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On the odd part of tame kernels of biquadratic number fields

by

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1. Introduction. Let F be an algebraic number field, $\mathcal{O}_F$ the ring of integers in F, and K_2 the Milnor K-functor. For a quadratic number field, the 2-primary part of $K_2\mathcal{O}_F$ has been intensively studied (see [5]–[10]). For an odd prime p, it is interesting to study the p-primary part of the tame kernels of number fields; some results have been found in [1]–[4].

In this paper we investigate the odd part of the tame kernel $K_2\mathcal{O}_F$ for a biquadratic field F, where $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Section 2 studies the p-Sylow subgroup of the class group of the cyclotomic extension of F. Let p be a regular prime. In Section 3, we prove some results connecting the p-rank of the tame kernel $K_2\mathcal{O}_F$ with the p-rank of the class groups of some subfields of the number field $F(\zeta_p)$. In particular, when $p = 3, 5$, we get some results connecting the p-rank of $K_2\mathcal{O}_F$ with the p-rank of the class groups of some quadratic fields. As an application, we calculate the 3-rank of the tame kernels $K_2\mathcal{O}_F$ when $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $-50 < d_1, d_2 < 50$. For any odd prime p and a positive integer $n \geq 2$, in Section 4, we prove explicitly some relations between the groups $(\mu_{p^n} \otimes Cl(\mathcal{O}_{F(\zeta_{p^n})}[1/p]))_{\mathrm{Gal}(F(\zeta_{p^n})/F)}$ and $K_2\mathcal{O}_F/p^n$ by using a map of Keune.

2. Biquadratic number fields. In this section, we give more information on cyclotomic extensions of biquadratic fields. We fix the following notation.

Let $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, $F_1 = \mathbb{Q}(\sqrt{d_1})$, $F_2 = \mathbb{Q}(\sqrt{d_2})$, with d_1 and d_2 squarefree. We assume that $(d_1, d_2) = 1$, p is an odd prime, n a positive integer, ζ_{p^n} a primitive root of unity of degree p^n, $L = F(\zeta_{p^n})$, $G = \mathrm{Gal}(L/F)$. Clearly, G is cyclic as a subgroup of the cyclic group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$. Let $\mathfrak{p}$ be a prime ideal of F dividing p, and let $Z_{\mathfrak{p}}$ be the decomposition group of $\mathfrak{p}$ in the abelian extension L/F. Denote by $e_{\mathfrak{p}}$ the ramification degree of

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$\mathfrak{p}$ in L/F. For a number field $*$, we denote by A_* and $A_*[1/p]$ the p-Sylow subgroups of $Cl(\mathcal{O}_*)$ and $Cl(\mathcal{O}_*[1/p])$ respectively.

We consider the following cases.

1) $d_1 = p^* = (-1)^{(p-1)/2}p$. Then F_1 is the unique quadratic subfield of $\mathbb{Q}(\zeta_{p^n})$, and p is totally ramified in F_1. By the assumption, $p \nmid d_2$. We consider two subcases.

a) $\left(\frac{d_2}{p}\right) = 1$. Then p splits in F_2. It follows that $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_2^2$. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, the prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of F are totally ramified in L/F, and $e_{\mathfrak{p}_1} = e_{\mathfrak{p}_2} = (L:F)$ and $Z_{\mathfrak{p}_1} = Z_{\mathfrak{p}_2} = G$.

b) $\left(\frac{d_2}{p}\right) = -1$. Then p is inert in F_2, hence $p\mathcal{O}_F = \mathfrak{p}^2$. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, the prime ideal $\mathfrak{p}$ of F is totally ramified in L/F with $e_{\mathfrak{p}} = (L:\ F)$ and $Z_{\mathfrak{p}} = G$.

2) $d_1 \neq p^*$ and $p \mid d_1$. Then p is totally ramified in F_1. Let $d_1 = p^* d_1'$. By the assumption, $p \nmid d_2$. We consider four subcases.

a) $\left(\frac{d_1'}{p}\right) = 1$ and $\left(\frac{d_2}{p}\right) = 1$. Then p splits in $\mathbb{Q}(\sqrt{d_1'})$ and F_2, so p splits in $\mathbb{Q}(\sqrt{d_1'}, \sqrt{d_2})$ and $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_2^2$. Let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{p^*}) = \mathbb{Q}(\sqrt{d_1'}, \sqrt{d_2}, \sqrt{p^*})$. Then the prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ split in E/F. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, we have $e_{\mathfrak{p}_1} = e_{\mathfrak{p}_2} = (L:E) = (L:F)/2$. The decomposition fields of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in L/F are both E, and $Z_{\mathfrak{p}_1} = Z_{\mathfrak{p}_2} = \mathrm{Gal}(L/E) = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\sqrt{p^*}))$.

b) $\left(\frac{d_1'}{p}\right) = 1$ and $\left(\frac{d_2}{p}\right) = -1$. Then p splits in $\mathbb{Q}(\sqrt{d_1'})$ and is inert in F_2, so $p\mathcal{O}_F = \mathfrak{p}^2$. Hence the prime ideal $\mathfrak{p}$ splits in E/F. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, the ramification degree $e_{\mathfrak{p}}$ equals $(L:E) = (L:F)/2$. The decomposition field of $\mathfrak{p}$ in L/F is E and $Z_{\mathfrak{p}} = \mathrm{Gal}(L/E) = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\sqrt{p^*}))$.

c) $\left(\frac{d_1'}{p}\right) = -1$ and $\left(\frac{d_2}{p}\right) = 1$. Then p splits in F_2 and is inert in $\mathbb{Q}(\sqrt{d_1'})$, so $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_2^2$. Hence the prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are inert in E/F. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, we have $e_{\mathfrak{p}_1} = e_{\mathfrak{p}_2} = (L:E) = (L:F)/2$. The decomposition fields of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in L/F are both F, and $Z_{\mathfrak{p}_1} = Z_{\mathfrak{p}_2} = \mathrm{Gal}(L/F) = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$.

d) $\left(\frac{d_1'}{p}\right) = -1$ and $\left(\frac{d_2}{p}\right) = -1$. Then p is inert in $\mathbb{Q}(\sqrt{d_1'})$ and F_2, so $p\mathcal{O}_F = \mathfrak{p}^2$. Since $\left(\frac{d_1' d_2}{p}\right) = 1$, p splits in $\mathbb{Q}(\sqrt{d_1' d_2})$. Hence the prime ideal $\mathfrak{p}$ splits in E/F. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, we have $e_{\mathfrak{p}} = (L:E) = (L:F)/2$. The decomposition field of $\mathfrak{p}$ in L/F is E, and $Z_{\mathfrak{p}} = \mathrm{Gal}(L/E) = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\sqrt{p^*}))$.

3) $p \nmid d_1$ and $p \nmid d_2$. Then p is unramified in F. Since p is totally ramified in $\mathbb{Q}(\zeta_{p^n})$, it is totally ramified in L/F. Hence the ramification degree $e_{\mathfrak{p}}$ of the prime ideal $\mathfrak{p} \mid p$ of F is $(L:F)$. The decomposition field of $\mathfrak{p}$ in L/F is F, and $Z_{\mathfrak{p}} = \mathrm{Gal}(L/F) = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$.

From the above conclusions, we deduce the following lemma:

LEMMA 2.1. *Let $L = F(\zeta_p)$. Assume that S'_F is the set of prime ideals of F which divide p and split completely in L. Then:*

(i) $|S'_F| = 2$ *if* $p = 3$, $d_1 \equiv 3 \pmod 9$ *and* $d_2 \equiv 1 \pmod 3$,
(ii) $|S'_F| = 1$ *if* $p = 3$, $3 \mid d_1$ *and* $d_2 \equiv 2 \pmod 3$,
(iii) $|S'_F| = 0$ *otherwise.*

LEMMA 2.2. *Assume that p is an odd prime and n a positive integer. Let $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, $K = \mathbb{Q}(\zeta_{p^n} + \zeta_{p^n}^{-1})$, $L = F(\zeta_{p^n})$. Let $L_1 = K(\sqrt{d_1})$, $L_2 = K(\sqrt{d_1 d_2})$, $L_3 = K((\zeta_{p^n} - \zeta_{p^n}^{-1})\sqrt{d_1 d_2})$, $L_4 = K(\sqrt{d_2})$, $L_5 = K((\zeta_{p^n} - \zeta_{p^n}^{-1})\sqrt{d_2})$, $L_6 = K((\zeta_{p^n} - \zeta_{p^n}^{-1})\sqrt{d_1})$, $L_7 = K(\zeta_{p^n} - \zeta_{p^n}^{-1})$. Assume further that p is regular and $F \cap \mathbb{Q}(\zeta_{p^n}) = \mathbb{Q}$. Then*

$$A_L = A_{L_1} \oplus A_{L_2} \oplus A_{L_3} \oplus A_{L_4} \oplus A_{L_5} \oplus A_{L_6}.$$

Proof. Let $G = \mathrm{Gal}(L/K)$, so we have $\tau_1, \tau_2, \sigma \in G$ such that

$$\tau_1 : \sqrt{d_1} \mapsto -\sqrt{d_1},\ \sqrt{d_2} \mapsto \sqrt{d_2},\ \zeta_{p^n} \mapsto \zeta_{p^n},$$
$$\tau_2 : \sqrt{d_1} \mapsto \sqrt{d_1},\ \sqrt{d_2} \mapsto -\sqrt{d_2},\ \zeta_{p^n} \mapsto \zeta_{p^n},$$
$$\sigma : \sqrt{d_1} \mapsto \sqrt{d_1},\ \sqrt{d_2} \mapsto \sqrt{d_2},\ \zeta_{p^n} \mapsto \zeta_{p^n}^{-1}.$$

Also, $\langle\sigma, \tau_2\rangle = \mathrm{Gal}(L/L_1)$, $\langle\sigma, \tau_1\tau_2\rangle = \mathrm{Gal}(L/L_2)$, $\langle\sigma\tau_1, \tau_1\tau_2\rangle = \mathrm{Gal}(L/L_3)$, $\langle\sigma, \tau_1\rangle = \mathrm{Gal}(L/L_4)$, $\langle\sigma\tau_2, \tau_1\rangle = \mathrm{Gal}(L/L_5)$, $\langle\sigma\tau_1, \tau_2\rangle = \mathrm{Gal}(L/L_6)$, $\langle\tau_1, \tau_2\rangle = \mathrm{Gal}(L/L_7)$. Then we have idempotents in $\mathbb{Z}_p[G]$:

$$\eta_0 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right)\left(\frac{1+\sigma}{2}\right), \quad \eta_1 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right)\left(\frac{1+\sigma}{2}\right),$$
$$\eta_2 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1+\sigma}{2}\right), \quad \eta_3 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right),$$
$$\eta_4 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1+\sigma}{2}\right), \quad \eta_5 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right),$$
$$\eta_6 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right), \quad \eta_7 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right).$$

It is easy to verify $1 = \eta_0 + \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \eta_6 + \eta_7$.

We have $\eta_0 A_L \subseteq N_{L/K} A_L = 0$ since p is regular. Furthermore, $\eta_1 A_L = \frac{1}{8}(1-\tau_1) N_{L/L_1} A_L$, so $\eta_1 A_L \subseteq A_{L_1}$. On the other hand, for any $a \in A_{L_1}$, we have $\sigma a = a$, $\tau_2 a = a$, $(1+\tau_1)a = 0$, i.e., $\tau_1 a = -a$, since $1+\tau_1 = N_{L_1/K}$ and p is regular. Hence, $\eta_1 a = \frac{1}{8}(1-\tau_1)(1+\tau_2)(1+\sigma)a = a$, and so $\eta_1 A_L = A_{L_1}$. Similarly, $\eta_4 A_L = A_{L_4}$ and $\eta_7 A_L = A_{L_7}$.

Also,

$$\eta_2 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1+\sigma}{2}\right) = \frac{1}{8}(1-\tau_1)(1+\tau_1\tau_2)(1+\sigma),$$
$$\eta_3 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right) = \frac{1}{8}(1-\tau_1)(1+\tau_1\tau_2)(1+\sigma\tau_1),$$
$$\eta_5 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right) = \frac{1}{8}(1-\tau_2)(1+\tau_1)(1+\sigma\tau_2),$$
$$\eta_6 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right)\left(\frac{1-\sigma}{2}\right) = \frac{1}{8}(1-\tau_1)(1+\tau_2)(1+\sigma\tau_1).$$

By the above proof, $\eta_i A_L = A_{L_i}$, $i = 2,3,5,6$. Since p is regular, we have $A_{L_7} = 0$, and the conclusion follows.

3. p-rank. In the following, we assume that $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. Let $L = F(\zeta_p)$ with $\Gamma = \mathrm{Gal}(L/\mathbb{Q}) \cong G \times T$, where $G = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $T = \mathrm{Gal}(F/\mathbb{Q})$.

Let ω be the Teichmüller character of the group $(\mathbb{Z}/p)^*$. We fix a primitive root $g \pmod p$ and let $\sigma := \sigma_g$ ($\sigma_g(\zeta_p) = \zeta_p^g$). We have the following idempotents:

$$\varepsilon_j = \frac{1}{p-1}\sum_{a=1}^{p-1}\omega^j(a)\sigma_a^{-1} = \frac{1}{p-1}\sum_{k=0}^{p-2}\omega^{jk}(g)\sigma^{-k}, \qquad 0 \le j \le p-2.$$

Let $\lambda : Cl(\mathcal{O}_L) \to Cl(\mathcal{O}_L[1/p])$ be the homomorphism induced by the imbedding $\mathcal{O}_L \to \mathcal{O}_L[1/p]$. Evidently λ is a surjective homomorphism of Γ-modules.

Let E be the maximal unramified p-extension of the field L with an elementary abelian Galois group $H := \mathrm{Gal}(E/L)$. Since $\mu_p := \langle\zeta_p\rangle \subset L$, E/L is a Kummer extension, i.e., $E = L(B^{1/p})$, for some subgroup B of L^* containing L^{*p}. Let $B_0 := B/L^{*p}$. For every $b \in L^*$ and $b_0 = bL^{*p}$, we have $b_0 \in B_0$ iff b is singular primary, i.e., $(b) = \mathfrak{a}^p$ for some ideal $\mathfrak{a}$ of L, and $x^p \equiv b \pmod{p(1-\zeta_p)}$ for some $x \in L^*$. Consequently, we have a homomorphism of Γ-modules:

$$\phi : B_0 \to (A_L)_p, \qquad \phi(b_0) = Cl(\mathfrak{a}).$$

Let U_L' be the group of singular primary units of L. Then $\mathrm{Ker}\,\phi \cong U_L'/U_L^p$ (see [2]).

Lemma 3.1 ([2]). *Let F be a number field with $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$, and let p be an odd prime. Then*

$$p\text{-rank}(K_2\mathcal{O}_F) = p\text{-rank}(\varepsilon_{p-2}A_L[1/p]) + |S_F'|.$$

LEMMA 3.2. *The mapping $\lambda : \varepsilon_{p-2}Cl(\mathcal{O}_L) \to \varepsilon_{p-2}Cl(\mathcal{O}_L[1/p])$ is an isomorphism if one of the following conditions is satisfied:*

1) $p \nmid d_1$ *and* $p \nmid d_2$,
2) $d_1 = pd_1'$, $\left(\frac{d_1'}{p}\right) = -1$ *and* $\left(\frac{d_2}{p}\right) = 1$,
3) $p \neq 3$, $p \mid d_1$ *and* $\left(\frac{d_2}{p}\right) = -1$.

Proof. Let $\mathfrak{P}$ be the unique prime ideal over p in L. The group $\operatorname{Ker}\lambda$ is generated by the class containing the prime ideal $\mathfrak{P}$ of L. If 1) and 2) are satisfied, then the prime ideal $\mathfrak{p} \mid p$ of F does not split in L by 1) and 3) of Section 2. So we have $\sigma(\mathfrak{P}) = \mathfrak{P}$. Therefore, $\mathfrak{P} \in \varepsilon_0 Cl(\mathcal{O}_L)$.

Suppose now condition 3) is satisfied. Then by 2b) and 2d) of Section 2, in L we have $(1-\zeta_p) = \mathfrak{P}_1\mathfrak{P}_2$, where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are different prime ideals of E. Consequently $\sigma(\mathfrak{P}_1) = \mathfrak{P}_1$ or $\mathfrak{P}_2$. Thus for $a = Cl(\mathfrak{P}_1)$ we have $\sigma(a) = \pm a$. Evidently a generates $\operatorname{Ker}\lambda$. Assume that $a^m \in \varepsilon_{p-2}Cl(\mathcal{O}_L)$ for some m. Then

$$\begin{aligned} a^m = \varepsilon_{p-2}a^m &= \frac{1}{p-1}\sum_{k=0}^{p-2}\omega^{k(p-2)}(g)\sigma^{-k}a^m \\ &= \frac{1}{p-1}\sum_{k=0}^{p-2}\omega^{k(p-2)}(g)(\pm 1)^{km}a^m \\ &= \frac{1}{p-1}\,\frac{1-(\omega^{p-2}(g)(\pm 1)^m)^{p-1}}{1-\omega^{p-2}(g)(\pm 1)^m}a^m. \end{aligned}$$

Since $p \neq 3$, $p-2 \neq \frac{1}{2}(p-1)$. It follows that $\omega^{p-2}(g)(\pm 1)^m \neq 1$. Therefore $a^m = 0$. This completes the proof.

THEOREM 3.3. *Assume that p is a regular prime. Let $K = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$, $L = F(\zeta_p)$, $L_1 = K(\sqrt{d_1})$, $L_2 = K(\sqrt{d_1d_2})$, $L_3 = K((\zeta_p - \zeta_p^{-1})\sqrt{d_1d_2})$, $L_4 = K(\sqrt{d_2})$, $L_5 = K((\zeta_p - \zeta_p^{-1})\sqrt{d_2})$, $L_6 = K((\zeta_p - \zeta_p^{-1})\sqrt{d_1})$.*

(i) *If $p > 3$, then*

$$\begin{aligned} p\text{-rank}(K_2\mathcal{O}_F) = {} & p\text{-rank}(\varepsilon_{p-2}A_{L_3}[1/p]) + p\text{-rank}(\varepsilon_{p-2}A_{L_5}[1/p]) \\ & + p\text{-rank}(\varepsilon_{p-2}A_{L_6}[1/p]) \end{aligned}$$

provided $d_1 = pd_1'$, $\left(\frac{d_1'}{p}\right) = 1$ and $\left(\frac{d_2}{p}\right) = 1$, while

$$\begin{aligned} p\text{-rank}(K_2\mathcal{O}_F) = {} & p\text{-rank}(\varepsilon_{p-2}A_{L_3}) + p\text{-rank}(\varepsilon_{p-2}A_{L_5}) \\ & + p\text{-rank}(\varepsilon_{p-2}A_{L_6}) \end{aligned}$$

otherwise.

(ii) *If* $3 \nmid d_1$ *and* $3 \nmid d_2$, *or* $d_1 \equiv 6 \pmod 9$ *and* $d_2 \equiv 1 \pmod 3$, *then*

$$3\text{-rank}(K_2\mathcal{O}_F) = 3\text{-rank}(A_{L_3}) + 3\text{-rank}(A_{L_5}) + 3\text{-rank}(A_{L_6}),$$

where $L_3 = \mathbb{Q}(\sqrt{-3d_1d_2})$, $L_5 = \mathbb{Q}(\sqrt{-3d_2})$, $L_6 = \mathbb{Q}(\sqrt{-3d_1})$.

(iii) *If* $d_1 \equiv 3 \pmod 9$ *and* $d_2 \equiv 1 \pmod 3$, *then*

$$3\text{-rank}(K_2\mathcal{O}_F) = 3\text{-rank}(A_{L_3}) + 3\text{-rank}(A_{L_5}) + 3\text{-rank}(A_{L_6}) + 2,$$

where $L_3 = \mathbb{Q}(\sqrt{-3d_1d_2})$, $L_5 = \mathbb{Q}(\sqrt{-3d_2})$, $L_6 = \mathbb{Q}(\sqrt{-3d_1})$.

(iv) *If* $d_1 \equiv 6 \pmod 9$ *and* $d_2 \equiv 2 \pmod 3$, *then*

$$3\text{-rank}(K_2\mathcal{O}_F) = 3\text{-rank}(A_{L_3}) + 3\text{-rank}(A_{L_5}) + 3\text{-rank}(A_{L_6}) + 1,$$

where $L_3 = \mathbb{Q}(\sqrt{-3d_1d_2})$, $L_5 = \mathbb{Q}(\sqrt{-3d_2})$, $L_6 = \mathbb{Q}(\sqrt{-3d_1})$.

(v) *If* $d_1 \equiv 3 \pmod 9$ *and* $d_2 \equiv 2 \pmod 3$, *then*

$$3\text{-rank}(K_2\mathcal{O}_F) = 3\text{-rank}(A_{L_3}[1/3]) + 3\text{-rank}(A_{L_5}) + 3\text{-rank}(A_{L_6}) + 1,$$

where $L_3 = \mathbb{Q}(\sqrt{-3d_1d_2})$, $L_5 = \mathbb{Q}(\sqrt{-3d_2})$, $L_6 = \mathbb{Q}(\sqrt{-3d_1})$.

Proof. (i) By the assumption and Lemma 2.2, we have

$$A_L = A_{L_1} \oplus A_{L_2} \oplus A_{L_3} \oplus A_{L_4} \oplus A_{L_5} \oplus A_{L_6}.$$

We recall that σ is a generator of $\mathrm{Gal}(L/F)$. Then

$$\sigma^{(p-1)/2}\varepsilon_{p-2} = \omega(g^{(p-1)/2})^{p-2}\varepsilon_{p-2} = -\varepsilon_{p-2}.$$

On the other hand, it is easy to check that $\sigma^{(p-1)/2}$ acts trivially on A_{L_1}, A_{L_2} and A_{L_4}. Therefore, the result follows from Lemmas 2.1, 3.1 and 3.2.

(ii) By the proof of (i), we have

$$3\text{-rank}(K_2\mathcal{O}_F) = 3\text{-rank}(\varepsilon_1 A_{L_3}) + 3\text{-rank}(\varepsilon_1 A_{L_5}) + 3\text{-rank}(\varepsilon_1 A_{L_6}),$$

where $L_3 = \mathbb{Q}(\sqrt{-3d_1d_2})$, $L_5 = \mathbb{Q}(\sqrt{-3d_2})$, $L_6 = \mathbb{Q}(\sqrt{-3d_1})$. It is easy to see that $\mathrm{Gal}(L_i/\mathbb{Q}) = \langle\sigma\rangle$ for $i = 3, 5, 6$. So $A_{L_i} = \varepsilon_1 A_{L_i}$ for $i = 3, 5, 6$, completing the proof.

(iii) By Lemmas 2.1, 2.2 and 3.1, we have

$$\begin{aligned}3\text{-rank}(K_2\mathcal{O}_F) = {} & 3\text{-rank}(\varepsilon_1 A_{L_3}[1/3]) + 3\text{-rank}(\varepsilon_1 A_{L_5}[1/3]) \\ & + 3\text{-rank}(\varepsilon_1 A_{L_6}[1/3]) + 2,\end{aligned}$$

where $L_3 = \mathbb{Q}(\sqrt{-3d_1d_2})$, $L_5 = \mathbb{Q}(\sqrt{-3d_2})$, $L_6 = \mathbb{Q}(\sqrt{-3d_1})$. Since $d_1 \equiv 3 \pmod 9$ and $d_2 \equiv 1 \pmod 3$, we have $\left(\frac{-d_1/3}{3}\right) = -1$ and $\left(\frac{-d_1d_2/3}{3}\right) = -1$. Hence 3 is inert is L_3 and L_6. Clearly, 3 is ramified in L_5. So $A_{L_i}[1/3] = A_{L_i}$, where $i = 3, 5, 6$. Therefore, the result follows from the proof of (ii).

The proofs of (iv) and (v) are similar to that of (iii).

Example. As an application of Theorem 3.3, we calculate the 3-rank of $K_2\mathcal{O}_F$ when $-50 < d_1, d_2 < 50$. The results are given in the tables of Section 5.

Let $\mathrm{Gal}(F_1/\mathbb{Q}) = \langle\tau_1\rangle$ and $\mathrm{Gal}(F_2/\mathbb{Q}) = \langle\tau_2\rangle$. Then $T := \mathrm{Gal}(F/\mathbb{Q}) = \langle\tau_1, \tau_2\rangle$. We consider the following idempotents:

$$\theta_0 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right), \quad \theta_1 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right),$$
$$\theta_2 = \left(\frac{1+\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right), \quad \theta_3 = \left(\frac{1-\tau_1}{2}\right)\left(\frac{1-\tau_2}{2}\right).$$

LEMMA 3.4. *For $p = 4s+1$, we have*

(1) $$\theta_1\varepsilon_2 = \frac{1}{p-1}\left(\frac{1-\tau_1}{4}\right)\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}N_{L/E_1},$$

(2) $$\theta_2\varepsilon_2 = \frac{1}{p-1}\left(\frac{1-\tau_2}{4}\right)\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}N_{L/E_2},$$

(3) $$\theta_3\varepsilon_2 = \frac{1}{p-1}\left(\frac{1-\tau_1}{4}\right)\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}N_{L/E_3},$$

where E_1, E_2 and E_3 are the subfields of L fixed by $\langle\tau_2, \sigma^s\tau_1\rangle$, $\langle\tau_1, \sigma^s\tau_2\rangle$ and $\langle\tau_1\tau_2, \sigma^s\tau_1\rangle$ respectively.

Proof. Since $\omega^{2s}(g) = -1$ and $\sigma^{4s} = 1$, we get

$$\begin{aligned}\varepsilon_2 &= \frac{1}{p-1}\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}(1+\omega^{2s}(g)\sigma^{-s}+\omega^{4s}(g)\sigma^{-2s}+\omega^{6s}(g)\sigma^{-3s})\\ &= \frac{1}{p-1}\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}(1-\sigma^s+\sigma^{2s}-\sigma^{3s})\\ &= \frac{1}{p-1}\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}(1-\sigma^s)(1+\sigma^{2s}).\end{aligned}$$

For (1),

$$\begin{aligned}\theta_1\varepsilon_2 &= \left(\frac{1-\tau_1}{2}\right)\left(\frac{1+\tau_2}{2}\right)\frac{1}{p-1}\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}(1-\sigma^s)(1+\sigma^{2s})\\ &= \frac{1}{p-1}\left(\frac{1-\tau_1}{4}\right)\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}(1+\tau_2)(1+\tau_1\sigma^s)(1+\sigma^{2s})\\ &= \frac{1}{p-1}\left(\frac{1-\tau_1}{4}\right)\sum_{k=0}^{s-1}\omega^{2k}(g)\sigma^{-k}N_{L/E_1}.\end{aligned}$$

The equalities (2) and (3) can be proved similarly.

LEMMA 3.5 ([2]). *p-rank$(\varepsilon_k A_L) = p$-rank$(\varepsilon_{p-k} B_0)$ for $0 \le k \le p$, where we identified $\varepsilon_{p-1} = \varepsilon_0$ and $\varepsilon_p = \varepsilon_1$.*

LEMMA 3.6. *For $0 \le k \le 3$ and $0 \le j \le p-2$, we have*

$$p\text{-rank}(\theta_k \varepsilon_{p-j} A_L) \le p\text{-rank}(\theta_k \varepsilon_j A_L) + p\text{-rank}(\theta_k \varepsilon_j (U'_L/U^p_L)).$$

Proof. Since $\phi : B_0 \to (A_L)_p$ is a homomorphism of Γ-modules, we have

$$\phi : \theta_k \varepsilon_j B_0 \to \theta_k \varepsilon_j (A_L)_p \quad \text{for } 0 \le k \le 3,\ 0 \le j \le p-2.$$

In view of $\operatorname{Ker} \phi \cong U'_L/U^p_L$ we get

$$(\operatorname{Ker} \phi) \cap (\theta_k \varepsilon_j B_0) \cong \theta_k \varepsilon_j (U'_L/U^p_L),$$

and hence

$$p\text{-rank}(\theta_k \varepsilon_j B_0) \le p\text{-rank}(\theta_k \varepsilon_j A_L) + p\text{-rank}(\theta_k \varepsilon_j (U'_L/U^p_L)).$$

Therefore the result follows from Lemma 3.5.

THEOREM 3.7. *Let $5 \nmid d_1 d_2$. If F is a totally real field, then*

$$5\text{-rank}(K_2\mathcal{O}_F) \le 5\text{-rank}(A_{E_1}) + 5\text{-rank}(A_{E_2}) + 5\text{-rank}(A_{E_3}) + 3,$$

and if F is a CM field, then

$$5\text{-rank}(K_2\mathcal{O}_F) \le 5\text{-rank}(A_{E_1}) + 5\text{-rank}(A_{E_2}) + 5\text{-rank}(A_{E_3}) + 1,$$

where $E_1 = \mathbb{Q}(\sqrt{5d_1})$, $E_2 = \mathbb{Q}(\sqrt{5d_2})$, $E_3 = \mathbb{Q}(\sqrt{5d_1 d_2})$.

Proof. By Lemmas 2.1, 3.1 and 3.2, 5-rank$(K_2\mathcal{O}_F) = 5$-rank$(\varepsilon_3 A_L)$. Since $\theta_0 \varepsilon_3 A_L \subset A_{\mathbb{Q}(\zeta_5)}$, Lemmas 3.4 and 3.6 yield

$$5\text{-rank}(K_2\mathcal{O}_F) \le \sum_{k=1}^{3} (5\text{-rank}(A_{E_k}) + 5\text{-rank}(U_{E_k}/U^5_{E_k})).$$

By the Dirichlet unit theorem, we conclude that 5-rank$(U_{E_k}/U^5_{E_k}) = 1$ resp. 0 if E_k is a real resp. imaginary quadratic field. This completes the proof.

4. p^n-rank. In this section, we use the same notation as in Section 2.

LEMMA 4.1 ([3]). *Let F be a number field, and for any odd prime p and a positive integer n, let $\Gamma = \operatorname{Gal}(F(\zeta_{p^n})/F)$. We have an exact sequence*

$$(*) \quad 0 \to (\mu_{p^n} \otimes Cl(\mathcal{O}_{F(\zeta_{p^n})}[1/p]))_\Gamma \xrightarrow{\iota} K_2\mathcal{O}_F/p^n \xrightarrow{\lambda} \bigoplus_{\mathfrak{p}|p} (\mu_{p^n})_{Z_\mathfrak{p}} \xrightarrow{c} (\mu_{p^n})_\Gamma \to 0.$$

THEOREM 4.2. *Let* $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ *and* $L = F(\zeta_{p^n})$, *where* $n \geq 2$.

(i) *If* $p > 3$, *then* $K_2\mathcal{O}_F/p^n \cong (\mu_{p^n} \otimes Cl(\mathcal{O}_L[1/p]))_\Gamma$.

(ii) $K_2\mathcal{O}_F/3^n \cong (\mu_{3^n} \otimes Cl(\mathcal{O}_L[1/3]))_\Gamma$ *provided* $d_1 = -3$ *and* $\left(\frac{d_2}{3}\right) = -1$, *or* $3 \mid d_1$ *with* $\left(\frac{-d_1/3}{3}\right) = -1$ *and* $\left(\frac{d_2}{3}\right) = 1$, *or* $3 \nmid d_1$ *and* $3 \nmid d_2$.

(iii) *We have an exact sequence*

$$0 \to (\mu_{3^n} \otimes Cl(\mathcal{O}_L[1/3]))_\Gamma \to K_2\mathcal{O}_F/3^n \to \mu_3 \to 0$$

provided $d_1 = -3$ *and* $\left(\frac{d_2}{3}\right) = 1$, *or* $3 \mid d_1$ *with* $\left(\frac{-d_1/3}{3}\right) = 1$ *and* $\left(\frac{d_2}{3}\right) = -1$, *or* $3 \mid d_1$ *with* $\left(\frac{-d_1/3}{3}\right) = -1$ *and* $\left(\frac{d_2}{3}\right) = -1$.

(iv) *We have an exact sequence*

$$0 \to (\mu_{3^n} \otimes Cl(\mathcal{O}_L[1/3]))_\Gamma \to K_2\mathcal{O}_F/3^n \to \mu_3 \oplus \mu_3 \to 0$$

provided $3 \mid d_1$ *with* $\left(\frac{-d_1/3}{3}\right) = 1$ *and* $\left(\frac{d_2}{3}\right) = 1$.

Proof. Since μ_{p^n} is a cohomologically trivial Z-module for every subgroup Z of Γ, we get

$$(\mu_{p^n})_{Z_\mathfrak{p}} \cong (\mu_{p^n})^{Z_\mathfrak{p}} = \mu_{p^n} \cap L^{Z_\mathfrak{p}},$$

where $L^{Z_\mathfrak{p}}$ is the decomposition field of $\mathfrak{p}$ in the extension L/F. From the conclusions of Section 2, this field is F or $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{p^*})$. Obviously, the pth root of 1, where $p > 3$, does not belong to this field. So (i) is proved by Lemma 4.1.

From the conclusions of Section 2, we have the following results:

If $d_1 = -3$ and $\left(\frac{d_2}{3}\right) = 1$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} \cong \mu_3 \oplus \mu_3$ and $(\mu_{3^n})_\Gamma \cong \mu_3$.

If $d_1 = -3$ and $\left(\frac{d_2}{3}\right) = -1$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} \cong \mu_3$ and $(\mu_{3^n})_\Gamma \cong \mu_3$.

If $3 \mid d_1 \neq -3$, $\left(\frac{-d_1/3}{3}\right) = 1$ and $\left(\frac{d_2}{3}\right) = 1$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} \cong \mu_3 \oplus \mu_3$ and $(\mu_{3^n})_\Gamma = \{1\}$.

If $3 \mid d_1 \neq -3$, $\left(\frac{-d_1/3}{3}\right) = 1$ and $\left(\frac{d_2}{3}\right) = -1$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} \cong \mu_3$ and $(\mu_{3^n})_\Gamma = \{1\}$.

If $3 \mid d_1 \neq -3$, $\left(\frac{-d_1/3}{3}\right) = -1$ and $\left(\frac{d_2}{3}\right) = 1$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} = \{1\}$ and $(\mu_{3^n})_\Gamma = \{1\}$.

If $3 \mid d_1 \neq -3$, $\left(\frac{-d_1/3}{3}\right) = -1$ and $\left(\frac{d_2}{3}\right) = -1$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} \cong \mu_3$ and $(\mu_{3^n})_\Gamma = \{1\}$.

If $3 \nmid d_1$ and $3 \nmid d_2$, then $\bigoplus_{\mathfrak{p}|3}(\mu_{3^n})_{Z_\mathfrak{p}} = \{1\}$ and $(\mu_{3^n})_\Gamma = \{1\}$.

Now (ii)–(iv) follow from Lemma 4.1.

5. Tables

Table 1. (d_1, d_2) with 3-rank$(K_2\mathcal{O}_F) = 1$

$(-47,-43)$	$(-47,-31)$	$(-47,-26)$	$(-47,-22)$	$(-47,-13)$	$(-47,-5)$	$(-47,29)$	$(-47,43)$	$(-46,-19)$
$(-46,-17)$	$(-46,-11)$	$(-46,-7)$	$(-46,29)$	$(-46,43)$	$(-43,-34)$	$(-43,-31)$	$(-43,-26)$	$(-43,-17)$
$(-43,-11)$	$(-43,-1)$	$(-43,14)$	$(-43,29)$	$(-41,-17)$	$(-41,-14)$	$(-41,-13)$	$(-41,-10)$	$(-41,-7)$
$(-41,-2)$	$(-41,13)$	$(-41,29)$	$(-41,43)$	$(-38,-37)$	$(-38,-35)$	$(-38,-29)$	$(-38,-23)$	$(-38,-13)$
$(-38,-11)$	$(-38,29)$	$(-38,43)$	$(-37,-35)$	$(-37,-34)$	$(-37,-23)$	$(-37,-14)$	$(-37,-2)$	$(-37,29)$
$(-37,43)$	$(-35,29)$	$(-35,43)$	$(-34,-23)$	$(-34,-7)$	$(-34,29)$	$(-34,43)$	$(-31,-29)$	$(-31,-26)$
$(-31,-10)$	$(-31,-2)$	$(-31,23)$	$(-31,29)$	$(-31,43)$	$(-29,-26)$	$(-29,-23)$	$(-29,-22)$	$(-29,-14)$
$(-29,-2)$	$(-29,-1)$	$(-29,43)$	$(-26,-19)$	$(-26,-7)$	$(-26,29)$	$(-26,43)$	$(-23,-22)$	$(-23,-17)$
$(-23,-14)$	$(-23,-11)$	$(-23,29)$	$(-23,31)$	$(-23,43)$	$(-22,-19)$	$(-22,29)$	$(-22,43)$	$(-19,-17)$
$(-19,-13)$	$(-19,29)$	$(-19,43)$	$(-17,-14)$	$(-17,-5)$	$(-17,29)$	$(-17,43)$	$(-14,-13)$	$(-14,29)$
$(-13,29)$	$(-13,41)$	$(-13,43)$	$(-11,-7)$	$(-11,29)$	$(-11,43)$	$(-10,29)$	$(-10,43)$	$(-7,29)$
$(-7,43)$	$(-5,29)$	$(-5,43)$	$(-2,29)$	$(-2,43)$	$(-1,29)$	$(-1,43)$	$(2,31)$	$(2,37)$
$(2,41)$	$(2,43)$	$(5,17)$	$(5,29)$	$(5,43)$	$(5,47)$	$(7,11)$	$(7,26)$	$(7,29)$
$(7,34)$	$(7,41)$	$(7,43)$	$(7,46)$	$(10,29)$	$(10,31)$	$(10,41)$	$(10,43)$	$(11,23)$
$(11,29)$	$(11,38)$	$(11,46)$	$(13,14)$	$(13,19)$	$(13,29)$	$(13,38)$	$(13,41)$	$(13,43)$
$(13,47)$	$(14,17)$	$(14,37)$	$(14,41)$	$(14,43)$	$(17,19)$	$(17,23)$	$(17,29)$	$(17,46)$
$(19,22)$	$(19,26)$	$(19,29)$	$(19,43)$	$(19,46)$	$(22,23)$	$(22,43)$	$(22,47)$	$(23,34)$
$(23,37)$	$(23,38)$	$(23,43)$	$(26,31)$	$(26,43)$	$(26,47)$	$(29,34)$	$(29,35)$	$(29,37)$
$(29,38)$	$(29,41)$	$(29,46)$	$(29,47)$	$(34,37)$	$(35,37)$	$(35,38)$	$(35,43)$	$(35,47)$
$(37,38)$	$(37,43)$	$(38,43)$	$(41,43)$	$(43,46)$	$(-39,-14)$	$(-39,-2)$	$(-39,43)$	$(-30,-17)$
$(-30,-11)$	$(-30,43)$	$(-21,43)$	$(6,13)$	$(6,19)$	$(6,43)$	$(15,22)$	$(15,34)$	$(15,43)$
$(-47,-21)$	$(-47,15)$	$(-41,-39)$	$(-41,42)$	$(-26,-21)$	$(10,33)$	$(13,42)$	$(31,42)$	$(-42,-37)$
$(-42,-31)$	$(-42,-19)$	$(-42,-13)$	$(-42,-1)$	$(-42,5)$	$(-42,11)$	$(-42,17)$	$(-42,23)$	$(-42,41)$
$(-42,47)$	$(-33,-31)$	$(-33,-19)$	$(-33,-13)$	$(-33,-10)$	$(-33,-7)$	$(-33,-1)$	$(-33,2)$	$(-33,5)$
$(-33,14)$	$(-33,17)$	$(-33,23)$	$(-33,26)$	$(-33,35)$	$(-33,38)$	$(-33,41)$	$(-33,47)$	$(-15,-13)$
$(-15,-7)$	$(-15,-1)$	$(-15,2)$	$(-15,11)$	$(-15,14)$	$(-15,17)$	$(-15,23)$	$(-15,26)$	$(-15,38)$
$(-15,41)$	$(-15,47)$	$(-6,-1)$	$(-6,5)$	$(-6,11)$	$(-6,17)$	$(-6,23)$	$(-6,35)$	$(-6,41)$
$(-6,47)$	$(3,5)$	$(3,11)$	$(3,14)$	$(3,17)$	$(3,23)$	$(3,26)$	$(3,35)$	$(3,38)$
$(3,41)$	$(3,47)$	$(21,23)$	$(21,26)$	$(21,38)$	$(21,41)$	$(21,47)$	$(30,41)$	$(30,47)$
$(39,41)$	$(39,47)$	$(-46,-33)$	$(-46,-15)$	$(-46,3)$	$(-46,21)$	$(-46,39)$	$(-43,-42)$	$(-43,-33)$
$(-43,-15)$	$(-43,-6)$	$(-43,3)$	$(-43,21)$	$(-43,30)$	$(-43,39)$	$(-37,-33)$	$(-37,-15)$	$(-37,-6)$
$(-37,3)$	$(-37,21)$	$(-37,30)$	$(-37,39)$	$(-34,-33)$	$(-34,-15)$	$(-34,3)$	$(-34,21)$	$(-34,39)$
$(-31,-15)$	$(-31,-6)$	$(-31,3)$	$(-31,21)$	$(-31,30)$	$(-31,39)$	$(-22,-15)$	$(-22,3)$	$(-22,21)$
$(-22,39)$	$(-19,-15)$	$(-19,-6)$	$(-19,3)$	$(-19,21)$	$(-19,30)$	$(-19,39)$	$(-13,-6)$	$(-13,3)$
$(-13,21)$	$(-13,30)$	$(-10,3)$	$(-10,21)$	$(-10,39)$	$(-7,-6)$	$(-7,3)$	$(-7,30)$	$(-7,39)$
$(-1,3)$	$(-1,21)$	$(-1,30)$	$(-1,39)$	$(2,3)$	$(2,21)$	$(2,39)$	$(5,21)$	$(5,39)$
$(11,21)$	$(11,30)$	$(11,39)$	$(14,39)$	$(17,21)$	$(17,30)$	$(17,39)$	$(23,30)$	$(23,39)$
$(35,39)$	$(38,39)$	$(-39,-37)$	$(-39,-34)$	$(-39,-31)$	$(-39,-10)$	$(-39,-7)$	$(-39,-1)$	$(-39,2)$
$(-39,5)$	$(-39,11)$	$(-39,14)$	$(-39,17)$	$(-39,23)$	$(-39,35)$	$(-39,38)$	$(-39,41)$	$(-39,47)$
$(-30,-19)$	$(-30,-13)$	$(-30,-7)$	$(-30,-1)$	$(-30,11)$	$(-30,17)$	$(-30,23)$	$(-30,41)$	$(-30,47)$
$(-21,-19)$	$(-21,-13)$	$(-21,-10)$	$(-21,-1)$	$(-21,2)$	$(-21,5)$	$(-21,11)$	$(-21,17)$	$(-21,23)$
$(-21,26)$	$(-21,38)$	$(-21,41)$	$(-21,47)$	$(6,11)$	$(6,17)$	$(6,23)$	$(6,35)$	$(6,41)$
$(6,47)$	$(15,17)$	$(15,23)$	$(15,26)$	$(15,38)$	$(15,41)$	$(15,47)$	$(33,35)$	$(33,38)$
$(42,47)$	$(-46,-39)$	$(-46,-21)$	$(-46,15)$	$(-43,-39)$	$(-43,-21)$	$(-43,6)$	$(-43,15)$	$(-43,42)$
$(-37,-21)$	$(-37,6)$	$(-37,15)$	$(-37,33)$	$(-37,42)$	$(-34,-21)$	$(-34,15)$	$(-34,33)$	$(-31,-30)$
$(-31,-21)$	$(-31,6)$	$(-31,15)$	$(-31,33)$	$(-31,42)$	$(-22,-21)$	$(-22,15)$	$(-19,6)$	$(-19,15)$
$(-19,33)$	$(-19,42)$	$(-13,6)$	$(-13,15)$	$(-13,33)$	$(-13,42)$	$(-10,33)$	$(-7,6)$	$(-7,15)$
$(-7,33)$	$(-1,6)$	$(-1,15)$	$(-1,33)$	$(-1,42)$	$(2,15)$	$(2,33)$	$(5,6)$	$(5,33)$
$(5,42)$	$(11,15)$	$(11,42)$	$(14,15)$	$(14,33)$	$(17,33)$	$(17,42)$	$(23,33)$	$(23,42)$
$(41,42)$								

Table 2. (d_1, d_2) with 3-rank$(K_2\mathcal{O}_F) = 2$

$(-14,43)$	$(2,29)$	$(11,43)$	$(14,29)$	$(17,43)$	$(22,29)$	$(23,29)$	$(26,29)$	$(29,31)$
$(29,43)$	$(31,43)$	$(34,43)$	$(43,47)$	$(-42,-41)$	$(-42,-29)$	$(-42,-23)$	$(-42,-17)$	$(-42,-11)$
$(-42,-5)$	$(-42,1)$	$(-42,13)$	$(-42,19)$	$(-42,31)$	$(-42,37)$	$(-33,-29)$	$(-33,-23)$	$(-33,-17)$
$(-33,-14)$	$(-33,-5)$	$(-33,-2)$	$(-33,1)$	$(-33,7)$	$(-33,10)$	$(-33,13)$	$(-33,19)$	$(-33,31)$
$(-33,34)$	$(-33,37)$	$(-15,-14)$	$(-15,-11)$	$(-15,-2)$	$(-15,1)$	$(-15,7)$	$(-15,13)$	$(-15,19)$
$(-15,22)$	$(-15,31)$	$(-15,34)$	$(-15,37)$	$(-15,46)$	$(-6,-5)$	$(-6,1)$	$(-6,7)$	$(-6,13)$
$(-6,19)$	$(-6,31)$	$(-6,37)$	$(-6,43)$	$(3,7)$	$(3,10)$	$(3,13)$	$(3,19)$	$(3,22)$
$(3,34)$	$(3,37)$	$(3,46)$	$(21,22)$	$(21,31)$	$(21,34)$	$(21,37)$	$(21,46)$	$(30,31)$
$(39,46)$	$(-47,-42)$	$(-47,-15)$	$(-47,-6)$	$(-47,3)$	$(-47,21)$	$(-47,30)$	$(-47,39)$	$(-41,-15)$
$(-41,-6)$	$(-41,3)$	$(-41,21)$	$(-41,30)$	$(-41,39)$	$(-38,-33)$	$(-38,-15)$	$(-38,3)$	$(-38,21)$
$(-38,39)$	$(-35,-33)$	$(-35,-6)$	$(-35,3)$	$(-35,39)$	$(-29,-15)$	$(-29,-6)$	$(-29,3)$	$(-29,21)$
$(-29,30)$	$(-29,39)$	$(-26,-15)$	$(-26,3)$	$(-26,21)$	$(-23,-15)$	$(-23,-6)$	$(-23,3)$	$(-23,21)$
$(-23,30)$	$(-23,39)$	$(-17,-15)$	$(-17,-6)$	$(-17,3)$	$(-17,21)$	$(-17,30)$	$(-17,39)$	$(-14,3)$
$(-14,39)$	$(-11,-6)$	$(-11,3)$	$(-11,21)$	$(-11,30)$	$(-11,39)$	$(-5,3)$	$(-5,21)$	$(-5,39)$
$(-2,3)$	$(-2,21)$	$(-2,39)$	$(1,3)$	$(1,21)$	$(1,30)$	$(1,39)$	$(7,30)$	$(7,39)$
$(10,21)$	$(10,39)$	$(13,21)$	$(13,30)$	$(19,21)$	$(19,30)$	$(31,39)$	$(34,39)$	$(37,39)$
$(33,43)$	$(42,43)$	$(-42,29)$	$(-33,29)$	$(-15,29)$	$(-6,29)$	$(3,29)$	$(21,29)$	$(29,30)$
$(29,39)$	$(-39,-22)$	$(-39,-19)$	$(-39,29)$	$(-30,29)$	$(-21,29)$	$(6,29)$	$(15,29)$	$(33,41)$
$(33,47)$	$(-46,33)$	$(-43,-30)$	$(-43,33)$	$(-37,-30)$	$(26,33)$	$(29,33)$	$(29,42)$	

Table 3. (d_1, d_2) with 3-rank$K_2\mathcal{O}_F = 3$

$(-42,43)$	$(-33,-47)$	$(-33,-41)$	$(-33,-26)$	$(-33,46)$	$(-15,43)$	$(-6,43)$	$(3,31)$	$(3,43)$
$(21,43)$	$(30,37)$	$(39,19)$	$(39,22)$	$(39,43)$				

Finally, there are only two pairs $(-33, 43)$ and $(30, 43)$ with 3-rank$(K_2\mathcal{O}_F) = 4$. For any (d_1, d_2) which does not appear in the above tables, the 3-primary part of $K_2\mathcal{O}_F$ is trivial.

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