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On the p-adic Leopoldt transform of a power series

by

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Let p be an odd prime number. Let X be the projective limit for the norm maps of the p-Sylow subgroups of the ideal class groups of $\mathbb{Q}(\zeta_{p^{n+1}})$, $n \geq 0$. Let $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and let θ be an even and non-trivial character of Δ. Then X is a $\mathbb{Z}_p[[T]]$-module and the characteristic ideal of the isotypic component $X(\omega\theta^{-1})$ is generated by a power series $f(T,\theta) \in \mathbb{Z}_p[[T]]$ such that (see for example [2])

$$\forall n \geq 1,\ n \equiv 0 \pmod{p-1}, \qquad f((1+p)^{1-n}-1,\theta) = L(1-n,\theta),$$

where $L(s,\theta)$ is the usual Dirichlet L-series. Therefore, it is natural and interesting to study the properties of the power series $f(T,\theta)$.

We denote by $\overline{f(T,\theta)} \in \mathbb{F}_p[[T]]$ the reduction of $f(T,\theta)$ modulo p. Then B. Ferrero and L. Washington ([3]) have proved

$$\overline{f(T,\theta)} \neq 0.$$

Note that, in fact, we have ([1])

$$\overline{f(T,\theta)} \notin \mathbb{F}_p[[T^p]].$$

W. Sinnott has proved the following ([8]):

$$\overline{f(T,\theta)} \notin \mathbb{F}_p(T).$$

But note that $\mathbb{F}_p[[T]] = \mathbb{F}_p[[(1+T)^a - 1]]$ for all $a \in \mathbb{Z}_p^*$. Therefore it is natural to introduce the notion of a *pseudo-polynomial* which is an element $F(T)$ in $\mathbb{F}_p[[T]]$ such that there exist an integer $r \geq 1$ and elements $c_1, \ldots, c_r \in \mathbb{F}_p$ and $a_1, \ldots, a_r \in \mathbb{Z}_p$ such that $F(T) = \sum_{i=1}^{r} c_i(1+T)^{a_i}$. An element of $\mathbb{F}_p[[T]]$ will be called a *pseudo-rational function* if it is the quotient of two pseudo-polynomials.

In this paper, we prove that $\overline{f(T,\theta)}$ is not a pseudo-rational function (Theorem 4.5(1)). This suggests the following question: is $\overline{f(T,\theta)}$ algebraic over $\mathbb{F}_p(T)$? We suspect that this is not the case but we have no proof for

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it. Note that, by the result of Ferrero and Washington, we can write

$$\overline{f(T,\theta)} = T^{\lambda(\theta)}U(T),$$

where $\lambda(\theta) \in \mathbb{N}$ and $U(T) \in \mathbb{F}_p[[T]]^*$. S. Rosenberg ([6]) has proved that

$$\lambda(\theta) \leq (4p(p-1))^{\phi(p-1)},$$

where ϕ is Euler's totient function. In this paper, we improve Rosenberg's bound (Theorem 4.5(2)):

$$\lambda(\theta) < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$

This implies that the lambda invariant of the field $\mathbb{Q}(\zeta_p)$ is less than $2\big(\frac{p-1}{2}\big)^{\phi(p-1)+1}$ (see Corollary 4.6 for the precise statement for an abelian number field). Note that this bound is certainly far from being sharp, because according to a heuristic argument due to Ferrero and Washington (see [5]), and to Greenberg's conjecture,

$$\lambda(\mathbb{Q}(\zeta_p)) = \sum_{\theta \in \widehat{\Delta},\, \theta \neq 1 \text{ and even}} \lambda(\theta) \leq \frac{\log(p)}{\log\log(p)}.$$

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1. Notations. Let p be an odd prime number, and K a finite extension of $\mathbb{Q}_p$. Let O_K be the valuation ring of K, and π a prime of K. We set $\mathbb{F}_q = O_K/\pi O_K$; it is a finite field of q elements characteristic p. Let T be an indeterminate over K, and set $\Lambda = O_K[[T]]$. Observe that $\Lambda/\pi\Lambda \simeq \mathbb{F}_q[[T]]$. Let $F(T) \in \Lambda \setminus \{0\}$. Then we can write in a unique way ([9, Theorem 7.3])

$$F(T) = \pi^{\mu(F)}P(T)U(T),$$

where $U(T)$ is a unit of Λ, $\mu(F) \in \mathbb{N}$, and $P(T) \in O_K[T]$ is a monic polynomial such that $P(T) \equiv T^{\lambda(F)} \pmod{\pi}$ for some integer $\lambda(F) \in \mathbb{N}$. If $F(T) = 0$, we set $\mu(F) = \lambda(F) = \infty$. An element $F(T) \in \Lambda$ is called a *pseudo-polynomial* (see also [6, Definition 2]) if there exist some integer $r \geq 1$, $c_1, \ldots, c_r \in O_K$ and $a_1, \ldots, a_r \in \mathbb{Z}_p$ such that

$$F(T) = \sum_{i=1}^{r} c_i(1+T)^{a_i}.$$

We denote the ring of pseudo-polynomials in Λ by A. For $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $F(T) \in \Lambda$, set

$$\gamma_\delta(F(T)) = \frac{1}{p-1} \sum_{\eta \in \mu_{p-1}} \eta^\delta F((1+T)^\eta - 1).$$

Then $\gamma_\delta : \Lambda \to \Lambda$ is an O_K-linear map and:

- for $\delta, \delta' \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\gamma_\delta \gamma_{\delta'} = 0$ if $\delta \neq \delta'$ and $\gamma_\delta^2 = \gamma_\delta$,
- $\sum_{\delta \in \mathbb{Z}/(p-1)\mathbb{Z}} \gamma_\delta = \mathrm{Id}_\Lambda$.

For $F(T) \in \Lambda$, we set

$$D(F(T)) = (1+T)\,\frac{d}{dT} F(T),$$
$$U(F(T)) = F(T) - \frac{1}{p} \sum_{\zeta \in \mu_p} F(\zeta(1+T) - 1) \in \Lambda.$$

Then $D, U : \Lambda \to \Lambda$ are O_K-linear maps. Observe that:

- $U^2 = U$,
- $DU = UD$,
- $\gamma_\delta U = U \gamma_\delta$ for all $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$,
- $D\gamma_\delta = \gamma_{\delta+1} D$ for all $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$.

If $F(T) \in \Lambda$, we denote its reduction modulo π by $\overline{F(T)} \in \mathbb{F}_q[[T]]$. If $f : \Lambda \to \Lambda$ is an O_K-linear map, we denote its reduction modulo π by $\overline{f} : \mathbb{F}_q[[T]] \to \mathbb{F}_q[[T]]$. For all $n \geq 0$, we set $\omega_n(T) = (1+T)^{p^n} - 1$.

Let B be a commutative and unitary ring. We denote by B^* the set of invertible elements of B.

We fix a topological generator κ of $1 + p\mathbb{Z}_p$. Let $x \in \mathbb{Z}_p$ and let $n \geq 1$. We denote by $[x]_n$ the unique integer $k \in \{0, \ldots, p^n - 1\}$ such that $x \equiv k \pmod{p^n}$. Let $\omega : \mathbb{Z}_p^* \to \mu_{p-1}$ be the *Teichmüller character*, i.e. $\omega(a) \equiv a \pmod{p}$ for all $a \in \mathbb{Z}_p^*$. For $x, y \in \mathbb{Z}_p$, we write:

- $x \sim y$ if there exists $\eta \in \mu_{p-1}$ such that $y = \eta x$,
- $x \equiv y \pmod{\mathbb{Q}^*}$ if there exists $z \in \mathbb{Q}^*$ such that $y = zx$.

The function $\log_p$ will denote the usual p-adic logarithm, and v_p the usual p-adic valuation on $\mathbb{C}_p$ such that $v_p(p) = 1$.

Let ϱ be a Dirichlet character of conductor f_ϱ. Recall that the Bernoulli numbers $B_{n,\varrho}$ are defined by the identity

$$\sum_{a=1}^{f_\varrho} \frac{\varrho(a) e^{aZ}}{e^{fZ} - 1} = \sum_{n \geq 0} \frac{B_{n,\varrho}}{n!}\, Z^{n-1},$$

where $e^Z = \sum_{n\geq 0} Z^n/n!$. If $\varrho = 1$, then for $n \geq 2$, $B_{n,1}$ is the nth Bernoulli number.

Let $x \in \mathbb{R}$. We denote by $[x]$ the biggest integer less than or equal to x. Finally, log will denote the usual logarithm.

2. Preliminaries. Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$. In this section, we will recall the construction of the p-adic Leopoldt transform Γ_δ (see [5, Theorem 6.2]) which is an O_K-linear map from Λ to Λ.

First, observe that $(\pi^n, \omega_n(T)) = \pi^n\Lambda + \omega_n(T)\Lambda$, $n \geq 1$, is a basis of neighbourhoods of zero in Λ:

LEMMA 2.1.

(1) $(\pi, T)^{2n} \subset (\pi^n, T^n) \subset (\pi, T)^n$ *for all* $n \geq 1$.
(2) $\omega_n(T) \in (p^{[n/2]}, T^{p^{[n/2]+1}})$ *for all* $n \geq 1$.
(3) *For* $N \geq 1$, *set* $n = [\log(N)/\log(p)]$. *Then*

$$T^N \in (p^{[n/2]}, \omega_{[n/2]+1}(T)).$$

Proof. Note that assertion (1) is obvious. Assertion (2) comes from the fact that

$$\forall k \in \{1, \dots, p^n\}, \qquad v_p\left(\frac{p^n!}{k!(p^n-k)!}\right) = n - v_p(k).$$

To prove assertion (3), it is enough to prove that for all $n \geq 0$, there exist $\delta_0^{(n)}(T), \dots, \delta_n^{(n)}(T) \in \mathbb{Z}[T]$ such that

$$T^{p^n} = \sum_{i+j=n} \omega_i(T) p^j \delta_j^{(n)}(T).$$

This is clear for $n = 0$. Assume that it is true for some n and let $r(T) \in \mathbb{Z}[T]$ be such that

$$\frac{\omega_{n+1}(T)}{\omega_n(T)} + pr(T) = T^{p^n(p-1)}.$$

Then

$$T^{p^{n+1}} = T^{p^n} \frac{\omega_{n+1}(T)}{\omega_n(T)} + pr(T)T^{p^n}.$$

Note that there exists $q(T) \in \mathbb{Z}[T]$ such that

$$\frac{\omega_{n+1}(T)}{\omega_n(T)} = \omega_n(T)^{p-1} + pq(T).$$

Thus

$$T^{p^{n+1}} = \omega_{n+1}(T)\delta_0^{(n)}(T) + \sum_{i+j=n,\, j\geq 1} (\omega_n(T)^{p-1} + pq(T))\omega_i(T) p^j \delta_j^{(n)}(T)$$
$$+ \sum_{i+j=n} \omega_i(T) p^{j+1} \delta_j^{(n)}(T) r(T).$$

Thus, there exist $\delta_0^{(n+1)}(T),\dots,\delta_{n+1}^{(n+1)}(T)\in\mathbb{Z}[T]$ such that

$$T^{p^{n+1}}=\sum_{i+j=n+1}\omega_i(T)p^j\delta_j^{(n+1)}(T).\ \blacksquare$$

The following lemma will be used (for a similar result see [6, Lemma 5]):

LEMMA 2.2. *Let $F(T)\in A$. Write $F(T)=\sum_{i=1}^r\beta_i(1+T)^{\alpha_i}$ with $\beta_1,\dots,\beta_r\in O_K$, $\alpha_1,\dots,\alpha_r\in\mathbb{Z}_p$, and $\alpha_i\neq\alpha_j$ for $i\neq j$. Let $N=\max\{v_p(\alpha_i-\alpha_j): i\neq j\}$. Let $n\geq 1$ be an integer. Then*

$$F(T)\equiv 0\ (\mathrm{mod}\ (\pi^n,\omega_{N+1}(T)))\ \Leftrightarrow\ \forall i=1,\dots,r,\ \beta_i\equiv 0\ (\mathrm{mod}\ \pi^n).$$

Proof. We have

$$F(T)\equiv\sum_{i=1}^r\beta_i(1+T)^{[\alpha_i]_{N+1}}\ (\mathrm{mod}\ \omega_{N+1}(T)).$$

Therefore $F(T)\equiv 0\ (\mathrm{mod}\ (\pi^n,\omega_{N+1}(T)))$ if and only if

$$\sum_{i=1}^r\beta_i(1+T)^{[\alpha_i]_{N+1}}\equiv 0\ (\mathrm{mod}\ \pi^n).$$

But $[\alpha_i]_{N+1}\neq[\alpha_j]_{N+1}$ for $i\neq j$. Therefore $\sum_{i=1}^r\beta_i(1+T)^{[\alpha_i]_{N+1}}\equiv 0\ (\mathrm{mod}\ \pi^n)$ if and only if $\beta_i\equiv 0\ (\mathrm{mod}\ \pi^n)$ for all $i=1,\dots,r$. ■

Observe that U, D, γ_δ are continuous O_K-linear maps by Lemma 2.1 and the following lemma:

LEMMA 2.3. *Let $F(T)\in\Lambda$ and $n\geq 0$.*

(1) $F(T)\equiv 0\ (\mathrm{mod}\ \omega_n(T))\Rightarrow\gamma_\delta(F(T))\equiv 0\ (\mathrm{mod}\ \omega_n(T))$.
(2) $F(T)\equiv 0\ (\mathrm{mod}\ \omega_n(T))\Rightarrow D(F(T))\equiv 0\ (\mathrm{mod}\ (p^n,\omega_n(T)))$.
(3) $n\geq 1$, $F(T)\equiv 0\ (\mathrm{mod}\ \omega_n(T))\Rightarrow U(F(T))\equiv 0\ (\mathrm{mod}\ \omega_n(T))$.

Proof. Assertions (1) and (2) are obvious. It remains to prove (3). Observe that, by [9, Proposition 7.2], for all $G(T)\in\Lambda$ we have

$$G(T)\equiv 0\ (\mathrm{mod}\ \omega_n(T))\ \Leftrightarrow\ \forall\zeta\in\mu_{p^n},\ G(\zeta-1)=0.$$

Now, let $F(T)\in\Lambda$ with $F(T)\equiv 0\ (\mathrm{mod}\ \omega_n(T))$. For all $\zeta\in\mu_{p^n}$, we get

$$U(F)(\zeta-1)=0.$$

Therefore $U(F(T))\equiv 0\ (\mathrm{mod}\ \omega_n(T))$. ■

Let $s\in\mathbb{Z}_p$. For $n\geq 0$, set

$$k_n(s,\delta)=[s]_{n+1}+\delta_np^{n+1}\in\mathbb{N}\setminus\{0\},$$

where $\delta_n\in\{1,\dots,p-1\}$ is such that $[s]_{n+1}+\delta_n\equiv\delta\ (\mathrm{mod}\ p-1)$. Observe that:

- $k_n(s,\delta)\equiv\delta\ (\mathrm{mod}\ p-1)$ and $k_n(s,\delta)\equiv s\ (\mathrm{mod}\ p^{n+1})$ for all $n\geq 0$,
- $k_{n+1}(s,\delta)>k_n(s,\delta)$ for all $n\geq 0$,
- $s=\lim_n k_n(s,\delta)$.

In particular,

$$\forall a \in \mathbb{Z}_p,\ \forall n \geq 0, \qquad a^{k_{n+1}(s,\delta)} \equiv a^{k_n(s,\delta)} \pmod{p^{n+1}}.$$

Now, let $F(T)\in A$. Write $F(T)=\sum_{i=1}^{r}\beta_i(1+T)^{\alpha_i}$ with $\beta_1,\ldots,\beta_r \in O_K$, $\alpha_1,\ldots,\alpha_r\in\mathbb{Z}_p$. We set

$$\Gamma_\delta(F(T)) = \sum_{\alpha_i\in\mathbb{Z}_p^*} \beta_i\omega^\delta(\alpha_i)(1+T)^{\log_p(\alpha_i)/\log_p(\kappa)},$$

where ω^δ is the δ power of the Teichmüller character. Thus, we have a surjective O_K-linear map $\Gamma_\delta : A \to A$.

Lemma 2.4. *Let $F(T) \in A$.*

(1) *Let $s \in \mathbb{Z}_p$. Then*

$$\forall n \geq 0, \qquad \Gamma_\delta(F)(\kappa^s - 1) \equiv D^{k_n(s,\delta)}(F)(0) \bmod p^{n+2}.$$

(2) *If $n \geq 1$ and $F(T) \equiv 0 \pmod{\omega_n(T)}$, then*

$$\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}.$$

Proof. For $a \in \mathbb{Z}_p^*$, write $a = \omega(a)\langle a\rangle$, where $\langle a\rangle \in 1 + p\mathbb{Z}_p$. Let

$$F(T) = \sum_{i=1}^{r} \beta_i(1+T)^{\alpha_i}, \qquad \beta_1,\ldots,\beta_r \in O_K,\ \alpha_1,\ldots,\alpha_r \in \mathbb{Z}_p.$$

We have

$$D^{k_n(s,\delta)}(F(T)) = \sum_{i=1}^{r} \beta_i\alpha_i^{k_n(s,\delta)}(1+T)^{\alpha_i}.$$

Thus

$$D^{k_n(s,\delta)}(F(T)) \equiv \sum_{\alpha_i\in\mathbb{Z}_p^*} \beta_i\omega^\delta(\alpha_i)\langle\alpha_i\rangle^s(1+T)^{\alpha_i} \pmod{p^{n+2}}.$$

But recall that

$$\Gamma_\delta(F)(\kappa^s - 1) = \sum_{\alpha_i\in\mathbb{Z}_p^*} \beta_i\omega^\delta(\alpha_i)\langle\alpha_i\rangle^s,$$

and assertion (1) follows easily.

Now, suppose that $F(T) \equiv 0 \pmod{\omega_n(T)}$ for some $n \geq 1$. Then

$$\forall a \in \{0,\ldots,p^n - 1\}, \qquad \sum_{\alpha_i\equiv a \,(\mathrm{mod}\, p^n)} \beta_i = 0.$$

This implies that

$$\forall a \in \{0,\ldots,p^{n-1} - 1\}, \qquad \sum_{\alpha_i\in\mathbb{Z}_p^*,\ \log_p(\alpha_i)/\log_p(\kappa)\equiv a \,(\mathrm{mod}\, p^{n-1})} \omega^\delta(\alpha_i)\beta_i = 0.$$

But recall that

$$\Gamma_\delta(F(T)) = \sum_{\alpha_i \in \mathbb{Z}_p^*} \beta_i \omega^\delta(\alpha_i)(1+T)^{\log_p(\alpha_i)/\log_p(\kappa)}.$$

Thus $\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}$. ∎

PROPOSITION 2.5. *Let $F(T) \in \Lambda$. Then there exists a unique power series $\Gamma_\delta(F(T)) \in \Lambda$ such that*

$$\forall s \in \mathbb{Z}_p,\ \forall n \geq 0, \qquad \Gamma_\delta(F)(\kappa^s - 1) \equiv D^{k_n(s,\delta)}(F)(0) \pmod{p^{n+1}}.$$

Proof. Let $(F_N(T))_{N\geq 0}$ be a sequence of elements in A such that

$$\forall N \geq 0, \qquad F(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$

Fix $N \geq 1$. Then

$$\forall m \geq N, \qquad F_m(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$

Therefore, by Lemma 2.4,

$$\forall m \geq N, \qquad \Gamma_\delta(F_m(T)) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.$$

This implies that the sequence $(\Gamma_\delta(F_N(T)))_{N\geq 1}$ converges in Λ to some power series $G(T) \in \Lambda$. Observe that, for all $N \geq 1$,

$$G(T) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)}.$$

In particular,

$$G(\kappa^s - 1) \equiv \Gamma_\delta(F_N)(\kappa^s - 1) \pmod{p^N}.$$

Thus, applying Lemma 2.4, we get

$$G(\kappa^s - 1) \equiv D^{k_{N-1}(s,\delta)}(F_N)(0) \pmod{p^N}.$$

But

$$D^{k_{N-1}(s,\delta)}(F(T)) \equiv D^{k_{N-1}(s,\delta)}(F_N(T)) \pmod{(p^N, \omega_N(T))}.$$

Therefore

$$G(\kappa^s - 1) \equiv D^{k_{N-1}(s,\delta)}(F)(0) \pmod{p^N}.$$

Now, set $\Gamma_\delta(F(T)) = G(T)$ to complete the proof. ∎

3. Some properties of the p-adic Leopoldt transform. We need the following fundamental result:

PROPOSITION 3.1. *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$, $F(T) \in \Lambda$, and $m, n \in \mathbb{N} \setminus \{0\}$. Then*

$$\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))} \Leftrightarrow \gamma_{-\delta} U(F(T)) \equiv 0 \pmod{(\pi^n, \omega_m(T))}.$$

Proof. A similar result has been obtained by S. Rosenberg ([6, Lemma 8]). We begin by proving that Γ_δ is a continuous O_K-linear map. By Lemma 2.1, this comes from the following assertion:

Let $F(T) \in \Lambda$. Let $n \geq 1$ and assume that $F(T) \equiv 0 \pmod{\omega_n(T)}$. Then

$$\Gamma_\delta(F(T)) \equiv 0 \pmod{\omega_{n-1}(T)}.$$

Indeed, let $(F_N(T))_{N\geq 0}$ be a sequence of elements in A such that

$$\forall N \geq 0, \quad F(T) \equiv F_N(T) \pmod{\omega_N(T)}.$$

By the proof of Proposition 2.5,

$$\forall N \geq 1, \quad \Gamma_\delta(F(T)) \equiv \Gamma_\delta(F_N(T)) \pmod{\omega_{N-1}(T)},$$

and Lemma 2.4 yields the assertion.

Now, since Γ_δ, $\gamma_{-\delta}$, U are continuous O_K-linear maps, it suffices to prove the statement of the proposition for $F(T) \in A$. Write $F(T) = \sum_{i=1}^{r} \beta_i (1+T)^{\alpha_i}$ with $\beta_1, \ldots, \beta_r \in O_K$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_p$. Let $I \subset \{\alpha_1, \ldots, \alpha_r\}$ be a set of representatives of the classes of $\alpha_1, \ldots, \alpha_r$ for the relation $\sim$. For $x \in I$, $x \not\equiv 0 \pmod{p}$, set

$$\beta_x = \sum_{\alpha_i \sim x} \beta_i \frac{\alpha_i}{x}.$$

We get

$$(p-1)\gamma_{-\delta} U(F(T)) = \sum_{\eta \in \mu_{p-1}} \sum_{x \in I,\, x \in \mathbb{Z}_p^*} \eta^{-\delta} \beta_x (1+T)^{\eta x}.$$

Now observe that

$$\Gamma_\delta(F(T)) = \Gamma_\delta \gamma_{-\delta} U(F(T)) = \sum_{x \in I,\, x \in \mathbb{Z}_p^*} \beta_x \omega^\delta(x)(1+T)^{\log_p(x)/\log_p(\kappa)}.$$

Therefore $\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))}$ if and only if, for all $a \in \{0, \ldots, p^{m-1}-1\}$,

$$\sum_{x \in I,\, x \in \mathbb{Z}_p^*,\, \log_p(x)/\log_p(\kappa) \equiv a \pmod{p^{m-1}}} \beta_x \omega^\delta(x) \equiv 0 \pmod{\pi^n}.$$

Now, observe that for each $a \in \{0, \ldots, p^m - 1\}$, there exists at most one $\eta \in \mu_{p-1}$ such that $[\eta x]_m = a$, and if such an η exists it is equal to $\omega(a)\omega^{-1}(x)$. Therefore $\Gamma_\delta(F(T)) \equiv 0 \pmod{(\pi^n, \omega_{m-1}(T))}$ if and only if

$$\forall a \in \{0, \ldots, p^m - 1\}, \quad \sum_{x \in I,\, x \in \mathbb{Z}_p^*,\, \exists \eta_x \in \mu_{p-1},\, [\eta_x x]_m = a} \beta_x \eta_x^{-\delta} \equiv 0 \pmod{\pi^n}.$$

This last property is equivalent to $\gamma_{-\delta} U(F(T)) \equiv 0 \pmod{(\pi^n, \omega_m(T))}$. ∎

Now, we can list the basic properties of Γ_δ:

Proposition 3.2. *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $F(T) \in \Lambda$.*

(1) *$\Gamma_\delta : \Lambda \to \Lambda$ is a surjective and continuous O_K-linear map.*
(2) *$\Gamma_\delta(F(T)) = \Gamma_\delta \gamma_{-\delta} U(F(T))$.*

(3) *For all* $a \in \mathbb{Z}_p^*$,

$$\mathit{\Gamma}_\delta(F((1+T)^a-1)) = \omega^\delta(a)(1+T)^{\log_p(a)/\log_p(\kappa)}\mathit{\Gamma}_\delta(F(T)).$$

(4) *Let* κ' *be another topological generator of* $1+p\mathbb{Z}_p$ *and let* $\mathit{\Gamma}'_\delta$ *be the* p*-adic Leopoldt transform associated to* κ' *and* δ. *Then*

$$\mathit{\Gamma}'_\delta(F(T)) = \mathit{\Gamma}_\delta(F)((1+T)^{\log_p(\kappa)/\log_p(\kappa')}-1).$$

(5) $\mu(\mathit{\Gamma}_\delta(F(T))) = \mu(\gamma_{-\delta}U(F(T)))$ *and*

$$\forall N \geq 1, \quad \lambda(\mathit{\Gamma}_\delta(F(T))) \geq p^{N-1} \ \Leftrightarrow\ \lambda(\gamma_{-\delta}U(F(T))) \geq p^N.$$

Proof. Assertions (1)–(4) come from the fact that $\mathit{\Gamma}_\delta$, $\gamma_{-\delta}$, U are continuous and that these assertions are true for pseudo-polynomials. Assertion (5) is a direct application of Proposition 3.1. ∎

Let us recall the following remarkable result due to W. Sinnott:

PROPOSITION 3.3 ([8, Proposition 1]). *Let* $r_1(T),\dots,r_s(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. *Let* $c_1,\dots,c_s \in \mathbb{Z}_p \setminus \{0\}$ *and suppose that*

$$\sum_{i=1}^{s} r_i((1+T)^{c_i}-1) = 0.$$

Then

$$\forall a \in \mathbb{Z}_p, \qquad \sum_{c_i \equiv a \,(\mathrm{mod}\, \mathbb{Q}^*)} r_i((1+T)^{c_i}-1) \in \mathbb{F}_q.$$

Let us give a first application of this result:

PROPOSITION 3.4. *Let* $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ *and* $F(T) \in K(T) \cap \mathit{\Lambda}$.

(1) *If* δ *is odd or* $\delta = 0$, *then*

$$\mu(\mathit{\Gamma}_\delta(F(T))) = \mu(U(F(T)) + (-1)^\delta U(F((1+T)^{-1}-1))).$$

(2) *If* δ *is even and* $\delta \neq 0$, *then*

$$\mu(\mathit{\Gamma}_\delta(F(T))) = \mu(U(F(T)) + U(F((1+T)^{-1}-1)) - 2U(F)(0)).$$

Proof. The case $\delta = 0$ has already been obtained by Sinnott ([7, Theorem 1]). We prove (1); the proof of (2) is quite similar. Observe that (1) is a consequence of Proposition 3.2 and the equality

$$\mu(\gamma_{-\delta}(F(T))) = \mu(F(T) + (-1)^\delta F((1+T)^{-1}-1)).$$

To prove this equality, observe that for any $r(T) \in \mathit{\Lambda}$,

$$\gamma_{-\delta}(r(T)) = (-1)^\delta \gamma_{-\delta}(r((1+T)^{-1}-1)).$$

Thus

$$2\gamma_{-\delta}(F(T)) = \gamma_{-\delta}(F(T) + (-1)^\delta F((1+T)^{-1}-1)).$$

We can assume that $F(T) + (-1)^\delta F((1+T)^{-1}-1) \neq 0$. Write

$$F(T) + (-1)^\delta F((1+T)^{-1}-1) = \pi^m G(T),$$

where $m \in \mathbb{N}$ and $G(T) \in \Lambda \setminus \pi\Lambda$. Note that $G(T) \in K(T)$. We have to prove that $\gamma_{-\delta}(G(T)) \not\equiv 0 \pmod{\pi}$. Suppose that $\gamma_{-\delta}(G(T)) \equiv 0 \pmod{\pi}$. By Proposition 3.3, there exists $c \in O_K$ such that

$$G(T) + (-1)^{\delta} G((1+T)^{-1} - 1) \equiv c \pmod{\pi}.$$

But we must have $c \equiv 0 \pmod{\pi}$. Since

$$G(T) = (-1)^{\delta} G((1+T)^{-1} - 1),$$

we get $G(T) \equiv 0 \pmod{\pi}$, which is a contradiction. ∎

LEMMA 3.5. *Let $F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Then $F(T)$ is a pseudo-polynomial if and only if $(1+T)^n F(T) \in \mathbb{F}_q[T]$ for some integer $n \geq 0$.*

Proof. Assume that $F(T)$ is a pseudo-polynomial. We can suppose that $F(T) \neq 0$. Write

$$F(T) = \sum_{i=1}^{r} c_i (1+T)^{a_i},$$

where $c_1, \ldots, c_r \in \mathbb{F}_q^*$, $a_1, \ldots, a_r \in \mathbb{Z}_p$ and $a_i \neq a_j$ for $i \neq j$. Since $F(T) \in \mathbb{F}_q(T)$ there exist $m, n \in \mathbb{N} \setminus \{0\}$ with $m > \max\{v_p(a_i - a_j) : i \neq j\}$ such that

$$(T^{q^n} - T)^{q^m} F(T) \in \mathbb{F}_q[T].$$

Thus

$$\sum_{i=1}^{r} c_i (1+T)^{a_i + q^{n+m}} - \sum_{i=1}^{r} c_i (1+T)^{a_i + q^m} \in \mathbb{F}_q[T].$$

Observe that

- $a_i + q^{n+m} \neq a_j + q^m$ for all $i, j \in \{1, \ldots, r\}$,
- $a_i + q^m = a_j + q^m \Leftrightarrow i = j$.

Write

$$\sum_{i=1}^{r} c_i (1+T)^{a_i + q^{n+m}} - \sum_{i=1}^{r} c_i (1+T)^{a_i + q^m} = \sum_{j=1}^{t} b_j (1+T)^{d_j},$$

where $b_1, \ldots, b_t \in \mathbb{F}_q^*$, $d_1, \ldots, d_t \in \mathbb{N}$, $d_i \neq d_j$ for each $i \neq j$. Then, by Lemma 2.2, for each $i \in \{1, \ldots, r\}$, there exists $j_i \in \{1, \ldots, t\}$ such that $c_i = d_{j_i}$ and $a_i + q^n = d_{j_i}$. In particular, $a_i + q^m \in \mathbb{N}$ for all i. Therefore $(1+T)^{q^m} F(T) \in \mathbb{F}_q[T]$. ∎

Let us give a second application of Proposition 3.3:

PROPOSITION 3.6. *Let $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. Suppose that there exist $r \in \{0, \ldots, (p-3)/2\}$, $c_1, \ldots, c_r \in \mathbb{Z}_p \setminus \{0\}$, $G_1(T), \ldots,$*

$G_r(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$ *and a pseudo-polynomial* $R(T) \in \mathbb{F}_q[[T]]$ *such that*

$$\overline{\gamma}_\delta(F(T)) = R(T) + \sum_{i=1}^{r} G_i((1+T)^{c_i} - 1).$$

Then there exists an integer $n \geq 0$ *such that*

$$(1+T)^n (F(T) + (-1)^\delta F((1+T)^{-1} - 1)) \in \mathbb{F}_q[T].$$

Proof. Note that if $\eta, \eta' \in \mu_{p-1}$, then $\eta \equiv \eta' \pmod{\mathbb{Q}^*} \Leftrightarrow \eta = \eta'$ or $\eta = -\eta'$. Since $r < (p-1)/2$, by Proposition 3.3, there exists $\eta \in \mu_{p-1}$ such that

$$\overline{\eta}^\delta F((1+T)^\eta - 1) + \overline{-\eta}^\delta F((1+T)^{-\eta} - 1) \text{ is a pseudo-polynomial.}$$

Therefore

$$F(T) + (-1)^\delta F((1+T)^{-1} - 1) \text{ is a pseudo-polynomial.}$$

It remains to apply Lemma 3.5. ■

Let $F(T) \in \Lambda$. We say that $F(T)$ is a *pseudo-rational function* if it is the quotient of two pseudo-polynomials. For example, for all $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_p^*$, $((1+T)^a - 1)/((1+T)^b - 1)$ is a pseudo-rational function. We finish this section by giving a generalization of [8, Theorem 1]:

Theorem 3.7. *Let* $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$ *and* $F(T) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]]$. *Then* $\overline{\Gamma}_\delta(F(T))$ *is a pseudo-rational function if and only if there exists some integer* $n \geq 0$ *such that*

$$(1+T)^n (\overline{U}(F(T)) + (-1)^\delta \overline{U}(F((1+T)^{-1} - 1))) \in \mathbb{F}_q[T].$$

Proof. Let $R(T) \in \mathbb{F}_q[[T]]$. Let $a_1, \ldots, a_r \in \mathbb{Z}_p$ and $c_1, \ldots, c_r \in \mathbb{F}_q$. Observe that, by Proposition 3.2,

$$\Big(\sum_{i=1}^{r} c_i (1+T)^{a_i}\Big) \overline{\Gamma}_\delta(R(T)) = \overline{\Gamma}_\delta\Big(\sum_{i=1}^{r} c_i R((1+T)^{\kappa^{a_i}} - 1)\Big).$$

Assume that $\overline{\Gamma}_\delta(F(T))$ is a pseudo-rational function. Then, by the above remark, there exist $c_1, \ldots, c_r \in \mathbb{F}_q^*$ and $a_1, \ldots, a_r \in \mathbb{Z}_p$, $a_i \neq a_j$ for $i \neq j$, such that

$$\overline{\Gamma}_\delta \overline{\gamma}_{-\delta} \overline{U}\Big(\sum_{i=1}^{r} c_i F((1+T)^{\kappa^{a_i}} - 1)\Big) \text{ is a pseudo-polynomial.}$$

This implies, again by Proposition 3.1, that

$$\overline{\gamma}_{-\delta} \overline{U}\Big(\sum_{i=1}^{r} c_i F((1+T)^{\kappa^{a_i}} - 1)\Big) \text{ is a pseudo-polynomial.}$$

Set

$$G(T) = \overline{U}(F(T)) + (-1)^\delta \overline{U}(F((1+T)^{-1} - 1)) \in \mathbb{F}_q(T) \cap \mathbb{F}_q[[T]].$$

Now, by Proposition 3.3, there exist $d_1, \ldots, d_l \in \mathbb{F}_q^*$, $b_1, \ldots, b_l \in \mathbb{Z}_p$, $b_i \neq b_j$ for $i \neq j$, $\eta_1, \ldots, \eta_l \in \mu_{p-1}$, with $\eta_i \kappa^{b_i} \equiv \eta_j \kappa^{b_j} \pmod{\mathbb{Q}^*}$ for all i, j, and $\eta_i \kappa^{b_i} \neq \eta_j \kappa^{b_j}$ for $i \neq j$, such that

$$\sum_{i=1}^{l} d_i G((1+T)^{\eta_i \kappa^{b_i}} - 1) \text{ is a pseudo-polynomial.}$$

For $i = 1, \ldots, l$, write

$$\eta_i \kappa^{b_i} = \eta_1 \kappa^{b_1} x_i,$$

where $x_i \in \mathbb{Q}^* \cap \mathbb{Z}_p^*$, and $x_i \neq x_j$ for $i \neq j$. Since $G(T) = (-1)^\delta G((1+T)^{-1} - 1)$, we can assume that $x_1, \ldots, x_l$ are positive. Now, we see that

$$\sum_{i=1}^{l} d_i G((1+T)^{x_i} - 1) \text{ is a pseudo-polynomial.}$$

Therefore, there exist $N_1, \ldots, N_l \in \mathbb{N} \setminus \{0\}$, $N_i \neq N_j$ for $i \neq j$, such that

$$\sum_{i=1}^{l} d_i G((1+T)^{N_i} - 1) \text{ is a pseudo-polynomial.}$$

Now, by Lemma 3.5, there exists some integer $N \geq 0$ such that

$$(1+T)^N \Big(\sum_{i=1}^{l} d_i G((1+T)^{N_i} - 1) \Big) \in \mathbb{F}_q[T].$$

Write

$$G(T) = \frac{P(T)}{(1+T)^n Q(T)},$$

where $n \geq 0$, $P(T), Q(T) \in \mathbb{F}_q[T]$, $Q(-1) \neq 0$, and $(P(T), (1+T)^n Q(T)) = 1$. Let $A(T), B(T) \in \mathbb{F}_q[T]$ be such that

$$G(T) = \frac{A(T)}{(1+T)^n} + \frac{B(T)}{Q(T)}.$$

Then there exists $M \geq 0$ such that

$$(1+T)^M \left(\sum_{i=1}^{l} d_i \, \frac{B((1+T)^{N_i} - 1)}{Q((1+T)^{N_i} - 1)} \right) \in \mathbb{F}_q[T].$$

But $(1+T, Q((1+T)^{N_i} - 1)) = 1$ for $i \in \{1, \ldots, l\}$. Therefore

$$\sum_{i=1}^{l} d_i \, \frac{B((1+T)^{N_i} - 1)}{Q((1+T)^{N_i} - 1)} \in \mathbb{F}_q[T].$$

Now assume that $\deg_T Q(T) \geq 1$. Write

$$B(T) = q(T) Q(T) + r(T),$$

where $q(T), r(T) \in \mathbb{F}_q[T]$ and $\deg_T r(T) < \deg_T Q(T)$. Observe that $r(T) \neq 0$. Hence

$$\sum_{i=1}^{l} d_i \frac{r((1+T)^{N_i}-1)}{Q((1+T)^{N_i}-1)} \in \mathbb{F}_q[T].$$

Recall that $-\deg_T$ is a discrete valuation on $\mathbb{F}_q(T)$. Since $N_i \neq N_j$ for $i \neq j$ and $d_1, \ldots, d_l \in \mathbb{F}_q^*$, we get

$$\deg_T \left(\sum_{i=1}^{l} d_i \frac{r((1+T)^{N_i}-1)}{Q((1+T)^{N_i}-1)} \right) < 0,$$

which is a contradiction. Thus $(1+T)^n G(T) \in \mathbb{F}_q[T]$. ∎

4. Application to Kubota–Leopoldt p-adic L-functions. Let θ be a Dirichlet character of the first kind, $\theta \neq 1$ and θ even. We denote by $f(T, \theta)$ the Iwasawa power series attached to the p-adic L-function $L_p(s, \theta)$ (see [9, Theorem 7.10]). Write

$$\theta = \chi \omega^{\delta+1},$$

where χ is of conductor $d \geq 1$ with $d \not\equiv 0 \pmod{p}$, and $\delta \in \mathbb{Z}/(p-1)\mathbb{Z}$. Set $\kappa = 1 + pd$ and $K = \mathbb{Q}_p(\chi)$. We define

$$F_\chi(T) = \frac{\sum_{a=1}^{d} \chi(a)(1+T)^a}{1-(1+T)^d}.$$

Let us give the basic properties of $F_\chi(T)$:

Lemma 4.1.

(1) *If $d \geq 2$, then $F_\chi(T) \in \Lambda$.*
(2) *If $d = 1$, then $\gamma_\alpha(F_\chi(T)) \in \Lambda$ for all $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$, $\alpha \neq 1$.*
(3) $U(F_\chi(T)) = F_\chi(T) - \chi(p) F_\chi((1+T)^p - 1)$.
(4) *If $d \geq 2$, then $F_\chi((1+T)^{-1} - 1) = \varepsilon F_\chi(T)$, where $\varepsilon = 1$ if χ is odd and $\varepsilon = -1$ if χ is even.*
(5) *If $d = 1$, then $F_\chi((1+T)^{-1} - 1) = -1 - F_\chi(T)$.*

Proof. (1), (4) and (5) are obvious.

(2) For $d = 1$, we have

$$F_\chi(T) = -1 + \frac{\sum_{a=0}^{p-1} (1+T)^a}{1-(1+T)^p}.$$

Set

$$G(T) = (1-(1+T)^p)\gamma_\alpha(F_\chi(T)).$$

Note that

$$\forall \eta \in \mu_{p-1}, \qquad \frac{1-(1+T)^p}{1-(1+T)^{\eta p}} \equiv \eta^{-1} \pmod{\omega_1(T)}.$$

Therefore

$$(p-1)G(T) \equiv \sum_{\eta \in \mu_{p-1}} \eta^{\alpha-1} \sum_{a=0}^{p-1} (1+T)^{\eta a} \pmod{\omega_1(T)}.$$

Thus

$$(p-1)G(T) \equiv \sum_{\eta \in \mu_{p-1}} \eta^{\alpha-1} \sum_{b=0}^{p-1} (1+T)^{b} \pmod{\omega_1(T)}.$$

Since $\alpha \neq 1$, we get $G(T) \equiv 0 \pmod{\omega_1(T)}$. Therefore $\gamma_\alpha(F_\chi(T)) \in \Lambda$.

(3) For $d = 1$, we have

$$U(F_\chi(T)) = \frac{\sum_{a=1}^{p-1} (1+T)^a}{1-(1+T)^p} = F_\chi(T) - F_\chi((1+T)^p - 1).$$

Now, let $d \geq 2$. Set $q_0 = pd$ and $\kappa = 1 + pd$. Note that

$$F_\chi(T) = \frac{\sum_{a=1}^{q_0} \chi(a)(1+T)^a}{1-(1+T)^{q_0}}.$$

Therefore

$$U(F_\chi(T)) = \frac{\sum_{a=1,\, a \not\equiv 0 \, (\mathrm{mod}\, p)}^{q_0} \chi(a)(1+T)^a}{1-(1+T)^{q_0}}.$$

But

$$F_\chi(T) - \chi(p) F_\chi((1+T)^p - 1) = \frac{\sum_{a=1}^{q_0} \chi(a)(1+T)^a}{1-(1+T)^{q_0}} - \chi(p) \frac{\sum_{a=1}^{d} \chi(a)(1+T)^{pa}}{1-(1+T)^{q_0}},$$

and the lemma follows easily. ∎

Lemma 4.2. *Assume that $d \geq 2$. The denominator of $F_\chi(T)$ is $\phi_d(1+T)$ where $\phi_d(X)$ is the dth cyclotomic polynomial, and the same is true for $\overline{F_\chi(T)}$.*

Proof. Let $\zeta \in \mu_d$. If ζ is not a primitive dth root of unity, then, by [9, Lemma 4.7], we have

$$\sum_{a=1}^{d} \chi(a) \zeta^a = 0.$$

Recall that

$$F_\chi(T) = \frac{\sum_{a=1}^{d} \chi(a)(1+T)^a}{1-(1+T)^d} \quad \text{and} \quad (1+T)^d - 1 = \prod_{n \mid d} \phi_n(1+T).$$

Therefore the denominator of $F_\chi(T)$ is $\phi_d(1+T)$.

If ζ is a primitive dth root of unity, then, by [9, Lemma 4.8], we have

$$\sum_{a=1}^{d} \chi(a)\zeta^a \not\equiv 0 \pmod{\widetilde{\pi}}$$

for any prime $\widetilde{\pi}$ of $K(\mu_d)$. Hence the denominator of $\overline{F_\chi(T)}$ is $\overline{\phi_d(1+T)}$. ■

LEMMA 4.3. *The derivative of $\gamma_{-\delta}(F_\chi(T))$ is not a pseudo-polynomial modulo π.*

Proof. We first handle the case $d \geq 2$. By (3) and (4) of Lemma 4.1, and Lemma 4.2, we get

$$\forall n \geq 0, \qquad (1+T)^n(\overline{U}\,\overline{F_\chi(T)} + (-1)^\delta\,\overline{U}\,\overline{F_\chi((1+T)^{-1}-1)}) \notin \mathbb{F}_q[T].$$

Thus by Proposition 3.6, $\overline{\gamma}_{-\delta}\overline{U}(\overline{F_\chi(T)})$ is not a pseudo-polynomial. But observe that $\overline{U} = \overline{D^{p-1}}$. Thus $\overline{D}\,\overline{\gamma}_{-\delta}(\overline{F_\chi(T)})$ is not a pseudo-polynomial.

For $d = 1$, set

$$\widetilde{F_\chi(T)} = F_\chi(T) - 2F_\chi((1+T)^2-1) = 1 - \frac{1}{2+T}.$$

Observe that:

- $F_\chi((1\widetilde{+T)^{-1}}-1) = 1 - \widetilde{F_\chi(T)}$,
- $U(\widetilde{F_\chi(T)}) = \widetilde{F_\chi(T)} - F_\chi((1\widetilde{+T)^p}-1)$.

Therefore, as in the case $d \geq 2$, $\overline{\gamma}_{-\delta}\overline{U}(\overline{\widetilde{F_\chi(T)}})$ is not a pseudo-polynomial. Thus neither is $\overline{\gamma}_{-\delta}\overline{U}(\overline{F_\chi(T)})$, and one can conclude as in the case $d \geq 2$. ■

LEMMA 4.4.

$$\Gamma_\delta U \gamma_{-\delta}(F_\chi(T)) = f\left(\frac{1}{1+T} - 1, \theta\right).$$

Proof. We consider the case $d = 1$; the case $d \geq 2$ is quite similar. Set $T = e^Z - 1$. We get

$$\gamma_{-\delta}(F_\chi(T)) = \sum_{n\geq 0,\, n\equiv 1+\delta \,(\mathrm{mod}\, p-1)} \frac{B_n}{n!} Z^{n-1}.$$

Thus, by [9, Theorem 5.11], we get

$$\forall k \in \mathbb{N}, k \equiv \delta \pmod{p-1}, \qquad D^k\gamma_{-\delta}U(F_\chi)(0) = L_p(-k,\theta).$$

But, by Proposition 2.5, for $s \in \mathbb{Z}_p$ we have

$$\Gamma_\delta\gamma_{-\delta}U(F_\chi)(\kappa^s-1) = \lim_n D^{k_n(s,\delta)}\gamma_{-\delta}U(F_\chi)(0) = L_p(-s,\theta) = f(\kappa^{-s}-1,\theta),$$

and the lemma follows. ■

We can now prove our main result:

THEOREM 4.5.

(1) *$\overline{f(T,\theta)}$ is not a pseudo-rational function.*

(2) *$\lambda(f(T,\theta)) < \left(\frac{p-1}{2}\phi(d)\right)^{\phi(p-1)}$, where ϕ is Euler's totient function.*

Proof. (1) Suppose that $\overline{f(T,\theta)}$ is a pseudo-rational function. Then so are $\overline{f\left(\frac{1}{1+T}-1,\theta\right)}$ and $\overline{\Gamma}_\delta\overline{\gamma}_{-\delta}\overline{U}(\overline{F_\chi(T)})$. Recall that if $d \geq 2$ then, by Proposition 3.2, $\overline{\Gamma}_\delta\overline{F_\chi(T)} = \overline{\Gamma}_\delta\overline{\gamma}_{-\delta}\overline{U}(\overline{F_\chi(T)})$.

We first deal with the case $d \geq 2$. By Theorem 3.7, there exists an integer $n \geq 0$ such that

$$(1+T)^n(\overline{U}(\overline{F_\chi(T)}) + (-1)^\delta\overline{U}(\overline{F_\chi((1+T)^{-1}-1)}) \in \mathbb{F}_q[T].$$

This is a contradiction by Lemmas 4.1(3)–(4) and 4.2.

For $d = 1$, we work with

$$\widetilde{F_\chi(T)} = F_\chi(T) - 2F_\chi((1+T)^2-1) = 1 - \frac{1}{2+T}.$$

Then, by Proposition 3.2, $\overline{\Gamma}_\delta\overline{\gamma}_{-\delta}\overline{U}(\overline{\widetilde{F_\chi(T)}})$ is a pseudo-rational function. We get a contradiction as in the case $d \geq 2$.

(2) Our proof is inspired by a method introduced by S. Rosenberg ([6]). We first deal with the case $d = 1$. Note that we can assume that $\lambda(f(T,\theta)) \geq 1$. Now, by Lemma 4.3,

$$\mu(\gamma_{-\delta}(F_\chi(T))) = 0.$$

Furthermore,

$$\gamma_{-\delta}(F_\chi)(0) \equiv 0 \pmod{\pi}.$$

Therefore, by Lemma 4.1(3),

$$\lambda(\gamma_{-\delta}U(F_\chi(T))) = \lambda(\gamma_{-\delta}(F_\chi(T))).$$

Hence we have to evaluate $\lambda(\gamma_{-\delta}(F_\chi(T)))$. Set $F(T) = -1/T$. Since δ is odd, we have

$$\gamma_{-\delta}(F_\chi(T)) = \gamma_{-\delta}(F(T)).$$

Observe that $F((1+T)^{-1}-1) = 1 - F(T)$. Let $S \subset \mu_{p-1}$ be a set of representatives of $\mu_{p-1}/\{1,-1\}$. We have

$$(p-1)\gamma_{-\delta}(F(T)) = 2\sum_{\eta\in S}\eta^{-\delta}F((1+T)^\eta-1) - \sum_{\eta\in S}\eta^{-\delta}.$$

Set

$$G(T) = \Big(\prod_{\eta\in S}((1+T)^\eta-1)\Big)\gamma_{-\delta}(F(T)).$$

Then:

- $\mu(G(T)) = 0$,
- $\lambda(G(T)) = (p-1)/2 + \lambda(\gamma_{-\delta}(F(T)))$.

For $S' \subset S$, set $t(S') = \sum_{x\in S'} x$. We can write

$$G(T) = \sum_{S'\subset S} a_{S'}(1+T)^{t(S')},$$

where $a_{S'} \in O_K$. Define

$$N = \max\{v_p(t(S') - t(S'')) : S', S'' \subset S,\, t(S') \neq t(S'')\}.$$

Observe that if $t(S') \neq t(S'')$, then

$$p^{v_p(t(S')-t(S''))} \leq |N_{\mathbb{Q}(\mu_{p-1})/\mathbb{Q}}(t(S') - t(S''))|.$$

Thus

$$p^N < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$

But, by Lemma 2.2, $\lambda(G(T)) < p^{N+1}$. Thus, by Proposition 3.2, we get

$$\lambda(f(T,\theta)) = \lambda\left(f\left(\frac{1}{1+T} - 1, \theta\right)\right) < p^N < \left(\frac{p-1}{2}\right)^{\phi(p-1)}.$$

Now, we consider the general case, i.e. $d \geq 2$. Again we can assume that $\lambda(f(T,\theta)) \geq 1$. Thus as in the case $d = 1$, we get

$$\lambda(\gamma_{-\delta} U(F_\chi(T))) = \lambda(\gamma_{-\delta}(F_\chi(T))).$$

Now, since $d \geq 2$, we have $\deg_T F_\chi(T) < 0$, and, by Lemma 4.2, we can write

$$F_\chi(T) = \frac{\sum_{a=0}^{\phi(d)-1} r_a(1+T)^a}{\phi_d(1+T)},$$

where $r_a \in O_K$ for $a \in \{0, \ldots, \phi(d) - 1\}$. Let again $S \subset \mu_{p-1}$ be a set of representatives of $\mu_{p-1}/\{1, -1\}$. By Lemma 4.1,

$$(p-1)\gamma_{-\delta}(F_\chi(T)) = 2\sum_{\eta\in S} \eta^{-\delta} F_\chi((1+T)^\eta - 1).$$

Set

$$G(T) = \Big(\prod_{\eta\in S} \phi_d((1+T)^\eta)\Big)\gamma_{-\delta}(F_\chi(T)).$$

We have

$$G(T) = \sum_{a=0}^{\phi(d)-1} \sum_{\eta\in S} \sum_{S'\subset S\setminus\{\eta\}} \sum_{\underline{d}=(d_{\eta'})_{\eta'\in S'},\, d_{\eta'}\in\{0,\ldots,\phi(d)\}} b_{S',\underline{d}}(1+T)^{a\eta+\sum_{\eta'\in S'} d_{\eta'}\eta'},$$

where $b_{S',\underline{d}} \in O_K$. Note that again $\mu(G(T)) = 0$, and $\lambda(G(T)) = \lambda(\gamma_{-\delta}(F_\chi(T)))$. Now, for $a, b \in \{0, \ldots, \phi(d) - 1\}$, $\eta_1, \eta_2 \in S$, $S_1 \subset S \setminus \{\eta_1\}$, and $S_2 \subset S \setminus \{\eta_2\}$, set

$$V = a\eta_1 + \sum_{\eta\in S_1} d_\eta \eta - b\eta_2 - \sum_{\eta\in S_2} d'_\eta \eta,$$

where $d_\eta \in \{0, \dots, \phi(d)\}$ for all $\eta \in S_1$, and $d'_\eta \in \{0, \dots, \phi(d)\}$ for all $\eta \in S_2$.

If $\eta_1 = \eta_2$ then we can write

$$V = (a-b)\eta_1 + \sum_{\eta \in S'} u_\eta \eta,$$

where $|u_\eta| \in \{0, \dots, \phi(d)\}$ and $|S'| \le (p-3)/2$.

If $\eta_1 \ne \eta_2$, we can write

$$V = a'\eta_1 + b'\eta_2 + \sum_{\eta \in S'} u_\eta \eta,$$

where $|a'|, |b'|, |u_\eta| \in \{0, \dots, \phi(d)\}$, and $|S'| \le (p-5)/2$. Therefore, if $V \ne 0$, we get

$$p^{v_p(V)} < \left(\frac{p-1}{2}\,\phi(d)\right)^{\phi(p-1)}.$$

Now, we can conclude as in the case $d = 1$. ∎

Let E be a number field and let E_∞/E be the cyclotomic $\mathbb{Z}_p$-extension. For $n \ge 0$, let A_n be the p-Sylow subgroup of the ideal class group of the nth layer in E_∞/E. Then, by [9, Theorem 13.13], there exist $\mu_p(E) \in \mathbb{N}$, $\lambda_p(E) \in \mathbb{N}$ and $\nu_p(E) \in \mathbb{Z}$ such that for all sufficiently large n,

$$|A_n| = p^{\mu_p(E)p^n + \lambda_p(E)n + \nu_p(E)}.$$

Recall that it is conjectured that $\mu_p(E) = 0$; this has been proved by B. Ferrero and L. Washington ([3]) if E is an abelian number field.

COROLLARY 4.6. *Let F be an abelian number field of conductor N. Write $N = p^m d$, where $m \in \mathbb{N}$ and $d \ge 1$, $d \not\equiv 0 \pmod p$. Then*

$$\lambda_p(F) < 2\left(\frac{p-1}{2}\,\phi(d)\right)^{\phi(p-1)+1}.$$

Proof. Set $q_n = p^{n+1}d$ for $n \ge 0$. Then $F \subset \mathbb{Q}(\mu_{q_m})$. It is not difficult to deduce that (see the arguments in the proof of Theorem 7.15 in [9])

$$\lambda_p(F) \le \lambda_p(\mathbb{Q}(\mu_{q_m})).$$

But $\lambda_p(\mathbb{Q}(\mu_{q_m})) = \lambda_p(\mathbb{Q}(\mu_{q_0}))$, and, by [9, Proposition 13.32 and Theorem 7.13],

$$\lambda_p(\mathbb{Q}(\mu_{q_0})) \le 2 \sum_{\theta \text{ even},\, \theta \ne 1,\, f_\theta | q_0} \lambda(f(T,\theta)).$$

It remains to apply Theorem 4.5. ∎

Note that the bound of the last corollary is certainly far from being sharp even in the case $p = 3$ (see [4]).

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