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# Billiard and diophantine approximation

by

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**1. Introduction.** For a real number $x$, $[x]$ is the integral part of $x$, $\{x\}$ is the fractional part of $x$ and $\|x\| = \min(\{x\}, 1 - \{x\})$ is the distance of $x$ to the nearest integer.

Let $0 < \theta < 1$ and $0 \le \alpha < 1$. A fraction $p/q$, $q > 0$, is called a *best $\alpha$-approximation* to $\theta$ (*homogeneous* for $\alpha = 0$) (see [4]) if

$$\|q\theta - \alpha\| = |q\theta - \alpha - p|,$$

and if

$$\|j\theta - \alpha\| > \|q\theta - \alpha\| \quad \text{for } 0 < j < q.$$

Notice that if $\|q\theta - \alpha\| = 0$ for some $q \in \mathbb{N}$, then the set of all best $\alpha$-approximations to $\theta$ is finite. It is well known [4] that the best homogeneous approximations to $\theta$ are given by the continued fraction process. Namely, the convergents $p_n/q_n$ to $\theta$ are the best homogeneous approximations to $\theta$, for $n \ge 1$ if $0 < \theta \le 1/2$ and for $n \ge 2$ if $1/2 < \theta < 1$ (see Remark 3.3). The idea of a best inhomogeneous approximation ($\alpha > 0$) has been investigated by several authors, for example Khintchine [7], Barnes and Swinnerton-Dyer [2], Cassels [3], Sós [11], Cusick, Rockett and Szüsz [5] or Komatsu [8].

We say that an index $q$ is a *critical index* of a real-valued sequence $G(j)$, $j \in \mathbb{N}$, if

$$G(j) > G(q) \quad \text{for } 0 < j < q.$$

We say that sequences $G(j)$ and $H(j)$, $j \in \mathbb{N}$, are *diophantine equivalent* if they have the same set of critical indices and are equal on this set. Note that a fraction $p/q$ is a best $\alpha$-approximation to $\theta$ iff $q$ is a critical index of the sequence $\|j\theta - \alpha\|$, $j \in \mathbb{N}$. Hence, the sequence of the best $\alpha$-approximations to $\theta$ is determined by any sequence diophantine equivalent to the sequence $\|j\theta - \alpha\|$, $j \in \mathbb{N}$.

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A $(\theta,\alpha)$-*billiard sequence* (*homogeneous* for $\alpha = 0$) is a sequence $F(j) \in [0,1)$, $j \in \mathbb{N}$, which satisfies the following conditions:

$$\begin{aligned} &F(1) = \alpha/2, \\ &F(j) + F(j+1) = \theta \text{ or } 1+\theta \quad \text{for } j \text{ odd}, \\ &F(j) + F(j+1) = 1 \text{ or } 0 \quad \text{for } j \text{ even}. \end{aligned}$$

Note that if $\alpha$ and $\theta$ are rational, then $F(j)$ is a periodic sequence. We consider a billiard table rectangle with the bottom left vertex labelled $v_0$, and the others, in a clockwise direction, $v_1$, $v_2$ and $v_3$. The distance from $v_0$ to $v_1$ is $\theta/2$. We describe the position of points on the perimeter by their distance around the perimeter measured in a clockwise direction from $v_0$, so that $v_1$ is at position $\theta/2$, $v_2$ at $1/2$ and $v_3$ at $(\theta+1)/2$. If a billiard ball is sent out from position $F(1) = \alpha/2$ at an angle of $\pi/4$, then the ball will rebound against the sides of the rectangle consecutively at points $F(2), F(3), \ldots$.

Let $F(j)$, $j \in \mathbb{N}$, be a $(\theta,\alpha)$-billiard sequence. We define the following sequences:

$$A(j) = \begin{cases} \min\{\|F(j) - v_i\| : 0 \le i \le 3\} & \text{for } j > 1, \\ \min\{\|F(j) - v_i\| : i = 1,3\} & \text{for } j = 1, \end{cases}$$

$$B(j) = \min\{\|F(j) - v_i\| : 1 \le i \le 3\}, \quad j \in \mathbb{N},$$

$$C(j) = \min\{\|F(k) - F(l)\| : 1 \le k < l \le j+1\}, \quad j \in \mathbb{N}.$$

Since $\|x-y\| = \min(\{x-y\}, 1-\{x-y\})$ is the shortest perimeter distance between $x,y \in [0,1)$, the value $A(j)$, $j > 1$, is the distance between the rebound $F(j)$ and the set of vertices of the rectangle, and $C(j)$ is the minimal distance between any two rebounds $F(k)$ and $F(l)$ for $1 \le k < l \le j+1$.

In Theorem 3.1(1) we prove that $\|j\theta - \alpha\| = \|F(j) - F(j+1)\|$ for $j \in \mathbb{N}$. The main aim of this paper is to prove Theorem 3.2: the sequences $\|j\theta-\alpha\|$, $2A(j)$ are diophantine equivalent, and so too are the sequences $C(j)$, $\min(2A(j), \|F(j+1) - F(1)\|)$. In the homogeneous case all the above sequences are diophantine equivalent to the sequence $2B(j)$, $j \in \mathbb{N}$.

In Theorem 3.3 we prove that if $p/q$, $q > 1$, is a best $\alpha$-approximation to $\theta$, then the numbers $(-1)^p$, $(-1)^q$ determine the unique vertex $v(q)$ such that $\|q\theta - \alpha\| = 2\|F(q) - v(q)\|$. In Corollaries 3.2 and 3.3 we consider the homogeneous case: if $p_n/q_n$ and $a_n$, $n \in \mathbb{N}$, are the sequences of convergents and partial quotients to $\theta < 1/2$, then the sequences $v(q_n)$ and $\operatorname{sgn}(F(q_n) - v(q_n))$ are determined by the sequence $(-1)^{a_n}$, $n \in \mathbb{N}$. On the other hand,

$$a_n = \left[\frac{\|q_{n-1}\theta\|}{\|q_n\theta\|}\right] = \left[\frac{\|F(q_{n-1}) - v(q_{n-1})\|}{\|F(q_n) - v(q_n)\|}\right] \quad \text{for } n > 1.$$

The following theorem is known as the *Steinhaus conjecture* or the *three* *distance theorem*: there are at most three lengths when the unit circle is partitioned by the points $\{j\theta\}$ for $1 \le j \le n$. This theorem was first proved by Sós [11], [12] and then by Świerczkowski [14] and Surányi [13] (see also [1]). Surányi formulates this result in terms of $n$-Farey points. Ravenstein [10] gives solutions in terms of "best" and "second best" rational approximations to $\theta$. Geelen and Simpson [6] prove the following *five distance theorem*: there are at most five lengths when the unit circle is partitioned by the points $\{j\theta\}$ and $\{j\theta + \beta\}$ for $0 \le j \le n$.

In Theorem 2.1 we give an explicit formula for a $(\theta, \alpha)$-billiard sequence. It follows that the five (three) distance theorem is equivalent to the following: there are at most five lengths (three if $F(1) = 0$) when the perimeter of the rectangle is partitioned by a finite sequence of successive rebounds of a billiard ball.

**2. Billiard and the five distance theorem.** For real numbers $x, y$ we write

$$x \equiv y \quad \text{iff} \quad x - y \text{ is an integral number.}$$

THEOREM 2.1. *Let* $0 < \theta < 1$ *and* $0 \le \alpha < 1$. *A sequence* $F(j) \in [0, 1)$, $j \in \mathbb{N}$, *is a* $(\theta, \alpha)$-*billiard sequence iff it satisfies the following conditions*:

$$\begin{aligned} F(2n) &= \{n\theta - \alpha/2\} && \text{for } n \in \mathbb{N}, \\ F(2n+1) &= \{-n\theta + \alpha/2\} && \text{for } n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

*Proof.* If the above equalities are satisfied, then

$$\begin{aligned} F(2n-1) + F(2n) &\equiv -(n-1)\theta + \alpha/2 + n\theta - \alpha/2 = \theta, \\ F(2n) + F(2n+1) &\equiv n\theta - \alpha/2 - n\theta + \alpha/2 = 0. \end{aligned}$$

Since $0 \le F(2n-1) + F(2n) < 2$, we get $F(2n-1) + F(2n) = \theta$ or $1 + \theta$. Since $0 \le F(2n) + F(2n+1) < 2$, we get $F(2n) + F(2n+1) = 0$ or $1$.

Conversely, we prove, by induction, that the $(\theta, \alpha)$-billiard sequence satisfies the condition of Theorem 2.1:

$$\begin{aligned} F(2n) &\equiv -F(2n-1) + \theta \equiv (n-1)\theta - \alpha/2 + \theta = n\theta - \alpha/2, \\ F(2n+1) &\equiv -F(2n) \equiv -n\theta + \alpha/2. \ \blacksquare \end{aligned}$$

This theorem shows that the set of values of a $(\theta, \alpha)$-billiard sequence is the union of two sets of points placed consecutively around the circle an angle $\theta$ apart in two opposite directions. Hence the five (three) distance theorem is equivalent to the following corollary:

COROLLARY 2.1. *There are at most five lengths* (*three if* $F(1) = 0$) *when the perimeter of the rectangle is partitioned by a finite sequence of successive rebounds of a billiard ball. Here "length" means the distance around the perimeter between adjacent rebound points.*

For a family $\mathcal{L}$ of sets, a member $S \in \mathcal{L}$ is defined to be a *minimal* set if it does not contain any other member of $\mathcal{L}$.

COROLLARY 2.2. *The trajectory obtained by a finite sequence of successive rebounds of a billiard ball "draws" at most* 15 *incongruent and minimal rectangles*: 5 *squares and* $\binom{5}{2}$ *non-square rectangles* (*at most* $3+\binom{3}{2}=6$ *if* $F(1)=0$).

REMARK 2.1. Consider a general case, when the initial angle of the ball's motion is not $\pi/4$. By a linear transformation $L$ (compressing or stretching) we can change the billiard table rectangle, so that the general case is transformed to the $\pi/4$ case of Corollary 2.2. Any square which appears in Corollary 2.2 is transformed by $L^{-1}$ into a rhombus. By analogy, any pair of rectangles with the perpendicular sides of the same length is transformed into a pair of parallelograms which are mirror images of each other. Thus we get at most 25 incongruent and minimal parallelograms: 5 rhombi and 10 pairs of parallelograms which are mirror images of each other. There are also at most 5 incongruent triangles adjacent to the perimeter in the $\pi/4$ case and 10 in the general case.

**3. Billiard and the best approximations.** Let $0<\theta<1$, $0\le\alpha<1$, let $F(j)$ be the $(\theta,\alpha)$-billiard sequence, and let $A(j), B(j), C(j)$, $j\in\mathbb{N}$, be the sequences defined in the Introduction.

LEMMA 3.1. *For a real number* $x$,

$$\|x\| = \min\big(2\|\tfrac{1}{2}x\|, 2\|\tfrac{1}{2}(x-1)\|\big).$$

*Proof.* If $2n<x<2n+1$, $n\in\mathbb{Z}$, then

$$\{x\} = x-2n = 2\|\tfrac{1}{2}x\| \quad\text{and}\quad \{-x\} = 2n-(x-1) = 2\|\tfrac{1}{2}(x-1)\|.$$

If $2n-1<x<2n$, $n\in\mathbb{Z}$, then

$$\{x\} = x+1-2n = 2\|\tfrac{1}{2}(x+1)\| \quad\text{and}\quad \{-x\} = 2n-x = 2\|\tfrac{1}{2}x\|.$$

Hence

$$\|x\| = \min(\{x\},\{-x\}) = \min\big(2\|\tfrac{1}{2}x\|, 2\|\tfrac{1}{2}(x-1)\|\big). \ \blacksquare$$

Notice that $\|x\|=\|y\|$ iff $x\equiv y$ or $x\equiv -y$.

THEOREM 3.1.

(1) $\|j\theta-\alpha\| = \|F(j)-F(j+1)\|$.

(2) $\|j\theta-\alpha\| = \begin{cases} \min\big(2\|F(j)-\tfrac{1}{2}\theta\|, 2\|F(j)-\tfrac{1}{2}(1+\theta)\|\big) & \text{for } j \text{ odd},\\ \min\big(2\|F(j)\|, 2\|F(j)-\tfrac{1}{2}\|\big) & \text{for } j \text{ even}.\end{cases}$

*Proof.* By Theorem 2.1 we have the following equalities:

$$F(2n+1)-F(2n+2) \equiv -n\theta+\alpha/2-[(n+1)\theta-\alpha/2] = -(2n+1)\theta+\alpha,$$

$$F(2n) - F(2n+1) \equiv n\theta - \alpha/2 - (-n\theta + \alpha/2) = 2n\theta - \alpha.$$

Hence (1) follows.

Theorem 2.1 also yields the following equalities:

$$2F(2n+1) - \theta \equiv -(2n+1)\theta + \alpha, \quad 2F(2n) \equiv 2n\theta - \alpha.$$

Hence by Lemma 3.1 we obtain (2). ■

LEMMA 3.2. *Sequences $G(j)$ and $H(j)$, $j \in \mathbb{N}$, are diophantine equivalent iff for every $j \in \mathbb{N}$ there exist $j_1, j_2 \le j$ such that*

$$G(j_1) \le H(j) \quad \textit{and} \quad H(j_2) \le G(j).$$

*Proof.* Assume that $G(j)$ and $H(j)$, $j \in \mathbb{N}$, are diophantine equivalent. If $j$ is not a critical index, then there exists a critical index $q < j$ such that $H(q) = G(q) \le \min\{H(j), G(j)\}$. If $j$ is a critical index then both inequalities are satisfied with $j_1 = j_2 = j$.

Now we prove the converse. Let $q$ be a critical index for the sequence $G(j)$. Since $G(j_1) \le H(j_2) \le G(q)$ for some $1 \le j_1 \le j_2 \le q$, we have $j_1 = j_2 = q$ and $H(q) = G(q)$. If $q$ is not a critical index for the sequence $H(j)$, then we obtain the contradiction $G(i_1) \le H(i_2) \le H(q) = G(q)$ for some $1 \le i_1 \le i_2 < q$. By analogy, if $q$ is a critical index for $H(j)$, then $H(q) = G(q)$ and $q$ is a critical index for $G(j)$. ■

By the definition of a $(\theta, \alpha)$-billiard sequence we obtain the following:

REMARK 3.1. For $j$ even,

$$F(j) - \tfrac{1}{2}\theta \equiv \tfrac{1}{2}\theta - F(j-1) \quad \text{and} \quad F(j) - \tfrac{1}{2}(\theta+1) \equiv \tfrac{1}{2}(\theta+1) - F(j-1).$$

For $j > 1$ odd,

$$F(j) \equiv -F(j-1) \quad \text{and} \quad F(j) - \tfrac{1}{2} \equiv \tfrac{1}{2} - F(j-1).$$

REMARK 3.2.

$$F(k) - F(l) \equiv \begin{cases} F(l-1) - F(k+1) & \text{for } 1 \le k < l \text{ of different parity,} \\ F(l+1) - F(k+1) & \text{for } 1 \le k < l \text{ of the same parity.} \end{cases}$$

LEMMA 3.3. *Let $j \in \mathbb{N}$.*

(1) *There exists $1 \le j_1 \le j$ such that*

$$C(j) = \min(\|F(j_1+1) - F(j_1)\|, \|F(j_1+1) - F(1)\|).$$

(2) *If $F(1) = 0$, then there exists $1 \le j_2 \le j$ such that*

$$C(j) = \|F(j_2+1) - F(j_2)\| \quad \textit{and} \quad \|F(j_2)\| \ne 0 \quad \textit{for } j_2 > 1.$$

*Proof.* By Remark 3.2 we have

$$\|F(k) - F(l)\| = \begin{cases} \left\|F\left(k + \frac{l-k+1}{2}\right) - F\left(l - \frac{l-k+1}{2}\right)\right\| & \text{for } 0 < l-k \text{ odd,} \\ \|F(1) - F(l-k+1)\| & \text{for } 0 < l-k \text{ even.} \end{cases}$$

Hence (1) follows.

Let $F(1)=0$. If $0<l-k$ is even, then by Remarks 3.2 and 3.1,

$$\begin{aligned}\|F(k)-F(l)\| &= \|F(1)-F(l-k+1)\| = \|F(1)-F(l-k)\|\\ &= \left\|F\left(1+\tfrac{l-k}{2}\right)-F\left(\tfrac{l-k}{2}\right)\right\|.\end{aligned}$$

Hence there exists the minimal number $1\le j_2\le j$ such that $\|F(j_2+1) - F(j_2)\| = C(j)$. If $j_2>1$ and $F(j_2)=0$, then by Remark 3.1 there exists an even $i\le j_2$ such that $F(i)=0$. Thus by Remark 3.2 we obtain a contradiction $0=\|F(i)-F(1)\| = \left\|F\left(\frac{1}{2}i+1\right)-F\left(\frac{1}{2}i\right)\right\| > C(j)$. This yields (2). ∎

THEOREM 3.2.

(1) *The sequences* $\|j\theta-\alpha\|$ *and* $2A(j)$, $j\in\mathbb{N}$, *are diophantine equivalent.*
(2) *The sequences* $C(j)$ *and* $\min(2A(j),\|F(j+1)-F(1)\|)$, $j\in\mathbb{N}$, *are diophantine equivalent.*
(3) *If* $F(1)=0$, *then the sequences* $\|j\theta\|$, $C(j)$ *and* $2B(j)$, $j\in\mathbb{N}$, *are diophantine equivalent.*

*Proof.* By Remark 3.1 we have:

$$\begin{aligned}&\min\left(2\left\|F(j)-\tfrac{1}{2}\theta\right\|, 2\left\|F(j)-\tfrac{1}{2}(1+\theta)\right\|\right)\\ &\qquad= \min\left(2\left\|F(j-1)-\tfrac{1}{2}\theta\right\|, 2\left\|F(j-1)-\tfrac{1}{2}(1+\theta)\right\|\right)\end{aligned}$$

for $j$ even, and

$$\min\left(2\|F(j)\|, 2\left\|F(j)-\tfrac{1}{2}\right\|\right) = \min\left(2\|F(j-1)\|, 2\left\|F(j-1)-\tfrac{1}{2}\right\|\right)$$

for $j>1$ odd. Hence by Theorem 3.1(2) we have

$$\text{(i)}\qquad \begin{cases} 2A(j) = \min(\|(j-1)\theta-\alpha\|, \|j\theta-\alpha\|) & \text{for } j>1,\\ 2A(1) = \|\theta-\alpha\|. \end{cases}$$

Thus by Lemma 3.2 condition (1) holds.

By Theorem 3.1(1) and (i) we have

$$\text{(ii)}\qquad C(j)\le \min(2A(j), \|F(j+1)-F(1)\|) \quad \text{for } j\in\mathbb{N}.$$

By Lemma 3.3(1) and Theorem 3.1, for every $j\in\mathbb{N}$ there exists $1\le j_1\le j$ such that

$$\begin{aligned}C(j) &= \min(\|F(j_1+1)-F(j_1)\|, \|F(j_1+1)-F(1)\|)\\ &= \begin{cases}\min\left(2\left\|F(j_1)-\tfrac{1}{2}\theta\right\|, 2\left\|F(j_1)-\tfrac{1}{2}(1+\theta)\right\|, \|F(j_1+1)-F(1)\|\right) & \text{for } j_1 \text{ odd},\\ \min\left(2\|F(j_1)\|, 2\left\|F(j_1)-\tfrac{1}{2}\right\|, \|F(j_1+1)-F(1)\|\right) & \text{for } j_1 \text{ even}\end{cases}\\ &\ge \min(2A(j_1), \|F(j_1+1)-F(1)\|).\end{aligned}$$

Hence by Lemma 3.2 and (ii) condition (2) is satisfied.

Let $F(1)=0$. By (ii) we have

$$\text{(iii)}\qquad C(j)\le 2A(j)\le 2B(j) \quad \text{for } j\in\mathbb{N}.$$

By Lemma 3.3(2) and Theorem 3.1, for every $j \in \mathbb{N}$ there exists $1 \le j_2 \le j$ such that

$$C(j) = \|F(j_2+1) - F(j_2)\| = \begin{cases} \min\big(2\big\|F(j_2) - \frac{1}{2}\theta\big\|, 2\big\|F(j_2) - \frac{1}{2}(1+\theta)\big\|\big) & \text{for } j_2 \text{ odd}, \\ \min\big(2\|F(j_2)\|, 2\big\|F(j_2) - \frac{1}{2}\big\|\big) & \text{for } j_2 \text{ even}. \end{cases}$$

Since $C(j) \le \|F(j_2)\| \ne 0$ for $j_2 > 1$, we have

$$C(j) \ge 2B(j_2).$$

Hence, by Lemma 3.2, (iii) and (1), condition (3) holds. ∎

By Theorem 3.2(3) we obtain the following Dirichlet approximation theorem [4].

Corollary 3.1. *Let $\theta$ and $\Theta > 1$ be real. Then there is an integer $q$ such that*

$$0 < q < \Theta, \qquad \|q\theta\| \le \Theta^{-1}.$$

*Proof.* If $F(1) = 0$ and $q_n$, $n \in \mathbb{N}$, is the increasing sequence of all critical indices of the sequence $C(j)$, $j \in \mathbb{N}$, then by Theorem 3.2(3), $\|q_n\theta\| = C(q_n)$. Consider $n$ such that $q_n < \Theta \le q_{n+1}$. Since $C(j) \le (j+1)^{-1}$ for $j \in \mathbb{N}$, we have

$$\|q_n\theta\| = C(q_n) \le C(q_{n+1} - 1) \le q_{n+1}^{-1} \le \Theta^{-1}. \ ∎$$

Remark 3.3 (see [4]). Let integers $p_n, q_n, a_n$ be defined by

$$\text{CF}(1) \qquad \begin{cases} p_0 = 1, \ q_0 = 0, \\ p_1 = 0, \ q_1 = 1, \end{cases} \qquad \begin{cases} p_{n+1} = a_n p_n + p_{n-1}, \\ q_{n+1} = a_n q_n + q_{n-1} \quad \text{for } n \ge 1, \end{cases}$$

where

$$a_n = \left[\frac{|q_{n-1}\theta - p_{n-1}|}{|q_n\theta - p_n|}\right]$$

if $q_n\theta \ne p_n$, and the process stops with $p_n$, $q_n$ if $q_n\theta = p_n$. Then the $p_n/q_n$ are the best homogeneous approximations to $\theta$ for $n \ge 1$ if $0 < \theta \le 1/2$ and for $n \ge 2$ if $1/2 < \theta < 1$. Further,

$$\text{CF}(2) \qquad (-1)^n(q_n\theta - p_n) \ge 0,$$

$$\text{CF}(3) \qquad q_{n+1}p_n - q_n p_{n+1} = (-1)^n.$$

It is usual to speak of the $p_n/q_n$ as the $n$th *convergents* to $\theta$ and to call the $a_n$ the *partial quotients*. Since the $a_n$ are determined by $\theta$ and $\theta = \lim p_n/q_n$, we may write $\theta = [0; a_1, a_2, \ldots]$.

Remark 3.4. Let $0 < \theta < 1/2$. Homogeneous $\theta$ and $(1-\theta)$-billiard sequences have symmetrical interpretation in the billiard rectangle with sides of length $\frac{1}{2}\theta$ and $\frac{1}{2}(1-\theta)$. Hence by Theorem 3.2(3) the sequences $\|j\theta\|$ and $\|j(1-\theta)\|$, $j \in \mathbb{N}$, have the same set of critical indices. One can confirm this in terms of convergents to $\theta$ and to $1-\theta$. Let $p_n/q_n$ and $\overline{p}_n/\overline{q}_n$, $n \in \mathbb{N}$, be the $n$th convergents to $\theta$ and $1-\theta$ respectively. If $\theta = [0; a_1, a_2, \ldots]$, $a_1 > 1$, then by the Lagrange formula [9], $1 - \theta = [0; \overline{a}_1, \overline{a}_2, \ldots] = [0; 1, a_1 - 1, a_2, a_3, \ldots]$. Hence, by CF(1) and by induction we obtain

$$\begin{aligned} \overline{q}_2 &= \overline{a}_1 \overline{q}_1 + \overline{q}_0 = 1 = q_1, \\ \overline{q}_3 &= \overline{a}_2 \overline{q}_2 + \overline{q}_1 = (a_1 - 1) q_1 + 1 = a_1 = q_2, \\ \overline{q}_{n+1} &= \overline{a}_n \overline{q}_n + \overline{q}_{n-1} = a_{n-1} q_{n-1} + q_{n-2} = q_n \quad \text{for } n \geq 3. \end{aligned}$$

LEMMA 3.4. *If $q\theta - \alpha = p + 2d$, where $q \in \mathbb{N}$, $p \in \mathbb{N} \cup \{0\}$, then*

$$F(q) \equiv \begin{cases} d & \text{for } p \text{ and } q \text{ both even}, \\ \theta/2 - d & \text{for } p \text{ even and } q \text{ odd}, \\ 1/2 + d & \text{for } p \text{ odd and } q \text{ even}, \\ \frac{1}{2}(1+\theta) - d & \text{for } p \text{ and } q \text{ both odd}. \end{cases}$$

*Proof.* By Theorem 2.1, we obtain

$$\begin{aligned} F(q) &\equiv \tfrac{1}{2}(q\theta - \alpha) = p/2 + d \equiv d \quad \text{for } p \text{ and } q \text{ both even}, \\ F(q) &\equiv \tfrac{1}{2}(\theta - q\theta + \alpha) = \theta/2 - p/2 - d \equiv \theta/2 - d \quad \text{for } p \text{ even and } q \text{ odd}, \\ F(q) &\equiv \tfrac{1}{2}(q\theta - \alpha) = p/2 + d \equiv 1/2 + d \quad \text{for } p \text{ odd and } q \text{ even}, \\ F(q) &\equiv \tfrac{1}{2}(\theta - q\theta + \alpha) = \theta/2 - p/2 - d \\ &\equiv \tfrac{1}{2}(1+\theta) - d \quad \text{for } p \text{ and } q \text{ both odd}. \quad \blacksquare \end{aligned}$$

THEOREM 3.3. *Let $p/q$, $q > 1$ be a best $\alpha$-approximation to $\theta$.*

(1) *There exists exactly one vertex $v(q)$ with $\|q\theta - \alpha\| = 2\|F(q) - v(q)\|$, and*

$$v(q) = \begin{cases} v_0 & \text{for } p \text{ and } q \text{ both even}, \\ v_1 & \text{for } p \text{ even and } q \text{ odd}, \\ v_2 & \text{for } p \text{ odd and } q \text{ even}, \\ v_3 & \text{for } p \text{ and } q \text{ both odd}. \end{cases}$$

(2) *If $F(1) = 0$, then*

$$F(q) = \begin{cases} v_1 - d & \text{for } p \text{ even and } q \text{ odd}, \\ v_2 + d & \text{for } p \text{ odd and } q \text{ even}, \\ v_3 - d & \text{for } p \text{ and } q \text{ both odd}, \end{cases}$$

*where $2d = q\theta - p$.*

*Proof.* Let $q > 1$. Since $p/q$ is a best $\alpha$-approximation to $\theta$, $q$ is a critical index of the sequence $\|j\theta - \alpha\|$. By Theorem 3.2(1), $q$ is a critical index of the sequence $A(j)$, $j \in \mathbb{N}$, and $\|q\theta - \alpha\| = 2A(q)$. Hence there exists a vertex $v(q) \in \{0, \frac{1}{2}\theta, \frac{1}{2}, \frac{1}{2}(1+\theta)\}$ such that $\|q\theta - \alpha\| = 2\|F(q) - v(q)\|$. By Remark 3.1, $\|F(q)\| = \|F(q-1)\|$ and $\|F(q) - \frac{1}{2}\| = \|F(q-1) - \frac{1}{2}\|$ for $q$ odd, and $\|F(q)-\frac{\theta}{2}\| = \|F(q-1)-\frac{\theta}{2}\|$ and $\|F(q)-\frac{1}{2}(1+\theta)\| = \|F(q-1)-\frac{1}{2}(1+\theta)\|$ for $q$ even. Hence we have

$$\text{(i)} \qquad \begin{cases} v(q) \in \{\frac{\theta}{2}, \frac{1}{2}(1+\theta)\} & \text{for } q \text{ odd,} \\ v(q) \in \{0, \frac{1}{2}\} & \text{for } q \text{ even.} \end{cases}$$

Since $\|F(q) - \frac{\theta}{2}\| + \|F(q) - \frac{1}{2}(1+\theta)\| = \|F(q)\| + \|F(q) - \frac{1}{2}\| = \frac{1}{2}$ and $\|F(q) - v(q)\| = \frac{1}{2}\|q\theta - \alpha\| < \frac{1}{2}\|\theta - \alpha\| \le \frac{1}{4}$, $v(q)$ is uniquely determined by (i). Thus by Lemma 3.4 we obtain condition (1).

If $F(1) = 0$, then by CF(3) of Remark 3.3, $p, q$ are relatively prime. Since $|q\theta - p| = \|q\theta\| < \|\theta\|$, we obtain $\frac{1}{2}(1+\theta+|q\theta-p|) < 1$ and $\frac{1}{2}(\theta-|q\theta-p|) > 0$. Hence by Lemma 3.4 we obtain condition (2). ∎

Example 3.1. Let $F(1) = 0$ and $\theta = t/m$ be a fraction in lowest terms. Since $m$ is the last critical index of the sequence $\|jt/m\|$, $j \in \mathbb{N}$, Theorem 3.2(3) implies that $C(j) > 0$ and $B(j) > 0$ for $1 \le j < m$. Hence by Theorem 2.1 we obtain the following conditions (1) and (2). Theorem 3.3(2) yields (3).

(1) $\{F(1), F(2), \ldots, F(m)\} = \{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\}$,

(2) $\min\left(\|F(j) - \frac{t}{2m}\|, \|F(j) - \frac{1}{2}\|, \|F(j) - \frac{1}{2}(1 + \frac{t}{m})\|\right) \ge \frac{1}{2m}$, $1 \le j < m$,

(3) $F(m) = \begin{cases} \frac{t}{2m} & \text{for } t \text{ even,} \\ \frac{1}{2} & \text{for } m \text{ even,} \\ \frac{1}{2}(1 + \frac{t}{m}) & \text{for } t \text{ and } m \text{ both odd.} \end{cases}$

By condition CF(1) one may state the results of Theorem 3.3 in terms of partial quotients.

Corollary 3.2. *Let $F(1) = 0$ and $\theta = [0; a_1, a_2, \ldots] < 1/2$. If $q_n$ is the increasing sequence of all critical indices of the sequence $\|j\theta\|$, $j \in \mathbb{N}$, and $v(q_n)$, $n \in \mathbb{N}$, is the sequence of vertices such that $\|q_n\theta\| = 2\|F(q_n) - v(q_n)\|$, then*

(1) $v(q_1) = \theta/2$, $v(q_2) = \begin{cases} \frac{1}{2} & \textit{for } a_1 \textit{ even,} \\ \frac{1}{2}(1+\theta) & \textit{for } a_1 \textit{ odd,} \end{cases}$

(2) $v(q_{n+2}) = v(q_n)$ *iff* $a_{n+1}$ *is even,*

(3) $v(q_{n+1}) \ne v(q_n)$.

*Proof.* Let $p_n/q_n$ be the $n$th convergent to $\theta$. By CF(1), $p_1 = 0$, $p_2 = 1$ and $q_2 = a_1$. Hence by Theorem 3.3 we obtain (1).

If $a_{n+1}$ is even, then by CF(1), $p_{n+2} \equiv p_n$ and $q_{n+2} \equiv q_n$. Thus, by Theorem 3.3, $v(q_{n+2}) = v(q_n)$. If $a_{n+1}$ is odd, then CF(1) yields $p_{n+2} \equiv p_{n+1} + p_n$ and $q_{n+2} \equiv q_{n+1} + q_n$. Since $p_{n+1}$ or $q_{n+1}$ is odd, we have $p_{n+2} \not\equiv p_n$ or $q_{n+2} \not\equiv q_n$. Thus, $v(q_{n+2}) \ne v(q_n)$ by Theorem 3.3.

We prove (3) by induction. Notice that $v(q_2) \neq v(q_1)$. Assume that $v(q_{n+1}) \neq v(q_n)$. If $a_{n+1}$ is even, then $v(q_{n+2}) = v(q_n) \neq v(q_{n+1})$ by (2). If $a_{n+1}$ is odd, then by CF(1) we have $v(q_{n+2}) \neq v(q_{n+1})$ analogously as in (2). ■

Corollary 3.3. *Let $F(1) = 0$ and let $\theta < 1/2$ be irrational. If $q_n$ is an increasing sequence of critical indices of the sequence $\|j\theta\|$, $j \in \mathbb{N}$, and $v(q_n)$ is a sequence of vertices such that $\|q_n\theta\| = 2\|F(q_n) - v(q_n)\|$, then*

$$(*)\qquad \operatorname{sgn}(F(q_{n+1}) - v(q_{n+1})) = \operatorname{sgn}(F(q_n) - v(q_n)) \quad \textit{iff} \quad v(q_{n+1}) = 1/2 \textit{ or } v(q_n) = 1/2.$$

*Proof.* By Corollary 3.2(3), $v(q_{n+1}) \neq v(q_n)$. Hence by Theorem 3.3(2) and condition CF(2) we obtain $(*)$. ■

This corollary shows that a billiard ball, starting from vertex 0, changes the orientation of its trajectory at the points $F(q_n)$ by the rule $(*)$.

Example 3.2. Suppose $F(1) = 0$, $\theta = [0; a_1, a_2, \ldots] < 1/2$, $a_1$ is even and $a_n$ is odd for $n \geq 2$. If $q_n$, $n \in \mathbb{N}$, is the increasing sequence of all critical indices of the sequence $\|j\theta\|$, $j \in \mathbb{N}$, then by Corollary 3.2 the sequence $v(q_n)$, $n \in \mathbb{N}$, is 3-periodic:

$$v(q_1) = \theta/2, \qquad v(q_2) = 1/2, \qquad v(q_3) = \tfrac{1}{2}(1 + \theta),$$

and by Corollary 3.3 the sequence $\operatorname{sgn}(F(q_n) - v(q_n))$, $n \in \mathbb{N}$, is 6-periodic:

$$\operatorname{sgn}(F(q_1) - v(q_1)) = \operatorname{sgn}(F(q_2) - v(q_2)) = \operatorname{sgn}(F(q_3) - v(q_3)) = -1,$$
$$\operatorname{sgn}(F(q_4) - v(q_4)) = \operatorname{sgn}(F(q_5) - v(q_5)) = \operatorname{sgn}(F(q_6) - v(q_6)) = 1.$$

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