(z^{4^n})}{b(z^{4^n})}=\frac{a(z)g_{11}^{(n)}(z^e)-b(z)g_{12}^{(n)}(z^e)}{-a(z)g_{21}^{(n)}(z^e)+b(z)g_{22}^{(n)}(z^e)}\quad\text{for every } n=kn_0,\ k=1,2,\ldots.$$

By assumption, $a(z^{4^n})$ and $b(z^{4^n})$ are relatively prime and there exist polynomials $c^{(n)}(z)$ such that

$$a(z^{4^n})c^{(n)}(z)=a(z)g_{11}^{(n)}(z^e)-b(z)g_{12}^{(n)}(z^e),\tag{3.6}$$

$$b(z^{4^n})c^{(n)}(z)=-a(z)g_{21}^{(n)}(z^e)+b(z)g_{22}^{(n)}(z^e).\tag{3.7}$$

It follows that $c^{(n)}(z)$ divides the determinant of the matrix $A_n(z)$, that is,

$$c^{(n)}(z)\mid(-1)^nz^{(4^n-1)/3}.$$

We infer from (3.6) that

$$a(1)c^{(n)}(1)=a(1)g_{11}^{(n)}(1)-b(1)g_{12}^{(n)}(1)$$

and thus

$$|a(1)|=|a(1)g_{11}^{(n)}(1)-b(1)g_{12}^{(n)}(1)|.\tag{3.8}$$

Also, we infer from (3.7) that

$$|b(1)|=|-a(1)g_{21}^{(n)}(1)+b(1)g_{22}^{(n)}(1)|.\tag{3.9}$$

On the other hand, we deduce from (3.3)–(3.5) that for every pair of integers i and j in $\{1,2\}$, we have the following recurrence relation:

$$g_{ij}^{(n+1)}(1) = 3g_{ij}^{(n)}(1) - g_{ij}^{(n-1)}(1), \qquad \forall n \geq 2.$$

The polynomial associated with this recurrence is $X^2 - 3X + 1$. It has two roots

$$\theta_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \theta_2 = \frac{3-\sqrt{5}}{2}.$$

Consequently, for every pair of integers i and j in $\{1,2\}$ there exist real coefficients λ_{ij} and λ'_{ij} such that

$$g_{ij}^{(n)}(1) = \lambda_{ij}\theta_1^n + \lambda'_{ij}\theta_2^n$$

for every integer $n \geq 2$. We thus infer from (3.8) that

$$|a(1)| = |(a(1)\lambda_{11} - b(1)\lambda_{12})\theta_1^n + (a(1)\lambda_{11} - b(1)\lambda_{12})\theta_2^n|$$

and from (3.9) that

$$|b(1)| = |(-a(1)\lambda_{21} + b(1)\lambda_{22})\theta_1^n + (-a(1)\lambda_{21} + b(1)\lambda_{22})\theta_2^n|.$$

Since $\theta_1 > 1$ and $\theta_2 < 1$, we obtain $a(1) = b(1) = 0$. It follows that the polynomial $X - 1$ divides both $a(z)$ and $b(z)$. This is a contradiction since $a(z)$ and $b(z)$ are relatively prime polynomials. This ends the proof of Proposition 3.1. ∎

We are now going to show how Theorem 1.3 follows from Proposition 3.1.

Proof of Theorem 1.3. By Proposition 3.1, we immediately see that G takes transcendental values at every non-zero algebraic point that belongs to the complex open unit disc.

It thus remains to prove that the same holds for the function F. In [7, (5.4)], the authors proved that for every complex number q with $|q| < 1$,

$$G(q) = qF(q^2) + G(q^4). \tag{3.10}$$

Let q be a non-zero algebraic number with $|q| < 1$. Set $u = \sqrt{q}$. Thus, u is also a non-zero algebraic number with modulus less than 1 and we infer from (3.10) that

$$F(q) = (G(u) - G(u^4))/u.$$

By Proposition 3.1, the quantity on the right-hand side is transcendental, hence $F(q)$ is transcendental. This concludes the proof. ∎

4. Proofs of Theorems 1.1 and 1.2. Theorem 1.1 is a straightforward consequence of an old result of Mahler [9] that we recall below.

Let $d \geq 2$ be an integer and let $f(z)$ be an analytic function defined on the complex open unit disc. Let us assume that

$$f(z^d) = \frac{\sum_{k=0}^{m} a_k(z) f(z)^k}{\sum_{k=0}^{m} b_k(z) f(z)^k}, \tag{4.1}$$

where $m < d$ and $a_k(z), b_k(z) \in \mathbb{Z}[z]$. Let $\Delta(z)$ denote the resultant of $\sum_{k=0}^{m} a_k(z) u^k$ and $\sum_{k=0}^{m} b_k(z) u^k$ viewed as polynomials in u.

Theorem M. *Let q be an algebraic number such that $0 < |q| < 1$ and $\Delta(q^{d^n}) \neq 0$ for every non-negative integer n. Then $f(q)$ is transcendental.*

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We infer from the definition of $\mathcal{C}$ (see (1.1)) that for every algebraic number q with $|q| > 1$ we have

$$\mathcal{C}(q) = q + \frac{1}{\mathcal{C}(q^2)}. \tag{4.2}$$

We now define an analytic function f on the open unit disc by setting $f(z) := \mathcal{C}(z^{-1})$ for $0 < |z| < 1$ and $f(0) := 0$. For every complex number q with $0 < |q| < 1$, (4.2) gives

$$f(q) = \frac{1}{q} + \frac{1}{f(q^2)}.$$

This can be rewritten as the following Mahler type equation:

$$f(z^2) = \frac{z}{zf(z) - 1}.$$

Furthermore, $f(z)$ is a transcendental function over $\mathbb{Q}(z)$. There are several ways to confirm this claim; for instance, it follows from the fact that $f(1/2) = \mathcal{C}(2)$ is a transcendental number (see Proposition 7.1 in [7]). With the notation of Theorem M we obtain $\Delta(z) = z$ and thus $\Delta(q^{2^n}) \neq 0$ when $q \neq 0$. Consequently, Theorem M implies that $f(q)$ is transcendental for every algebraic number q with $0 < |q| < 1$. Hence $\mathcal{C}(q)$ is transcendental for every algebraic number q with $|q| > 1$, as claimed. ∎

We now prove Theorem 1.2 as a consequence of Proposition 3.1 and a result from [7].

Proof of Theorem 1.2. By Proposition 6.4 of [7], we know that $F(u) \neq 0$ and $G(u) \neq 0$ for every complex number u with $|u| < 1$. We thus infer from (3.10) that

$$\frac{F(u^2)}{G(u^4)} = \frac{1}{u}\left(\frac{G(u)}{G(u^4)} - 1\right) \tag{4.3}$$

for every algebraic number u with $0 < |u| < 1$.

Let q be a non-zero algebraic number with $|q| < 1$. Set $u := q^{3/2}$. Thus, u is also a non-zero algebraic number lying in the open unit disc, and (4.3) gives

$$\frac{qF(q^3)}{G(q^6)} = \frac{u^{2/3}F(u^2)}{G(u^4)} = \frac{1}{u^{1/3}}\left(\frac{G(u)}{G(u^4)} - 1\right).$$

Furthermore, by Proposition 3.1, the right-hand side is a transcendental number. Thus, $qF(q^3)/G(q^6)$ is transcendental. By Theorem DS, so is $\mathcal{C}_{\mathrm{ev}}(q)$, which ends the proof in that case.

We now infer from (3.10) that, for every non-zero complex number u with $|u| < 1$,

$$\frac{G(u)}{F(u^2)} = u\left(1 + \frac{G(u^4)}{G(u) - G(u^4)}\right). \tag{4.4}$$

Let q be an algebraic number with $0 < |q| < 1$. Set $u := q^3$. Thus, u is also a non-zero algebraic number lying in the open unit disc, and (4.4) gives

$$\frac{G(q^3)}{q^2F(q^6)} = \frac{G(u)}{u^{2/3}F(u^2)} = u^{1/3}\left(1 - \frac{G(u^4)}{G(u) - G(u^4)}\right).$$

Furthermore, by Proposition 3.1, the right-hand side is a transcendental number. Thus, $G(q^3)/q^2F(q^6)$ is transcendental. By Theorem DS, so is $\mathcal{C}_{\mathrm{od}}(q)$, which ends the proof. ∎

Acknowledgments. The author is supported by the ANR through the project "DyCoNum"-JCJC06 134288.

References

[1] B. Adamczewski and Y. Bugeaud, *On the complexity of algebraic numbers I. Expansions in integer bases*, Ann. of Math. 165 (2007), 547–565.

[2] J.-P. Allouche and J. O. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge Univ. Press, Cambridge, 2003.

[3] G. Christol, T. Kamae, M. Mendès France et G. Rauzy, *Suites algébriques, automates et substitutions*, Bull. Soc. Math. France 108 (1980), 401–419.

[4] M. Coons, *The transcendence of series related to Stern's diatomic sequence*, Int. J. Number Theory, to appear.

[5] H. Davenport and K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), 160–167.

[6] K. Dilcher and K. B. Stolarsky, *A polynomial analogue to the Stern sequence*, Int. J. Number Theory 3 (2007), 85–103.

[7] —, —, *Stern polynomials and double-limit continued fractions*, Acta Arith. 140 (2009), 119–134.

[8] L. Lorentzen and H. Waadeland, *Continued Fractions with Applications*, North-Holland, Amsterdam, 1992.

[9] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366; Corrigendum, ibid. 103 (1930), 532.

[10] K. Nishioka, *Mahler Functions and Transcendence*, Lecture Notes in Math. 1631, Springer, Berlin, 1996.

[11] M. Ratliff, *The Thue–Siegel–Roth–Schmidt Theorem for algebraic functions*, J. Number Theory 10 (1978), 99–126.

[12] D. S. Thakur, *Diophantine approximation exponents and continued fractions for algebraic power series*, ibid. 79 (1999), 284–291.

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Received on 10.4.2009
and in revised form on 20.7.2009 (6000)