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Iterated Cesàro averages, frequencies of digits, and Baire category

by

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1. Statement of results. Fix a positive integer $N \geq 2$. Throughout this paper, we will consider the unique, non-terminating, base N expansion of a number $x \in [0,1]$, written as

$$x = \frac{d_1(x)}{N} + \frac{d_2(x)}{N^2} + \cdots + \frac{d_n(x)}{N^n} + \cdots \qquad \text{with } d_i(x) \in \{0,1,\ldots,N-1\}.$$

For each digit $i \in \{0,1,\ldots,N-1\}$, we will write

$$\Pi_i(x;n) = \frac{1}{n}|\{1 \leq j \leq n \mid d_j(x) = i\}|$$

for the frequency of the digit i among the first n digits of x.

We recall that in a metric space X, a set S is called *residual* if its complement is of the first category. Also recall that we say that a *typical element x has property* P if the set $S = \{x \in X \mid x \text{ has property P}\}$ is residual. We refer the reader to Oxtoby [Ox] for more details. The limiting behaviour of the frequencies $\Pi_i(x;n)$ has been investigated extensively during the past many years. For example, Volkmann [Vo] and Šalát [Ša1] proved that for a typical x, we have

$$\limsup_{n\to\infty} \Pi_i(x;n) = 1 \quad \text{and} \quad \liminf_{n\to\infty} \Pi_i(x;n) = 0$$

for all $0 \leq i \leq N-1$. Their proofs can also be found in various textbooks; see, for example, Billingsley [Bi, p. 16] and Hlawka [Hl, p. 77].

During the past 50 years these results have been extended and generalized further in many different directions [APT, CZ, Ol, Ša2, Si, SŠ]. For example, recently Olsen [Ol] proved Theorem A below. To state this result we need to introduce some notation. Let Δ_N denote the family of

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N-dimensional probability vectors, i.e.

$$\Delta_N = \Big\{(p_0, p_1, \ldots, p_{N-1}) \,\Big|\, p_i \geq 0, \sum_{i=0}^{N-1} p_i = 1\Big\},$$

and write

$$\boldsymbol{\Pi}(x;n) = (\Pi_0(x;n), \Pi_1(x;n), \ldots, \Pi_{N-1}(x;n))$$

for the vector of frequencies of digits. Then clearly $\boldsymbol{\Pi}(x;n) \in \Delta_N$, and consequently the set of accumulation points of the sequence $(\boldsymbol{\Pi}(x;n))_n$ is a subset of Δ_N. Somewhat surprisingly, in [Ol] it is proved that for a typical $x \in [0,1]$, the set of accumulation points of $(\boldsymbol{\Pi}(x;n))_n$ is all of Δ_N.

Theorem A [Ol]. *The set*

$$\{x \in [0,1] \mid \textit{the set of accumulation points of } (\boldsymbol{\Pi}(x;n))_{n=1}^{\infty} \textit{ equals } \Delta_N\}$$

is residual.

Given a divergent sequence, forming its Cesàro averages may succeed in producing a convergent sequence, and one might expect that iterating this method would eventually give a convergent sequence when applied to the sequence $(\boldsymbol{\Pi}(x;n))_n$. In this paper we will prove a somewhat unexpected result that strengthens Theorem A considerably. Namely, we show that for a typical x, the set of accumulation points of all higher order Cesàro averages of the sequence $(\boldsymbol{\Pi}(x;n))_n$ equals Δ_N. To state this result precisely, we make the following definitions. Let

$$\Pi_i^{(1)}(x;n) = \Pi_i(x;n),$$

and for $k \geq 2$, let

$$\Pi_i^{(k)}(x;n) = \frac{\sum_{j=1}^{n} \Pi_j^{(k-1)}(x;j)}{n}$$

denote kth iterated Cesàro average. Next let

$$\boldsymbol{\Pi}^{(k)}(x;n) = (\Pi_0^{(k)}(x;n), \Pi_1^{(k)}(x;n), \ldots, \Pi_{N-1}^{(k)}(x;n))$$

denote the vector of kth iterated Cesàro averages. Now we can state our main theorem.

Theorem 1.1. *The set*

$$R = \{x \in [0,1] \mid \textit{the set of accumulation points of } (\boldsymbol{\Pi}^{(k)}(x;n))_{n=1}^{\infty} \textit{ equals } \Delta_N \textit{ for all } k \in \mathbb{N}\}$$

is residual.

The proof of Theorem 1.1 is given in Section 2. Before presenting it, we make a few remarks concerning our result in different contexts. Below we will denote the Hausdorff dimension and the packing dimension by $\dim_{\mathrm{H}}$

and $\dim_P$ respectively. The reader is referred to [Fa] for the definitions of dimensions.

Using Theorem 1.1 it is easy to find the packing dimension of the set R.

COROLLARY 1.2. *The packing dimension of the set R equals* 1, *i.e.*

$$\dim_P R = 1.$$

Proof. Recall that if C is a compact subset of $\mathbb{R}$ and M is a subset of C with $\dim_P M < \dim_P C$, then M is of the first category in C (see [Ed, Exercise (1.8.4)]). Combined with Theorem 1.1, this implies that $\dim_P R = \dim_P([0,1]) = 1$. ∎

Now we compare this with the Hausdorff dimension of R. It follows from [OW] that $\dim_H R = 0$. Hence, in terms of dimensions, the size of R varies between "very big" and "very small" depending on the exact viewpoint, but it follows from Theorem 1.1 that R is always "very big" topologically.

This contrast between the topological and measure-theoretical viewpoints is also emphasized by Borel's Normal Number Theorem, which states that

$$\Pi_i(x;n) \to \frac{1}{N}$$

for Lebesgue almost all $x \in [0,1]$. It follows from Borel's Normal Number Theorem that for Lebesgue almost all $x \in [0,1]$, the sequence $(\Pi_i(x;n))_n$ has only one accumulation point (namely $1/N$). This contrasts vastly with the topological point of view. Namely, Theorem 1.1 states that for a typical x, the set of accumulation points of $(\Pi_i^{(k)}(x;n))_n$ equals the simplex of N-dimensional probability vectors for all $k \in \mathbb{N}$.

2. Proof of Theorem 1.1. Throughout the proof, we will work with a subset of $[0,1]$, namely

$$\mathbb{I} = [0,1] \setminus \{x \in [0,1] \mid x \text{ has a terminating } N\text{-adic expansion}\}.$$

To simplify the notation in our proof, we define the function $\varphi_1(x) = 2^x$ and $\varphi_m(x) = \varphi_1(\varphi_{m-1}(x))$ for $m \geq 2$. For brevity, write $\mathbb{D} = (\mathbb{Q}^N \cap \Delta_N) \setminus \{(1,0,0,\ldots,0)\}$. (We exclude one particular vector for technical reasons, which become apparent in the proof of Claim 2.) We define the property P as follows. We say that a sequence $(\boldsymbol{x}_n)_n$ in $\mathbb{R}^N$ has *property* P if for all $\boldsymbol{q} \in \mathbb{D}$, $m \in \mathbb{N}$, $i \in \mathbb{N}$, and $\epsilon > 0$, there exists $j \in \mathbb{N}$ satisfying:

(i) $j \geq i$,
(ii) $j/2^j < \epsilon$,
(iii) if $j < n < \varphi_m(j)$ then $|\boldsymbol{x}_n - \boldsymbol{q}| < \epsilon$.

Our proof of Theorem 1.1 will consist of three lemmas:

(1) First we will prove that the set

$$A = \{x \in \mathbb{I} \mid (\boldsymbol{\Pi}^{(1)}(x;n))_{n=1}^{\infty} \text{ has property P}\} \tag{2.1}$$

is residual.

(2) Then we will show that if $(\boldsymbol{\Pi}^{(k)}(x;n))_{n=1}^{\infty}$ has property P, then $(\boldsymbol{\Pi}^{(k+1)}(x;n))_{n=1}^{\infty}$ also has property P.

(3) Finally, we will show that $A \subseteq R$ (recall that R is defined in Theorem 1.1).

LEMMA 2.1. *The set A is residual* (*recall that A is defined in* (2.1)).

Proof. For fixed $h, m, i \in \mathbb{N}$ and $\boldsymbol{q} \in \mathbb{D}$, we define *property* $\mathrm{P}_{h,m,\boldsymbol{q},i}$, as follows. We say that a sequence $(\boldsymbol{x}_n)_n$ has property $\mathrm{P}_{h,m,\boldsymbol{q},i}$ if for every $\epsilon > 1/h$, there exists $j \in \mathbb{N}$ satisfying:

(i) $j \geq i$,
(ii) $j/2^j < \epsilon$,
(iii) $j < n < \varphi_m(2^j) \Rightarrow |\boldsymbol{x}_n - \boldsymbol{q}| < \epsilon$.

Let $G_{h,m,\boldsymbol{q},i} = \{x \in \mathbb{I} \mid (\boldsymbol{\Pi}^{(1)}(x;n))_{n=1}^{\infty} \text{ has property } \mathrm{P}_{h,m,\boldsymbol{q},i}\}$. Clearly,

$$\bigcap_{h\in\mathbb{N}}\bigcap_{m\in\mathbb{N}}\bigcap_{\boldsymbol{q}\in\mathbb{D}}\bigcap_{i\in\mathbb{N}} G_{h,m,\boldsymbol{q},i} = A.$$

CLAIM 1. *$G_{h,m,\boldsymbol{q},i}$ is open.*

Proof. Let $x \in G_{h,m,\boldsymbol{q},i}$. Since $x \in G_{h,m,\boldsymbol{q},i}$, there exists a positive integer j such that $j \geq i$, $j/2^j < 1/h$, and if $j < n < \varphi_m(2^j)$, then $|\boldsymbol{\Pi}^{(1)}(x;n) - \boldsymbol{q}| < 1/h$.

We now choose δ to equal $1/N^{\varphi_m(2^j)+1}$ if the 2^jth digit of x is neither 0 nor $N-1$, otherwise we choose it to be $1/N^a$ where a is any integer such that the $(a-1)$st and $(a-2)$nd digits are not both either 0 or $N-1$, and that $a > \varphi_m(2^j) + 1$. Then all $y \in B(x, \delta)$ have their first $\varphi_m(2^j)$ digits the same as x, and so $B(x,\delta) \subseteq G_{h,m,\boldsymbol{q},i}$. This completes the proof of Claim 1.

CLAIM 2. *$G_{h,m,\boldsymbol{q},i}$ is dense.*

Proof. Let $x \in \mathbb{I}$ and $\delta > 0$. We must now find $y \in B(x,\delta) \cap G_{h,m,\boldsymbol{q},i}$. Let $t \in \mathbb{N}$ be such that $1/N^t < \delta$. We can clearly choose a positive integer $s \in \mathbb{N}$ and $z_1, \ldots, z_s \in \{0, 1, \ldots, N-1\}$ such that if $z = z_1/N + z_2/N^2 + \cdots + z_s/N^s$ then $\boldsymbol{\Pi}^{(1)}(z;s) = \boldsymbol{q}$. Let

$$y = \frac{d_1(x)}{N} + \cdots + \frac{d_t(x)}{N^t} + \sum_{i=0}^{\infty}\left(\frac{z_1}{N^{t+is+1}} + \frac{z_2}{N^{t+is+2}} + \cdots + \frac{z_s}{N^{t+is+s}}\right).$$

Then $y \in B(x,\delta)$ (as y has the first t digits the same as x).

Next we show that $y \in G_{h,m,\boldsymbol{q},i}$. All z_i's cannot be 0, because we excluded the vector $(1, 0, \ldots, 0)$. Therefore, y has a non-terminating N-adic expansion. Let $\epsilon \geq 1/h$, and choose j such that

$$\frac{j}{2^j} < \epsilon \quad \text{and} \quad j \geq N \max\left(\max_{l \in \{0,1,\ldots,N-1\}} \left| \frac{N_l(z;s)(2+t/s) + N_l(y;t)}{\epsilon} \right|, i, t \right)$$

where $N_l(x;n) = |\{0 < j \leq n \mid d_j(x) = l\}|$. Fix a positive integer n with $j < n < \varphi_m(2^j)$ and observe that we can find integers r and b, with $0 \leq r < s$ and $0 \leq b < N_l(z;s)$, such that $n = t + \lfloor (n-t)/s \rfloor s + r$, and $N_l(y;n) = N_l(y;t) + \lfloor (n-t)/s \rfloor N_l(z;s) + b$. We now have

$$\begin{aligned}
|\boldsymbol{\Pi}^{(1)}(y;n) - \boldsymbol{q}| &= N \max_l \left| \Pi_l^{(1)}(y;n) - \frac{N_l(z;s)}{s} \right| \\
&= N \max_l \left| \frac{N_l(y;n)}{n} - \frac{n\frac{N_l(z;s)}{s}}{n} \right| \\
&= N \max_l \left| \frac{N_l(y;t) + \lfloor \frac{n-t}{s} \rfloor N_l(z;s) + b}{n} - \frac{\left(t + \lfloor \frac{n-t}{s} \rfloor s + r\right)\frac{N_l(z;s)}{s}}{n} \right| \\
&\leq N \max_l \left(\left| \frac{\lfloor \frac{n-t}{s} \rfloor N_l(z;s) - \frac{N_l(z;s)}{s}(t+r) - N_l(z;s)\lfloor \frac{n-t}{s} \rfloor}{n} \right| + \frac{N_l(y;t)+b}{n} \right) \\
&\leq N \max_l \left(\left| \frac{-\frac{N_l(z;s)}{s}(t+r)}{n} \right| + \frac{N_l(y;t) + N_l(z;s)}{n} \right) \\
&\leq N \max_l \left(\frac{\frac{N_l(z;s)}{s}(t+r) + N_l(y;t) + N_l(z;s)}{n} \right) \\
&\leq N \max_l \left(\frac{\frac{N_l(z;s)}{s}(t+s) + N_l(y;t) + N_l(z;s)}{j} \right) \\
&\leq N \max_l \left(\frac{N_l(z;s)(2+t/s) + N_l(y;t)}{j} \right) \leq \epsilon,
\end{aligned}$$

where the maximum is over $l \in \{0,1,\ldots,N-1\}$. This shows that $y \in G_{h,m,\boldsymbol{q},i}$, and completes the proof of Claim 2.

It follows from Claims 1 and 2 that A is the countable intersection of open and dense sets, and hence residual. This completes the proof of Lemma 2.1. ■

LEMMA 2.2. *If* $(\boldsymbol{\Pi}^{(k)}(x;n))_{n=1}^{\infty}$ *has property* P, *then* $(\boldsymbol{\Pi}^{(k+1)}(x;n))_{n=1}^{\infty}$ *also has property* P.

Proof. Let $(\boldsymbol{\Pi}^{(k)}(x;n))_{n=1}^{\infty}$ have property P, and fix $\epsilon > 0$, $\boldsymbol{q} \in \mathbb{D}$, $i \in \mathbb{N}$ and $m \in \mathbb{N}$. Since $(\boldsymbol{\Pi}^{(k)}(x;n))_{n=1}^{\infty}$ has property P, there exists $j' \in \mathbb{N}$ with $j' \geq i$, $j'/2^{j'} < \epsilon/3$, and such that if $j' < n < \varphi_{m+1}(2^{j'})$ then $|\boldsymbol{\Pi}^{(k)}(x;n) - \boldsymbol{q}| < \epsilon/3$. Let $j = 2^{j'}$. For all $j < n < \varphi_m(2^j)$ (i.e. $2^{j'} < n < \varphi_{m+1}(2^{j'})$), we have

$$\begin{aligned}|\boldsymbol{\Pi}^{(k+1)}(x;n)-\boldsymbol{q}| &= \left|\frac{\boldsymbol{\Pi}^{(k)}(x;1)+\boldsymbol{\Pi}^{(k)}(x;2)+\cdots+\boldsymbol{\Pi}^{(k)}(x;n)}{n}-\boldsymbol{q}\right| \\ &= \left|\frac{\boldsymbol{\Pi}^{(k)}(x;1)+\cdots+\boldsymbol{\Pi}^{(k)}(x;j')}{n}\right. \\ &\qquad \left.+\frac{\boldsymbol{\Pi}^{(k)}(x;j'+1)+\cdots+\boldsymbol{\Pi}^{(k)}(x;n)-(n-j')\boldsymbol{q}}{n}-\frac{j'\boldsymbol{q}}{n}\right| \\ &\le \frac{|\boldsymbol{\Pi}^{(k)}(x;1)+\cdots+\boldsymbol{\Pi}^{(k)}(x;j')|}{n} \\ &\qquad +\frac{|\boldsymbol{\Pi}^{(k)}(x;j'+1)-\boldsymbol{q}|+\cdots+|\boldsymbol{\Pi}^{(k)}(x;n)-\boldsymbol{q}|}{n}+\frac{|j'\boldsymbol{q}|}{n} \\ &\le \frac{j'}{n}+\frac{\epsilon}{3}\,\frac{n-j'}{n}+\frac{j'}{n}\le\frac{j'}{2^{j'}}+\frac{\epsilon}{3}+\frac{j'}{2^{j'}}\le\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.\end{aligned}$$

This completes the proof of Lemma 2.2. ∎

LEMMA 2.3. *The set A is a subset of R (recall that R is defined in Theorem 1.1).*

Proof. Let $x \in A$. By Lemma 2.2, we deduce that $(\boldsymbol{\Pi}^{(k)}(x;n))_n$ has property P for all k. We now want to show that $x \in R$, i.e. the set of accumulation points of $(\boldsymbol{\Pi}^{(k)}(x;n))_n$ equals Δ_N. It is clear that the set of accumulation points of $(\boldsymbol{\Pi}^{(k)}(x;n))_n$ is a subset of Δ_N. Hence, it suffices to show that each $\boldsymbol{p} \in \Delta_N$ is an accumulation point of $(\boldsymbol{\Pi}^{(k)}(x;n))_n$. Therefore, let $\boldsymbol{p} \in \Delta_N$. Fix $l \in \mathbb{N}$ and $\boldsymbol{q} \in \mathbb{D}$ such that $|\boldsymbol{p}-\boldsymbol{q}| \le 1/l$.

We first observe that we can find $n_l > l$ such that

$$|\boldsymbol{q}-\boldsymbol{\Pi}^{(k)}(x;n_l)| \le 1/l. \tag{2.2}$$

We now prove (2.2). Indeed, since $x \in A$, we conclude from Lemma 2.2 that $(\boldsymbol{\Pi}^{(k)}(x;n))_n$ has property P. In particular, we can find $j \in \mathbb{N}$ with $l \le j$ and such that if $j < n < \varphi_m(2^j)$ then $|\boldsymbol{\Pi}^{(k)}(x;n)-\boldsymbol{q}| < 1/l$. Hence if n_l is any integer with $j < n_l < \varphi_m(2^j)$ then $|\boldsymbol{\Pi}^{(k)}(x;n_l)-\boldsymbol{q}| < 1/l$.

Hence, the sequence $(n_l)_l$ satisfies n_l and

$$|\boldsymbol{p}-\boldsymbol{\Pi}^{(k)}(x;n_l)| \le |\boldsymbol{p}-\boldsymbol{q}|+|\boldsymbol{\Pi}^{(k)}(x;n_l)-\boldsymbol{q}| \le 2/l. \tag{2.3}$$

Since $n_l > l$, we can extract an increasing subsequence $(n_{l_u})_u$ of $(n_l)_l$. It now follows from (2.3) that $\boldsymbol{\Pi}^{(k)}(x;n_{l_u}) \to \boldsymbol{p}$. Hence $\boldsymbol{p}$ is an accumulation point of $(\boldsymbol{\Pi}^{(k)}(x;n_{l_u}))_{u=1}^{\infty}$. This completes the proof of Lemma 2.3. ∎

Proof of Theorem 1.1. It follows from Lemma 2.3 that $A \subseteq R$. By Lemma 2.1, A is residual in $\mathbb{I}$. Since it is easily seen that $[0,1] \setminus \mathbb{I}$ is a countable union of nowhere dense sets, A is residual in $[0,1]$. Hence, we conclude that R is residual. ∎

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